Ruelle operator and transcendental entire maps

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Abstract

If f is a transcendental entire function with only algebraic singularities we calculate the Ruelle operator of f. Moreover, we prove both (i) if f has a summable critical point, then f is not structurally stable under certain topological conditions and (ii) if all critical points of f belonging to Julia set are summable, then there exists no invariant lines fields in the Julia set.

1. Introduction

If f is a transcendental entire map we denote by f^n, $n \in \mathbb{N}$, the n-th iterate of f and write the Fatou set as $F(f) = \{z \in \mathbb{C};$ there is some open set U containing z in which $\{f^n\}$ is a normal family $\}$. The complement of $F(f)$ is called the Julia set $J(f)$. We say that f belongs to the class S_q if the set of singularities of f^{-1} contains at most q points. Two entire maps g and h are topologically equivalent if there exist homeomorphisms $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that $\psi \circ g = h \circ \phi$.

If we denote by M_f, $f \in S_q$ the set of all entire maps topologically equivalent to f we can define on M_f as in [1] a structure of $(q+2)$ dimensional complex manifold.

Fatou's conjecture states that the only structurally stable maps on M_f are the hyperbolic ones. This conjecture is false in the case when there is an invariant line field in the Julia set of f, our result is a partial answer to this conjecture for transcendental entire maps with only finite number of algebraic singularities.

P. Makienko [6, 7] and G.M. Levin [5] have studied the Ruelle operator and the invariant line fields for rational maps, the idea of this work is to study an application of the proposed approach given in [7] for transcendental entire functions in class S_q, where the singularities of f^{-1} are only algebraic.

Assumptions on maps. From now on we will assume that

1. f is transcendental entire and that the singularities of f^{-1} are algebraic and finite and all critical points are simple (that is $f''(c) \neq 0$).

2. It follows from a very well known result of complex variables that there exist a decomposition
$$\frac{1}{f'(z)} = \sum_{i=1}^{\infty} \left(\frac{b_i}{z - c_i} - p_i(z) \right) + h(z),$$

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where p_i are polynomials, $h(z)$ is an entire function, $\{c_i\}$ are the critical points of f, and $b_i = \frac{1}{f''(c_i)}$ are constants depending on f. Now we assume that the series

$$\sum_{i=1}^{\infty} \frac{b_i}{c_i^3}$$

is absolutely convergent.

Note that elements of generic subfamily of the family $P_1(z) + P_2(sin(P_3(z)))$ satisfy to assumptions above, here $P_i(z)$ are polynomials.

Let $F_{n,m}$ the space of forms of the kind $\phi(z)Dz^m D\overline{z}^n$. Consider two formal actions of f on $F_{n,m}$, say $f^*_{n,m}$ and $f_{*n,m}$, on a function ϕ at the point z by the formulas

$$f^*_{n,m}(\phi) = \sum \phi(\xi_i)(\xi_i')^n(\overline{\xi_i'})^m = \sum_{y\in f^{-1}(z)} \frac{\phi(y)}{(f'(y))^n(\overline{f'(y)})^m},$$

and

$$f_{*n,m}(\phi) = \phi(f)\cdot(f')^n\cdot\overline{(f')^m},$$

where $n, m \in \mathbb{Z}$ and $\xi_i, i = 1, ..., d$ are the branches of the inverse map f^{-1}. As in [6] we define

1. The operator $f^* = f^*_{2,0}$ as the Ruelle operator of f.
2. The operator $|f^*| = f^*_{1,1}$ as the modulus of the Ruelle operator.
3. The operator $B_f = f_{*-1,1}$ as the Beltrami operator of f.

Let c_i be the critical points of f and $Pc(f) = \overline{\cup_i \cup_{n\geq 0} f^n(f(c_i))}$ be the postcritical set.

Lemma 1. *Let $Y \subset \widehat{\mathbb{C}}$ be completely invariant measurable subset respect to f. Then*

1. *$f^* : L_1(Y) \to L_1 Y$ is linear endomorphism "onto" with $\|f^*\|_{L_1(Y)} \leq 1$;*
2. *Beltrami operator $B_f : L_\infty(Y) \to L_\infty(Y)$ is dual operator to f^*;*
3. *if $Y \subset \left\{\widehat{\mathbb{C}}\backslash\overline{\cup_i f^{-i}(Pc(f))}\right\}$ is an open subset and let $A(Y) \subset L_1(Y)$ be subset of holomorphic functions, then $f^*(A(Y)) \subset A(Y)$;*
4. *fixed points on the modulus of Beltrami operator define a non- negative absolutely continuous invariant measure in $\mathbb{C}$.*

Observe that all items above follow from definitions.

Definition. The space of quasi-conformal deformations of a given map f, denoted by $qc(f)$, is defined as.

$$qc(f) = \big\{g \in M_f : \text{ there is a quasiconformal automorphism } h_g \text{ of the Riemann sphere } \widehat{\mathbb{C}} \text{ such that } g = h_g \circ f \circ h_g^{-1}\big\}/A_{\text{ff}}(\mathbb{C}),$$

where $A_{ff}(\mathbb{C})$ is the affine group.

Definition. For $f \in S_q$ structurally stable the space of all grand orbits of f on $\widehat{\mathbb{C}} \backslash \overline{\{\cup_i f^i(P_c(f))\}}$ forms a disconnected Riemann surface, say $S(f)$, of finite quasi-conformal type, see for details [9].

Definition. A point $a \in f$ is called "summable" if and only if either

1. the set $X_a(f) = \overline{\{\cup_n f^n(f(a))\}}$ is bounded and the series
$$\sum_{i=0} \frac{1}{(f^i)\prime(f(a))}$$
is absolutely convergent or
2. the set $X_a(f)$ is unbounded and the series
$$\sum_{i=0} \frac{1}{(f^i)\prime(f(a))} \text{ and } \sum_{i=0} \frac{|f^n(f(a))|| \ln |f^n(f(c))||}{(f^i)\prime(f(a))}$$
are absolutely convergent.

Definition. Let X be the space of transcendental entire maps $f \in S_q$, fixing $0, 1$, with summable critical point $c \in J(f)$ and either

1. $f^{-1}(f(c))$ is not in $X_c(f)$,
2. $X_c(f)$ does not separate the plane,
3. $m(X_c) = 0$, where m is the Lebesgue measure,
4. $c \in \partial D \subset J(f)$, where D is a component of $F(f)$.
Note that (4) includes the maps with completely invariant domain.

The main results of this work are Theorems A and B for transcendental entire maps. In [7] the theorems were proved for rational maps. The big differences between them is that for transcendental entire maps infinity is an essential singularity and there are not poles.

Theorem A. *Let $f \in X$. If f has a summable critical point, then f is not structurally stable map.*

Definition. Denote by W the space of transcendental entire maps in S_q such that:

1. There is no parabolic points for $f \in W$.
2. All critical point are simple (that is $f''(c) \neq 0$) and the forward orbit of any critical point c is infinite and does not intersect the forward orbit of any other critical point.
3. f satisfies (1) to (5) in the above definition, for all critical points of f.

Conditions (1) and (2) are required for simplicity of the proof but they are not relevant.

Definition. We call a transcendental entire map f summable if all critical points belonging to the Julia set are summable.

Theorem B. *If $f \in W$ is summable, then there exists no invariant line fields on $J(f)$.*

Remark: A theorem of McMullen [8] states that for the full family $f_{a,b} = a + b \sin z$ has $m(J(f)) > 0$, then the arguments of J. Rivera Letelier [10] make non sense in this case .

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2. Bers map

Let $\phi \in L_\infty(\mathbb{C})$ and let $B_f(\phi) = \phi(f)\frac{\overline{f'}}{f'} : L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C})$ be the Beltrami operator. Then the open unit ball B of the space $Fix(B_f) \subset L_\infty(\mathbb{C})$ of fixed points of B_f is called *the space of Beltrami differentials for* f and describe all quasi-conformal deformations of f.

Let $\mu \in Fix(B_f)$, then for any λ with $|\lambda| < \frac{1}{\|\mu\|}$ the element $\mu_\lambda = \lambda\mu \in B \subset Fix(B_f$. Let h_λ be quasi-conformal maps corresponding to Beltrami differentials μ_λ with $h_\lambda(0, 1, \infty) = (0, 1, \infty)$. Then the map

$$\lambda \to f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1} \in M_f$$

is a conformal map. If $f_\lambda(z) = f(z) + \lambda G_\mu(z) + ...$, then differentiation respect to λ in the point $\lambda = 0$ gives the following equation

$$F_\mu(f(z)) - f'(z)F_\mu(z) = G_\mu(z),$$

where $F_\mu(z) = \frac{\partial f_\lambda(z)}{\partial \lambda}|_{\lambda=0}$.

Remark 1. Due to quasiconformal map theory (see for example [4]) for any $\mu \in L_\infty(\mathbb{C})$ with $\|\mu\|_\infty < \epsilon$ and small ϵ, there exists the following formula for quasi-conformal f_μ fixing $0, 1, \infty$.

$$f_\mu(z) = z - \frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(xi) d\xi \wedge d\bar{\xi}}{\xi(\xi - 1)(\xi - z)} + C(\epsilon, f)\|\mu\|_\infty^2,$$

where $|z| < f$ and $C(\epsilon, f)$ is constant does not depending on μ. Then

$$F_\mu(z) = \frac{\partial f_\lambda}{\partial \lambda}_{|\lambda=0} = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(xi) d\xi \wedge d\bar{\xi}}{\xi(\xi - 1)(\xi - z)}.$$

Hence we can define the linear map $\beta : Fix(B_f) \to H^1(f)$ by the formula, where $H^1(f)$ is defined below

$$\beta(\mu) = F_\mu(f(z)) - f'(z)F_\mu(z).$$

We call β *the Bers map* as an analogy with Kleinian group (see for example [3]).

Let $A(S(f))$ be the space of quadratic holomorphic integrable differentials on disconnected surface $S(f)$. Let $HD(S(f))$ be the space of harmonic differentials on $S(f))$. In every chart every element $\alpha \in HD(S(f))$ has a form $\alpha = \frac{\overline{\phi} dz^2}{\rho^2 |dz|^2}$, where $\phi dz^2 \in A(S(f))$ and $\rho|dz|$ is the Poincare metric. Let $P : \overline{\overline{\mathbb{C}}} \setminus \{\overline{\cup_i f^{-i}(Pc(f))}\} \to S(f)$ be the projection. Then the pull back $P_* : HD(S(f)) \to Fix(B_f)$ defines a linear injective map.

The space $HD(f) = P_*(A(S(f)))$ is called *the space of harmonic differentials.* For any element $\alpha \in HD(f)$ the support $supp(\alpha) \in F(f)$. Then $dim(HD(f)) = dim(A(S(f)))$.

Let $J_f = Fix(B_f)_{|J(f)}$ be the space of invariant Beltrami differentials supported by Julia set.

Now define $H(f) = \{\varphi : \Delta \to C_f$ such that $\varphi(\lambda) = f + \lambda f_1 + \lambda^2 f_2 + ...,$ for λ very small$\}$, and $C_f = \{g \in M_f : g(0) = 0$ and $g(1) = 1\} \subset M_f$.

We can define an equivalence relation $\sim$ on $H(f)$ in the following way, $\varphi_1 \sim \varphi_2$ if and only if $\frac{\partial(\varphi_1 - \varphi_2)}{\partial\lambda}|_{\lambda=0} = 0$.

Definition. $H^1(f) = H(f)/\sim$.

Observe that (i) $H^1(f)$ is linear complex space and (ii) there exists an injection Ψ such that $\Psi : H^1(f) \to T_f(C_f) =$ Complex tangent space.

Theorem 2.1. *Let f be structurally stable transcendental entire map. Then $\beta : HD(f) \times J_f \to H^1(f)$ is an isomorphism.*

In structurally unstable cases β restricted on $HD(f) \times J_f$ is always injective.

Proof. The map f is structurally stable hence $dim(qc(f)) = dim(HD(f) \times J_f) = dim(H^1(f)) = dim(M_f/A_{\text{ff}}(\mathbb{C})) = q$. If we show that β is *onto*, then we are done.

Let f_1 be any element of $H^1(f)$. There exists a function $\varphi(\lambda)$ such that for λ sufficiently small $\varphi(\lambda) \subset C_f$ (since f is structurally stable). Then $\varphi(\lambda) = f_\lambda$ is a holomorphic family of transcendental entire maps, thus $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$, where h_λ is a holomorphic family of quasi-conformal maps. Hence

$$f_1(z) = V(f(z)) - f'V(z),$$

where $V = \frac{\partial h_\lambda}{\partial \lambda}|_{\lambda=0}$. The family of the complex dilatations $\mu_\lambda(z) = \frac{\overline{\partial} h_\lambda(z)}{\partial h_\lambda(z)} \in Fix(f)$ forms a meromorphic family of Beltrami differentials. If $\mu_\lambda(z) = \lambda\mu_1(z) + \lambda^2\mu_2(z) + ...,$ where $\mu_i(z) \in Fix(f)$. Then

$$\frac{\partial h_\lambda}{\partial \lambda}_{|\lambda=0} = -\frac{z(z-1)}{\pi} \iint \frac{\mu_1(\xi) d\xi d\bar{\xi}}{\xi(\xi-1)(\xi-z)} = F_{\mu_1}(z)$$

and hence $F_{\mu_1}(f(z)) - f'(z)F_{\mu_1}(z) = f_1$.

If we let $\nu = \mu_{1|F(f)}$, then we can state the following claim.

Claim. *There exists an element $\alpha \in HD(S(f))$ such that $\beta(\alpha) = \beta(\nu)$.*

Proof of the claim. We will use here quasi-conformal theory (see for example the books of I. Kra [3] and S.L. Krushkal [4] and the papers of C. McMullen and D. Sullivan [8], [9]). Let ω be the Beltrami differential on $S(f)$ generated by ν (that is $P_*(\omega) = \nu$). Let $< \psi, \phi >$ be the Petersen scalar product on $S(f)$, where $\phi, \psi \in A(S(f))$ and

$$< \psi, \phi >= \iint_{S(f)} \rho^{-2} \overline{\psi} \phi,$$

where ρ is hyperbolic metric on disconnected surface $S(f)$. Then by (for example) Lemmas 8.1 and 8.2 of chapter III in [3] this scalar product defines a Hilbert space structure on $A(S(f))$. Then there exists an element $\alpha' \in HD(S(f))$ such that equality

$$\iint_{S(f)} \omega\phi = \iint_{S(f)} \alpha'\phi$$

holds for all $\phi \in A(S(f))$.

Now let $A(O)$ be space of all holomorphic integrable functions over O, where $O = \{\overline{\mathbb{C}\backslash\cup_i f^{-i}(Pc(f))}\} \subset F(f)$. Then the push forward operator $P^* : A(O) \to A(S(f))$ is dual to the pull back operator P_*. Hence element $P_*(\alpha')$ satisfies the next condition

$$\iint_O \nu g = \iint_O P_*(\alpha')g,$$

for any $g \in A(O)$.

All above means that $\iint P_*(\alpha)\gamma_a(z) = \iint \nu\gamma_a(z)$ for all $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}, a \in J(f)$. Hence the transcendental entire maps $\beta(P_*(\alpha))(a) = \beta(\nu)(a)$ on $J(f)$ and we have the desired result with $\alpha = P_*(\alpha')$. Thus the claim and the theorem are proved.

3. Calculation of the Ruelle operator

Let us recall that from above there exist a decomposition

$$\frac{1}{f'(z)} = \sum_{i=1}^{\infty}\left(\frac{b_i}{z-c_i} - p_i(z)\right) + h(z), \tag{1}$$

where p_i are polynomials, $h(z)$ is an entire function, $\{c_i\}$ are the critical points of f, and $b_i = \frac{1}{f''(c_i}$ and the series $\sum \frac{b_i}{c_i^3}$ is absolutely convergent.

In order to use Bers' density theorem and the infinitesimal formula of quasi-conformal maps, see Remark 1. We will work with linear combinations of the following functions.

$$\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)} \in L_1(\mathbb{C}),$$

where $a \in \mathbb{C} \setminus \{0, 1\}$.

Proposition 3.1. *Let $\gamma_a(z)$ as above. If f is any transcendental entire map with simple critical points, then*

$$f^*(\gamma_a(z)) = \frac{\gamma_{f(a)}(z)}{f'(a)} + \sum_{i=1} b_i\gamma_a(c_i)\gamma_{f(c_i)}(z).$$

The coefficients b_i and c_i comes from (1).

Proof. Let $\varphi \in C^\infty(S)$, where $S = \mathbb{C} \setminus \{0, 1\}$, with compact support, denoted by $supp(\varphi)$ and $\varphi(0) = \varphi(1) = 0$. Now consider the following:

$$\int_{\mathbb{C}} \varphi_{\bar{z}} f^*\gamma_a(z) dz \wedge d\bar{z} = \int_{\mathbb{C}} \frac{(\varphi_{\bar{z}} \circ f)\bar{f}'}{f'(z)}\gamma_a(z) dz \wedge d\bar{z} = \int_{\mathbb{C}} \frac{(\varphi \circ f)_{\bar{z}}}{f'(z)}\gamma_a(z) dz \wedge d\bar{z}$$

the first equality is by the duality with the Beltrami operator, see Lemma 1 in Section 1.

Let us denote by $\psi = \varphi(f)$, so $supp(\psi) = f^{-1}supp(\varphi)$ and is the union $\bigcup K_i$ of compact sets if there is not asymptotic values on it. Hence applying the decomposition in (1) of $1/f'(z)$ we have

$$\int_{\mathbb{C}} \frac{(\varphi\circ f)_{\bar z}}{f'(z)}\gamma_a(z)dz\wedge d\bar z = \sum\int_{\mathbb{C}}\psi_{\bar z}\frac{a(a-1)b_i}{(z-c_i)z(z-1)(z-a)}dz\wedge d\bar z -$$
$$-\sum\int_{\mathbb{C}}\psi_{\bar z}p_i(z)\gamma_a(z)dz\wedge d\bar z + \int_{\mathbb{C}}\psi_{\bar z}h(z)\gamma_a(z)dz\wedge d\bar z =$$
$$=\sum b_i\int_{\mathbb{C}}\psi_{\bar z}\frac{a(a-1)}{z(z-1)}\left(\frac{1}{a-c_i}\right)\left(\frac{1}{z-a}-\frac{1}{z-c_i}\right)dz\wedge d\bar z - \sum\int_{\mathbb{C}}\psi_{\bar z}p_i(z)\gamma_a(z)dz\wedge d\bar z +$$
$$+\int_{\mathbb{C}}\psi_{\bar z}h(z)\gamma_a(z)dz\wedge d\bar z =$$
$$=\sum\frac{b_i}{a-c_i}\left(\int_C\psi_{\bar z}\gamma_a(z) - \frac{a(a-1)}{c_i(c_i-1)}\int_{\mathbb{C}}\psi_{\bar z}\gamma_{c_i}(z)\right)dz\wedge d\bar z -$$
$$-\sum\int_{\mathbb{C}}\psi_{\bar z}p_i(z)\gamma_a(z)dz\wedge d\bar z + \int_{\mathbb{C}}\psi_{\bar z}h(z)\gamma_a(z)dz\wedge d\bar z. \tag{2}$$

On the other hand making some calculations and applying Green's formula we have the following equalities.

$$\int_{\mathbb{C}}\psi_{\bar z}\frac{a(a-1)}{z(z-1)(z-a)}dz\wedge d\bar z = (a-1)\left(\int_{\partial supp\varphi}\frac{\psi_{\bar z}}{z} - a\int_{\partial supp\varphi}\frac{\psi_{\bar z}}{z-1} + \int_{\partial supp\varphi}\frac{\psi_{\bar z}}{z-a}\right)dz =$$

$$= (a-1)\psi(0) - a(\psi(1)) + \psi(a). \tag{3}$$

Since $\psi(0) = \varphi f(0) = \varphi(0) = 0$, also $\psi(1) = \varphi f(1) = 0$. Hence $\varphi(f(a)) = \psi(a) = (3)$.

Applying again Green's formula we have:

$$(3) = \varphi(f(a)) = \varphi(f(a)) + (f(a)-1)\varphi(0) - f(a)\varphi(1) = \int_{\mathbb{C}}\frac{\varphi_{\bar z}}{z-f(a)}dz\wedge d\bar z +$$

$$\int_{\mathbb{C}}\frac{(f(a)-1)\varphi_{\bar z}}{z}dz\wedge d\bar z - \int_{\mathbb{C}}\frac{f(a)\varphi_{\bar z}}{z-1}dz\wedge d\bar z = \int_{\mathbb{C}}\varphi_{\bar z}\gamma_{f(a)}(z)dz\wedge d\bar z$$

this proves

$$\int_{\mathbb{C}}\varphi_{\bar z}(f)\gamma_a(z)dz\wedge d\bar z = \int_{\mathbb{C}}\varphi_{\bar z}\gamma_{f(a)}(z)dz\wedge d\bar z. \tag{4}$$

Applying (4) on (2) we obtain

$$(2) = \sum\frac{b_i}{a-c_i}\left(\int_{\mathbb{C}}\varphi_{\bar z}\gamma_{f(a)}(z) - \frac{a(a-1)}{c_i(c_i-1)}\int_{\mathbb{C}}\varphi_{\bar z}\gamma_{f(c_i)}(z)\right)dz\wedge d\bar z -$$
$$-\sum\int_{\mathbb{C}}\varphi_{\bar z}(f)p_i(z)\gamma_{f(a)}(z)dz\wedge d\bar z + \int_{\mathbb{C}}\varphi_{\bar z}(f)h(z)\gamma_{f(a)}(z)dz\wedge d\bar z = \tag{5}$$

$$\int_{\mathbb{C}} \varphi_{\bar{z}} \gamma_{f(a)}(z) \left(\sum \frac{b_i}{a - c_i} - p_i(a) + h(a) \right) dz \wedge d\bar{z} + \sum \frac{a(a-1)b_i}{c_i(c_i-1)(c_i-a)} \int_{\mathbb{C}} \varphi_{\bar{z}} \gamma_{f(c_i)}(z) dz \wedge d\bar{z} =$$

$$\int_{\mathbb{C}} \varphi_{\bar{z}} \gamma_{f(a)}(z) \left(\frac{1}{f'(a)} \right) dz \wedge d\bar{z} + \sum \frac{a(a-1)b_i}{c_i(c_i-1)(c_i-a)} \int_{\mathbb{C}} \varphi_{\bar{z}} \gamma_{f(c_i)}(z) dz \wedge d\bar{z}.$$

Since $\gamma_a(c_i) = \frac{a(a-1)}{c_i(c_i-1)(c_i-a)}$, we have

$$\int_{\mathbb{C}} \varphi_{\bar{z}} f^* \gamma_a dz \wedge d\bar{z} = \int_{\mathbb{C}} \varphi_{\bar{z}} \left[\frac{\gamma_{f(a)}(z)}{f'(a)} + \sum b_i \gamma_a(c_i) \gamma_{f(c_i)}(z) \right] dz \wedge d\bar{z}.$$

Hence

$$\int_{\mathbb{C}} \varphi_{\bar{z}} (f^*(\gamma_a(z)) - \left[\frac{\gamma_{f(a)}(z)}{f'(z)} + \sum b_i \gamma_a(c_i) \gamma_{f(c_i)}(z) \right] dz \wedge d\bar{z} = 0.$$

This is true for each φ, so the function inside the integral is by Weyl's lemma an holomorphic function on $\mathbb{C} \setminus \{0, 1\}$ which is integrable. In our case, this implies that

$$f^* \gamma_a(z) = \frac{\gamma_{f(a)}(z)}{f'(a)} + \sum_{i=1} b_i \gamma_a(c_i) \gamma_{f(c_i)}(z).$$

4. Formal Relations of Ruelle Poincare Series

In this section we want to study properties of series of the form

$$\sum\nolimits_{n=0}^{\infty} x^n f^{*n}(\gamma_a(z))$$

where f^{*n} denotes the n-th iteration of the Ruelle operator. Observe from Section 2 that

$$f^{*0}(\gamma_a(z)) = \gamma_a(z)$$

$$f^*(\gamma_a(z)) = \frac{1}{f'(a)} \gamma_{f(a)}(z) + \sum b_i \gamma_a(c_i) \gamma_{f(c_i)}(z)$$

$$f^{*2}(\gamma_a(z)) = \frac{1}{f'(a)} (\frac{\gamma_{f^2(a)}(z)}{f'(f(a))} + \sum b_i \gamma_{f(a)}(c_i) \gamma_{f(c_i)}(z) + \sum b_i \gamma_a(c_i) f^* \gamma_{f(c_i)}(z) =$$

$$\frac{\gamma_{f^2(a)}(z)}{(f^2)'(a)} + \sum b_i \left(\frac{1}{f'(a)} \gamma_{f(a)}(c_i \gamma_{f(c_i)}(z) + \gamma_a(c_i) f^* \gamma_{f(c_i)} \right)$$

$$f^{*3}(\gamma_a(z)) = \frac{1}{(f^3)'(a)}\gamma_{f^3(a)}(z) +$$

$$\sum b_i \left(\frac{\gamma_{f^2(a)(c_i)}\gamma_{f(c_i)}(z)}{(f^2)'(a)} + \frac{\gamma_{f(a)}(c_i)}{f'(a)} f * (\gamma_{f(c_i)}(z)) + \gamma_a(c_i) f^{*2}(\gamma_{f(c_i)}(z) \right)$$

in general we have

$$f^{*n}(\gamma_a(z)) = \frac{1}{(f^n)'(a)}\gamma_{f^n(a)}(z) + \sum_i b_i c^i_{n-1}$$

for some coefficients c^i_j, determined by the *Cauchy's product* of two series $A = \sum a_i$ and $B = \sum b_i$ where $C = A \bigotimes B = \sum c_n$ and $c_n = \sum_{i=0}^n a_i b_{n-i}$.

Now define $S(a,z) = \sum_{n=0}^{\infty} f^{*n}(\gamma_a(z))$, $A(a,z) = \sum_{n=0}^{\infty} \frac{1}{(f^n)'(a)}\gamma_{f^n(a)}(z)$ and $S(f(c_i),z) \bigotimes A(a,z) = C^i = \sum_j c^i_j$. Thus we have

$$S(a,z) = A(a,z) + \sum b_i \sum_{n=0}^{\infty} c^i_{n-1} =$$

$$A(a,z) + \sum b_i [S(f(c_i),z) \bigotimes A(a,c_i)].$$

Define $S(x,a,z) = \sum x^n f^{*n}(\gamma_a(z))$ and $A(x,a,z) = \sum \frac{x^n}{(f^n)'(a)}\gamma_{f^n(a)}(z)$, since

$$x^n f^{*n}(\gamma_a(z)) = \frac{x^n}{(f^n)'(a)}\gamma_{f^n(a)}(z) + x \sum b_i c^i_{n-1} x^{n-1}$$

then $S(x,a,z) = A(x,a,z) + x \sum b_i \sum_{n=1}^{\infty} c^i_{n-1} x^{n-1} = A(x,a,z) + \sum b_i [S(x,f(c_i),z) \bigotimes A(x,a,c_i)]$ by the Cauchy's lemma on power series formula it can be written as $S(x,a,z) = A(x,a,z) + \sum b_i [S(x,f(c_i),z) A(x,a,c_i)]$ for all x in the disc of convergence of the series.

Lemma 2. *For all $|x| < 1$, $S(x,a,z) \subset L_1(\mathbb{C})$.*

Proof.

$$\int |S(x,a,z)| \leq \sum |x^n| \int |f^{*n}(\gamma_a(z))| = \sum |x^n| ||f^{*n}(\gamma_a(z))|| \leq ||\gamma_a(z)|| \sum |x^n| = \frac{||\gamma_a(z)||}{1-|x|} < \infty.$$

Lemma 3. *If a is summable with $a \in \mathbb{C}$, then $A(x,a,z) \in L_1(\mathbb{C})$ for all $|x| < 1$.*

Proof.

$$\int |A(x,a,z)| \leq \sum \frac{|x^n|}{|(f^n)'(a)|} ||\gamma_{f^n(a)}(z)||,$$

now by the properties of potential function we have

$$\int |\gamma_t(z)| \leq K|t||ln|t||$$

hence

$$\int |A(x,a,z)| \leq K\sum \frac{|x^n|}{|(f^n)'(a)|}|f^n(a)||lnf^n(a)| \leq Kmax_{y\in\bigcup f^n(a)}|y||ln|y||\sum \left|\frac{x^n}{(f^n)'(a)}\right| < \infty,$$

if the series $\{f^n(a)\}$ is bounded. If not apply that the series $\sum \frac{f^n(a)|ln|f^n(a)||}{(f^n)'(a)}$ absolutely converges. Then $A(x,a,z) \in L_1(C)$.

Corollary 1. *Under conditions of Lemma 3 we have* $lim_{x\to 1}||A(x,a,z) - A(1,a,z)|| = 0$ *in* $L_1(\mathbb{C})$.

Proof. Observe that we can choose N such that $2\sum_{i\geq N}|\frac{1}{(f^n)'(a)}| \leq \epsilon/2$, let δ such that $|1-x| < \delta$. We have that $|1-x^N|\sum_{i<N}|\frac{1}{(f^n)'(a)}| \leq \epsilon/2$. Hence

$$\sum\left|\frac{x^n-1}{(f^n)'(a)}\right| \leq \sum_{n<N}\left|\frac{x^n-1}{(f^n)'(a)}\right| + \sum_{n\geq N}\left|\frac{x^n-1}{(f^n)'(a)}\right| \leq \epsilon/2 + 2\sum_{n\geq N}\left|\frac{1}{(f^n)'(a)}\right| \leq \epsilon$$

so $lim_{x\to 1}\sum|\frac{x^n-1}{(f^n)'(a)}| = 0$, but

$$lim_{x\to 1}||A(x,a,z) - A(1,a,z)|| \leq Kmax_{y\in\bigcup f^n(a)}(|y||ln|y||)\sum \frac{x^n-1}{(f^n)'(a)},$$

if sequence $\{f^n(a)\}$ is bounded, otherwise use the absolute convergence of $\sum \frac{f^n(a)\ln((f^n)'(a)}{(f^n)'(a)}$, hence the corollary is proved.

Lemma 4. *If* $\sum \frac{1}{(f^n)'(a)}$ *converges absolutely, then* $\lim_{x\to 1}|A(x,a,c_i) - A(a,c_i)| = 0$.

Proof. Consider $X_a = \bar{\bigcup}_{n>0} f^n(a)$. If c_i is not in $X_a(f)$, then $A(a,c_i) = \sum \frac{1}{(f^n)'(a)}\gamma_{f^n(a)}(c_i)$ has no poles. Since $|\gamma_{f^n(a)}(c_i)| < 1/d^3$, for some constant d, it is bounded in Y_a.

If c_i is in $X_a(f)$, let $D_\epsilon = |z - c_i| < \epsilon$ so for $z \in D_\epsilon$, we can use the equality $f'(z) = (z-c_i)f''(c_i) + O(|z-c_i|^2)$ and obtain

$$\frac{1}{|f^{n_i} - c_i|} \leq \frac{|f''(c_i))| + O(|f^{n_i} - c_i|)}{|f'(f^{n_i}(a))|} \leq K\frac{1}{|f'(f^{n_i}(a))|},$$

hence

$$\left|\gamma_{f^{n_i}(a)}(c_i)\right| \leq K\frac{1}{|f'(f^{n_i}(a))|}\frac{|f^{n_i}(a)||f^{n_i}(a)-1|}{|c_i(c_i-1)|} \leq K_1\frac{1}{|f'(f^{n_i}(a))|},$$

where K and K_1 are constant depending only on ϵ and the points c_i. As result for all $|x| \leq 1$ we have

$$\left|\sum_i \frac{x^{n_i}}{(f^{n_i})'(a)}\gamma_{f^{n_i}(a)}(c_i)\right| \leq K_1 \sum_i \frac{|x|^{n_i}}{|(f^{n_i+1})'(a)|} < \infty.$$

This proves the Lemma. So we have that the following equality holds

$$S(x,a,z) = A(x,a,z) + x\sum b_i S(x,a,d_i)A(x,a,c_i).$$

5. Ruelle Operator and Line Fields

Let f be a transcendental entire map, we say that f admits an *invariant line field* if there is a measurable Beltrami differential μ on the complex plane $\mathbb{C}$ such that $B_f\mu = \mu$ a.e. $|\mu| = 1$ on a set of positive measure and μ vanishes else were. If $\mu = 0$ outside the Julia set $J(f)$, we say that μ is *carried on the Julia set*. See [8] for results of holomorphic line fields.

In Section 2 we consider the set $Fix(B_f) = \{\mu \in L_\infty(C) : B_f(\mu) = \mu\}$, with B_f being the Beltrami operator. Consider now the following integrals

$$\int_{\mathbb{C}} \mu S(x,a,z)dz\wedge d\bar{z} = \int \mu A(x,a,z)dz\wedge d\bar{z} + x\sum b_i A(x,a,c_i)\int \mu S(x,f(c_i),z)dz\wedge d\bar{z}.$$

The above equation is equal to the following expression, by the properties of the potential F_μ

$$\sum x^n \int \mu (f^*)^n\gamma_a(z) = \sum\frac{x^n}{(f^n)'(a)}F_\mu(f^n(a)) + x\sum b_i A(x,a,c_i)\int \mu\sum x^n(f^*)^n(\gamma_{f(c_i)}(z)) =$$

$$\sum\frac{x^n}{(f^n)'(a)}F_\mu(f^n(a)) + \frac{x}{1-x}\sum b_i A(x,a,c_i)F_\mu(f^n(c_i)).$$

By invariance of the Ruelle operator we have

$$\sum x^n\int \mu(f^*)^n\gamma_a(z) = \frac{\int \mu\gamma_a(z)}{1-x} = \frac{F_\mu(a)}{1-x} =$$

$$\sum\frac{1}{(f^n)'(a)}x^n F_\mu(f^n(a)) + \frac{x}{1-x}\sum b_i A(x,a,c_i)F_\mu(f^n(c_i)).$$

Hence

$$F_\mu(a) = (1-x)\sum\frac{1}{(f^n)'(a)}x^n F_\mu(f^n(a)) + x\sum b_i A(x,a,c_i)F_\mu(f^n(c_i)). \tag{6}$$

By Corollary 2 and Lemma 4 we can pass to the limit $x \to 1$ in (5), as a result we have

$$F_\mu(a) = \sum b_i A(1, a, c_i) F_\mu(f^n(c_i)).$$

By hypothesis d_1 is summable, so for $f \in S_q$

$$F_\mu(d_1)\left(1 - \sum_{\substack{c_k \\ f(c_k)=d_1}} b_k A(d_1, c_k)\right) = \sum_{i=2}^{q} F_\mu(d_i)\left(\sum_{\substack{c_l \\ f(c_l)=d_i}} b_l A(d_1, c_l)\right).$$

If we denote $\Psi_i = \sum_{c_k} b_k A(d_1, c_k)$ for $f(c_k) = d_i$, then we can rewrite the above equation as:

$$F_\mu(d_1)(1 - \Psi_1) = \sum_{i=2}^{q} F_\mu(d_i)\Psi_i. \tag{7}$$

Definition. We say that (6) is a trivial relation if and only if $\Psi_i = 0$, $i = 2, 3 \ldots q$ and $\Psi_1 = 1$.

For transcendental entire maps there are, in general, many critical points c_k which are mapped to the critical value d_1, even if the function is structurally stable.

Proposition 5.2. *If (6) is a non trivial relation, then f is unstable.*

Proof. By Hypothesis the set of $\{f(c_i)\}$ is finite. By equation (5) the Bers operator β induces an isomorphism

$$\beta^* : HD(f) \times J_f \to \mathbb{C}^q.$$

with coordinates $\beta^*(\mu) = \{F_\mu(d_1) \ldots F_\mu(d_q)\}$.

If (6) is a non trivial relation, then the relation gives a non trivial equation on the image of β^*, where the image of β^* is a subset of the set of solutions of this equation. Then $dim(HD(f) \times J_f) = dim(\text{image of } \beta^*) < q$. Thus the proposition is proved.

6. Fixed Point Theory

In this section we want to prove Theorems A and B which were stated in the introduction. In order to prove the theorems we will give a series of results.

Proposition 6.3. *If (6) is a trivial relation, then $f^*A(d_1, z) = A(d_1, z)$.*

Proof. Let us remember that $A(d_1, z) = \sum \frac{1}{(f^n)'(d_1)} \gamma_{f^n(d_1)}(z)$ and

$$f^*(\gamma_a(z)) = \frac{1}{f'(a)} \gamma_{f(a)}(z) + \sum b_i \gamma_a(c_i) \gamma_{f(c_i)}(z),$$

and so

$$f^*(A(d_1,z)) = A(d_1,z) - \gamma_{d_1}(z) + \sum\nolimits_i \gamma_{f(c_i)}(z)\left(\sum_{\substack{c_l \\ f(c_l)=d_i}} b_l A(d_1,c_l)\right) = A(d_1,z)$$

Denote by $Z = \overline{\bigcup_i f^i(d_1)}$ and $Y = \mathbb{C} \setminus Z$

Proposition 6.4. *If $\varphi = A(d_1,z) \neq 0$ on Y, then (6) is a non trivial relation.*

Before we prove the above proposition we will prove a series of results which will help us to prove the proposition.

Consider the modulus of the Ruelle operator: $|f^*|\alpha = \sum \alpha(\xi_i)|\xi_i'|^2$ where ξ_i are the inverse branches of z under the map f.

Lemma 5. $|f^*||\varphi| = |\varphi|$.

By hypothesis we have $\varphi = f^*\varphi = \sum \varphi(\xi_i)(\xi_i')^2$ For fixed i, denote by $\alpha_i(z) = \varphi(\xi_i)(\xi_i')^2$ and $\beta_i = \varphi - \alpha_i$. We have the following claim:

Claim. $|\alpha_i + \beta_i| = |\alpha_i| + |\beta_i|$, *for almost every point.*

Proof.

$$||\varphi|| = ||f^*\varphi|| = \int |\alpha_i + \beta_i| \leq \int |\alpha_i| + \int ||\beta_i| \leq ||\varphi||,$$

which implies that $\int |\alpha_i + \beta_i| = \int |\alpha_i| + \int |\beta_i|$. Now let $A = \{z : |\alpha_i(z) + \beta_i(z)| < |\alpha_i(z)| + |\beta_i(z)|\}$ with $m(A) \geq 0$, where m is the Lebesgue measure. Then $\int_{(\mathbb{C}\setminus A)\bigcup A} |\alpha_i + \beta_i| \int_A |\alpha_i + \beta_i| + \int_{\mathbb{C}-A} |\alpha_i + \beta_i| < \int_A |\alpha_i(z)| + |\beta_i(z)| + \int_{\mathbb{C}\setminus A} |\alpha_i(z)| + |\beta_i(z)|$, which is a contradiction, thus the claim is proved.

Now by induction on the claim, we have that $\sum|\alpha_i| = |\sum \alpha_i|$ and so $f^*|\varphi| = |\varphi|$. This proves Lemma 5.

Remark 2. The measure $\sigma(a) = \int \int_A |\phi(z)|$ is a non negative invariant absolutely continue probability measure, where $A \subset \widehat{\mathbb{C}}$ is a measurable set.

Definition. A measurable set $A \in \widehat{\mathbb{C}}$ is called back wandering if and only if $m(f^{-n}(A) \cap f^{-k}(A)) = 0$, for $k \neq n$.

Corollary 2. *If $\varphi \neq 0$ on Y, then (i) $J(f) = \widehat{\mathbb{C}}$, (ii) $m(Z) = 0$ and (iii) $\frac{\bar{\varphi}}{|\varphi|}$ defines an invariant Beltrami differential.*

Proof. (i) Every non periodic point of the Fatou set has a back wandering neighborhood. By Remark 2 we have that $\varphi = 0$. Thus $J(f) = \widehat{\mathbb{C}}$ and $\varphi \neq 0$ on every component of Y.

(ii) If $m(Z) > 0$, then $m(f^{-1}(Z)) > 0$ so $m(f^{-1}(Z) - Z) > 0$ since $f^{-1}(Z) \neq Z$, $Z \neq \mathbb{C}$, denote by $Z_1 = f^{-1}(Z) - Z$. Then Z_1 is back wandering thus $\varphi = 0$. Therefore, $m(Z) = 0$.

(iii) By using notations and the proof of Lemma 5 we have $f_i(x) = \frac{\alpha_i}{\beta_i} = \frac{\varphi}{\alpha_i} - 1$ so $\varphi = (1 + f_i(x))\alpha_i = (1 + f_i(x)(\varphi(\xi_i(x))(\xi_i')^2(x)$, with $x \in supp(\alpha_i)$. Consider $t = \xi_i(x)$, with $t \in \xi_i(supp(\alpha_i))$. Then $\varphi(f(t))(f')^2(t) = (1 + f_i(f(t)))\varphi(t)$. Hence,

$$\frac{\bar{\varphi}(x)}{|\varphi(x)|} = \frac{(1 + f_i(x))\bar{\varphi}(\xi_i(x))\left(\bar{\xi_i'}\right)^2(x)}{(1 + f_i(x))|\varphi(\xi_i(x)||(\xi_i')^2(x)|},$$

and so

$$\mu = \frac{\bar{\varphi}}{|\varphi|} = \frac{\bar{\varphi}(\xi_i)\bar{\xi_i'}}{|\varphi(\xi_i)|\xi_i'}$$

as result $\mu = \mu(\xi_i)\frac{\bar{\xi_i'}}{\xi_i'}$ is an invariant line field. Thus the corollary is proved.

Lemma 6. $\frac{\beta_j}{\alpha_j} = k_j \geq 0$ *is a non negative function.*

Proof. We have $|1 + \frac{\beta_j}{\alpha_j}| = 1 + \left|\frac{\beta_j}{\alpha_j}\right|$, then if $\frac{\beta_j}{\alpha_j} = \gamma_1^j + i\gamma_2^j$ we have

$$\left(1 + (\gamma_1^j)\right)^2 + (\gamma_2^j)^2 = \left(1 + \sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}\right)^2 = 1 + (\gamma_1^j)^2 + (\gamma_2^j)^2 + 2\sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}.$$

Hence $\gamma_2^j = 0$ and $\frac{\alpha_j}{\beta_j} = \gamma_1^j$ is a real-valued function but $\frac{\alpha_j}{\beta_j}$ is meromorphic function. So $\gamma_1^j = k_j$ is constant on every connected component of Y and the condition $|1 + k_j| = 1 + |k_j|$ shows $k_j \geq 0$.

Proof of Proposition 6.4.

Proof. Let us show first that all postcritical values are in Z. Assume that there is some $d_i \in Y$, then by the Lemma 6, $\varphi(z) = (1 + k_j)\varphi(\xi_j(z))\xi'^2_j(z)$. Assume that the branch $\xi_j(z)$ is such that tends to c_i when z tends to d_i. Then ξ'^2_j tends to ∞ and so $\varphi(c_i) = 0$. Also for every k_j we have that $\varphi(c_i) = (1 + k_j)\varphi(\xi_j(c_i))\xi'^2_j(c_i)$, so $\varphi(\xi_j(c_i)) = 0$. This implies that if c is a preimage of a critical point, then $\varphi(c) = 0$, since $J = \mathbb{C}$ then $\varphi = 0$ in Y, which is a contradiction.

Let us show now that $Z = \bigcup f^i(d_1)$. We will use a McMullen argument like in [8]. By Lemma 5 and Corollary 3, $\mu = \frac{\bar{\varphi}}{|\varphi|}$ is an invariant line field. That implies that φ is dual to μ and it is defined up to a constant. We will construct a meromorphic function ψ, dual to μ and such that ψ has finite number of poles on each disc D_R of radius R centered at 0.

For that suppose that for $z \in \mathbb{C}$ there exists a branch g of a suitable f^n, such that $g(U_z) \in Y$, where U_z is a neighborhood of z. Then define $\psi(\xi) = \varphi(g(\xi))(g')^2(\xi)$, for all

$\xi \in U_z$. Note that $\psi(\xi)$ is dual to μ and has no poles in U_z. If there is no such branch g, then ξ is in the postcritical set, and there is a branch covering F from a neighborhood of ξ to U_z, then define $\psi(\xi) = F^*(\varphi)$, with F^* the Ruelle operator of F. The map ψ is a meromorphic function dual to μ in U_z and has finite number of poles.

By the discussion above it is possible to construct a meromorphic function dual to μ in any compact disc D_R. If we make R tends to ∞, we have a meromorphic function ψ defined in $\mathbb{C}$ dual to μ and with a discrete set of poles.

Observe now that such ψ is holomorphic in Y, then Z is discrete and so $Z = \bigcup f^i(d_1)$ as we claim. Since every postcritical set is in Z, that implies that f is unstable and this proves the proposition.

The following propositions can be found in [7]. For completeness we prove them.

Proposition 6.5. *Let $a_i \in \mathbb{C}, a_i \neq a_j$, for $i \neq j$ be points such that $Z = \overline{\cup_i a_i} \subset \mathbb{C}$ is a compact set. Let $b_i \neq 0$ be complex numbers such that the series $\sum b_i$ is absolutely convergent. Then the function $l(z) = \sum_i \frac{b_i}{z-a_i} \neq 0$ identically on $Y = \mathbb{C}\backslash Z$ in the following cases*

1. *the set Z has zero Lebesgue measure*
2. *if diameters of components of $\mathbb{C}\backslash Z$ uniformly bounded below from zero and*
3. *If O_j denote the components of Y, then $\cup_i a_i \in \cup_j \partial O_j$.*

Proof Assume that $l(z) = 0$ on Y. Let us calculate derivative $\overline{\partial} l$ in sense of distributions, then $\omega = \overline{\partial} l = \sum_i b_i \delta_{a_i}$ and by standard arguments

$$l(z) = -\int \frac{d\omega(\xi)}{\xi - z}.$$

Such as $a_i \neq a_j$ for $i \neq j$, then measure $\omega = 0$ iff all coefficients $b_i = 0$.

Let us check (1). Otherwise in this case we have that the function l is locally integrable and $l = 0$ almost everywhere and hence $\omega = \overline{\partial} l = 0$ in sense of distributions and hence $\omega = 0$ as a functional on space of all continuous functions on Z which is a contradiction with the arguments above.

2) Assume that $l = 0$ identically out of Z. Let $R(Z) \subset C(Z)$ denote the algebra of all uniform limits of rational functions with poles out of Z in the sup $-$topology, here $C(Z)$ as usually denotes the space of all continuous functions on Z with the sup $-$norm. Then measure ω denote a lineal functional on $R(Z)$. The items (2) and (3) are based on the generalized Mergelyan theorem (see [2]) which states *If diameters of all components of $\mathbb{C}\backslash Z$ are bounded uniformly below from 0, then every continuous function holomorphic on interior of Z belongs to $R(Z)$.*

Let us show that ω annihilates the space $R(Z)$. Indeed let $f(z) \in R(Z)$ be a transcendental entire map and γ enclosing Z close enough to Z such that $f(z)$ does not have poles in interior of γ. Then such that $l = 0$ out of Z we only apply Fubini's theorem

$$\int r(z) d\omega(z) = \int d\omega(z) \frac{1}{2\pi i} \int_\gamma \frac{r(\xi) d\xi}{\xi - z} = \frac{1}{2\pi i} \int_\gamma r(\xi) d\xi \int \frac{d\omega(z)}{\xi - z} = \frac{1}{2\pi i} \int_\gamma r(\xi) l(\xi) d\xi = 0.$$

Then by generalized Mergelyan theorem we have $R(Z) = C(Z)$ and $\omega = 0$. Contradiction.

Now let us check (3). We **claim** *that $l = 0$ almost everywhere on $\cup_i \partial O_i$.*

Proof of the claim. Let $E \subset \cup_i \partial O_i$ be any measurable subset with positive Lebesgue measure. Then the function $F_E(z) = \iint_E \frac{dm(\xi)}{\xi - z}$ is continuous on $\mathbb{C}\backslash \cup_i O_i$ and is holomorphic onto interior of $\mathbb{C}\backslash\cup_i O_i$. Again by generalized Mergelyan theorem $F_E(z)$ can be approximated on $\mathbb{C}\backslash \cup_i O_i$ by functions from $R(\mathbb{C}\backslash \cup_i O_i)$ and hence by arguments above and by assumption we have $\int F_E(z) d\omega(z) = 0$. But again application of Fubini's theorem gives

$$0 = \int F_E(z) d\omega(z) = \int d\omega(z) \iint_E \frac{dm(\xi)}{\xi - z} = \iint_E dm(\xi) \int d\omega(z) \frac{1}{\xi - z} = \iint l(\xi) d(m(\xi)).$$

Hence for any measurable $E \subset \cup_i \partial O_i$ we have $\iint_E l(z) = 0$. The claim is proved. Now for any component $O \in Y$ and any measurable $E \subset \partial O$ we have $\iint_E l(z) = 0$. By assumption $l = 0$ almost everywhere on $\mathbb{C}$. Contradiction thus the proposition is proved.

Proposition 6.6. *If $f \in X$, then $A(d_1, z) = \varphi(z) \neq 0$ identically on Y in the following cases*

1. *if $f^{-1}(d_1) \notin X_{c_1}$,*
2. *if diameters of components of Y are uniformly bounded below from 0,*
3. *If $m\left(X_{c_1}\right) = 0$, where m is the Lebesgue measure on $\mathbb{C}$,*
4. *if $X_{c_1} \subset \cup_i \partial D_i$, where D_i are components of Fatou set.*

Proof Let us prove (1). If f is structurally stable then relation (3) is trivial.

Assume now that the set X_{c_1} is bounded. Then by Proposition 6.5 we have that $\varphi(z) = \frac{C_1}{z} + \frac{C_2}{z-1} + \sum \frac{1}{(f^i)'(d_1)(z - f^i(d_1))} = l(z) \neq 0$. Other cases follows directly from Proposition 6.5 also.

Now let X_{c_1} be unbounded. Let $y \in \mathbb{C}$ be a point such that the point $1 - y \in Y$, then the map $g(z) = \frac{yz}{z+y-1}$ maps X_{c_1} into $\mathbb{C}$. Let us consider the function $G(z) = \frac{1}{z} \sum_i \frac{(f^i(d_1)-1)}{(f^i)'(d_1)} - \frac{1}{z-1} \sum_i \frac{f^i(d_1)}{(f^i)'(d_1)} + \sum \frac{1}{(f^i)'(d_1)(z - g(f^i(d_1)))}$, then by proposition 6.5 $G(z) \neq 0$ identically on $g(Y)$.

Now we **Claim** that *Under condition of theorem A*

$$G(g(z)) g'(z) = \phi(z).$$

Proof of claim. Let us define $C_1 = \sum_i \frac{(f^i(d_1)-1)}{(f^i)'(d_1)}$ and $C_2 = \sum_i \frac{f^i(d_1)}{(f^i)'(d_1)}$ then we have

$$\frac{C_1}{g(z)} = \frac{C_1(z+y-1)}{yz} \text{ and } \frac{C_2}{g(z)-1} = \frac{C_2(z+y-1)}{(y-1)(z-1)}$$

and for any n

$$\frac{1}{g(z) - g(f^n(d_1))} = \frac{(z+y-1)(f^n(d_1)+y-1)}{y(y-1)(z - f^n(d_1))} = \frac{1}{y(y-1)} \left(\frac{(z+y-1)^2}{z - f^n(d_1)} + 1 - y - z \right),$$

then

$$G(g(z)) = \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)} + \sum \frac{1}{(f^i)'(d_1)(g(z) - g(f^i(d_1))} =$$
$$= \frac{1}{y(y-1)} \left((1-y-z) \sum \frac{1}{(f^i)'(d_1)} + (z+y-1)^2 \sum \frac{1}{(f^i)'(d_1)(z - f^i(d_1)} + \right.$$
$$\left. + \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)} \right) = *$$

and

$$* = \frac{1}{g'(z)}\left(\phi(z) - \frac{\sum_i \frac{f^i(d_1)-1}{(f^i)'(d_1)}}{z} + \frac{\sum_i \frac{f^i(d_1)}{(f^i)'(d_1)}}{z-1} + \frac{\sum \frac{1}{(f^i)'(d_1)}}{1-y-z} + \frac{C_1(y-1)}{z(z+y-1)} - \frac{C_2 y}{(z-1)(z+y-1)}\right) =$$

$$= \frac{\phi(z)}{g'(z)}.$$

Hence $\phi(z) = 0$ identically on Y if and only if $G(z) = 0$ identically on $g(Y)$. So by proposition 6.5 we complete this proposition.

Proof of Theorem A

Theorem A. *Let $f \in X$. If f has a summable critical point, then f is not structurally stable map.*

Proof. It follows from Proposition 6.6 that $\varphi \neq 0$ on Y, then (6) is non a trivial relation by Proposition 6.4, then applying Proposition 5.2 the map f is not stable. Therefore Theorem A is proved.

Corollary A. *Let f transcendental entire map with summable critical point $c \in J(f)$. If $\varphi \neq 0$ onto $\mathbb{C}\backslash X_c$, then f is an unstable map.*

Proof of Theorem B

Theorem B. *If $f \in W$ is summable, then there exists no invariant line fields on $J(f)$.*

Proof. As we observe in equation (6) in Section 4, each summable critical point restricts the image of the β operator. The image of the β operator , belongs to the common solutions of the equations

$$F_\mu(f(c_i)(1 - b_i A(c_i, f(c_i)) = \sum_{i \neq j} b_j F_\mu(f(c_j) A(c_j, f(c_j)$$

for all $c_i \in J(f)$, hence if this system is linearly independent, then the dimension of the image of β will be 0 and so we will have $J_f = \emptyset$. So we have to assume that the above system is linearly dependent.

That means in this case, that there are constants B_i such that the function $\varphi(z) = \sum B_i A(z, f(c_i))$ is a fixed point of the Ruelle operator f^*. As in the Lemmas above, the measures

$$\frac{\partial \varphi}{\partial \overline{z}} = \sum_i B_i \sum_n \frac{\delta_{f^n(f(c_i))}}{(f^n)'(f(c_i))} = 0.$$

Then $B_i = 0$, this proves Theorem B.

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