Principal configurations and umbilicity of submanifolds in $\mathbb{R}^N$

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Abstract

We consider the principal configurations associated to smooth vector fields ν normal to a manifold M immersed into a euclidean space and give conditions on the number of principal directions shared by a set of k normal vector fields in order to guaranty the umbilicity of M with respect to some normal field ν. Provided that the umbilic curvature is constant, this will imply that M is hyperspherical. We deduce some results concerning binormal fields and asymptotic directions for manifolds of codimension 2. Moreover, in the case of a surface M in $\mathbb{R}^N$, we conclude that if $N > 4$, it is always possible to find some normal field with respect to which M is umbilic and provide a geometrical characterization of such fields.

1 Introduction

Given an n-manifold M of codimension k in euclidean space, any normal field ν on M defines a system of curvature lines that we call principal configuration associated to ν. For each ν-principal direction we have an associated ν-principal curvature. A point p of M is said to be ν-umbilic if the n principal curvatures coincide at p. Such points are critical points of the ν-principal configuration. The manifold M is called ν-umbilic provided all its points are ν-umbilic. In this case we say that the ν-umbilic

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configuration is trivial (in the sense that all the curvature lines reduce to critical points). The behaviour of these systems in the case of surfaces immersed in $\mathbb{R}^4$ was analyzed in [17]. A generic surface immersed in $\mathbb{R}^4$ admits a family of principal configurations [14]. Nevertheless, it was proven in [16] that the ν-umbilicity of the surface is equivalent to having a fixed principal configuration associated to every normal field $\eta \neq \nu$.

Our aim in this paper is to generalize this result to the case of n-manifolds, $n \geq 2$, immersed in euclidean space with codimension $k \geq 2$. To do this, we observe that if S_{ν_i}, $i = 1, 2$, are the shape operators associated to normal fields ν_i, $i = 1, 2$, on M, the ν_i-configurations coincide if and only if S_{ν_1} and S_{ν_2} have the same eigenvectors. Based on this fact we can show that

If M is a generic n-manifold of codimension $k \geq 2$ with a unique non-trivial principal configuration, then there exists some normal field ν (locally defined at each point) on M such that M is ν-umbilical.

Moreover, this requirement can be weakened in the following sense: Depending on the relative values of the dimension n and the codimension k of M, it is sufficient to ask that k linearly independent normal fields on M share just a certain number $\delta(n, k) \leq n$ of curvature lines. This number decreases with k (see Section 3). In fact, a moment arrives in which it becomes zero, so we can conclude:

Provided M is immersed with high enough codimension, then it is always possible to find (locally at every point) some subbundle U of NM such that M is ν-umbilical for any section ν of the subbundle U.

In particular we have:

A generic n-manifold M immersed in $\mathbb{R}^{\frac{1}{2}n(n+3)}$ admits an everywhere locally defined normal field ν such that M is ν-umbilical.

We must drive the attention to the fact that the linear subspace, N_1M, spanned by the second fundamental form in the normal space, NM, at each point of an n-manifold M, has at most dimension $\frac{1}{2}n(n+1)$. This, clearly, implies that, provided M is immersed in $\mathbb{R}^N$ with $N > \frac{1}{2}n(n+3)$, there exists a subbundle of NM all whose sections make M umbilical with vanishing associated curvature. Now, the subtlety of our result consists in guaranteing the umbilicity with respect to normal fields with non necessarily vanishing associated curvature, even when M is immersed with codimension $\frac{1}{2}n(n+1)$.

We analyze with more detail the following two situations:

a) Submanifolds of codimension 2 (Section 4): In this case we see that having a unique non-trivial configuration is a *sufficient and necessary* condition for ν-umbilicity. We observe that this provides an exact generalization of our previous result for surfaces in 4-space [16].

b) Surfaces immersed in $\mathbb{R}^N, N \geq 5$ (Section 5): We study their ν-umbilicity properties in terms of the curvature ellipse at each point. The concept of curvature ellipse at a point of a surface in $\mathbb{R}^N$ was first introduced by [11] and has also been studied in [5] and [12]. We characterize geometrically the U-subbundle for which M is umbilical as the $(N-4)$-subbundle of NM which is orthogonal to the plane determined by the curvature ellipse at each point.

We point out that different properties of U-umbilic submanifolds have been studied by B. Y. Chen and K. Yano (see for instance [1], [2] and [3]). In particular, they have proven that umbilicity with respect to a parallel normal field is equivalent to hypersphericity.

2 Curvature lines associated to a normal vector field

Let M be a smooth oriented manifold of dimension n immersed in $\mathbb{R}^N,\ N = n + k$, with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^N$. For each $p \in M$, consider the decomposition $T_p\mathbb{R}^N = T_pM \oplus N_pM$, where N_pM is the orthogonal complement of T_pM in $\mathbb{R}^N$. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^N$. Given local vector fields $X,\ Y$ on M, let $\bar{X},\ \bar{Y}$ be some local extensions to $\mathbb{R}^N$. The tangent component of the Riemannian connection in $\mathbb{R}^N$ is the Riemannian connection of $M : \nabla_X Y = (\bar{\nabla}_{\bar{X}}\bar{Y})^\top$.

Let $\mathcal{X}(M)$ and $\mathcal{N}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map

$$\alpha : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{N}M,\ \alpha(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y.$$

This map is symmetric and bilinear.

Let $p \in M$ and $\nu \in \mathcal{N}(M),\ \nu \neq 0$, and define the function

$$H_\nu : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}(M),\ H_\nu(X,Y) =< \alpha(X,Y), \nu > .$$

This function is symmetric and bilinear. The second fundamental form of M with respect to ν at p is the associated quadratic form

$$II_\nu : \mathcal{X}(M) \to \mathcal{F}(M),\ II_\nu(X) = H_\nu(X,X).$$

Recall the shape operator

$$S_\nu : \mathcal{X}(M) \to \mathcal{X}(M),\ S_\nu(X) = -(\bar{\nabla}_{\bar{X}}\bar{\nu})^\top,$$

where $\bar{\nu}$ is a local extension to $\mathbb{R}^N$ of the normal vector field ν at p and $\top$ means the tangent component. This operator is self-adjoint and for any $X,\ Y \in T_pM$ satisfies the following equation: $< S_\nu(X), Y >= H_\nu(X,Y)$. So, the second fundamental form with respect to ν can be expressed by $II_\nu(X) =< S_\nu(X), X >$. Thus for each $p \in M$, there exists an orthonormal basis of eigenvectors of $S_\nu \in T_pM$. The corresponding eigenvalues $k_1, \dots,\ k_n$ are the ν-principal curvatures. Let $\mathcal{U}_\nu$ be the subset of *ν-preumbilic points* of M, i.e., $\mathcal{U}_\nu$ is made of all the points at which at least two ν-principal curvatures coincide. We denote $U_\nu(k_{i_1}, ..., k_{i_r}) = \{p \in U_\nu : k_{i_1}(p) = \dots = k_{i_r}(p)\},\ r = 2, ..., n$. A point lying in $U_\nu(k_1, ..., k_n)$ is called *ν-umbilic*. Given $p \in M - \mathcal{U}_\nu$, there are n ν-principal directions defined by the eigenvectors of S_ν. Provided $M - \mathcal{U}_\nu$ is open, this setting determines fields of directions on $M - \mathcal{U}_\nu$ which are smooth and integrable. The integrals of these fields are n families of orthogonal curves on $M - \mathcal{U}_\nu$, called *ν-principal lines of curvature*. These n orthogonal foliations of $M - \mathcal{U}_\nu$, together with the decomposition $\{U_\nu(k_{i_1}, ..., k_{i_r})\}$ of $\mathcal{U}_\nu$ form the ν-principal configuration of M. In fact, the points of $U_\nu(k_{i_1}, ..., k_{i_r})$ can be seen as the critical points for the i_j-th foliation, $j = 1, ..., r$, whereas, the ν-umbilic are critical points for the n foliations.

We observe that for most pairs (M,ν), where ν is a normal field over M, the subset $M-\mathcal{U}_\nu$ is open and dense in M. To see this we suppose that M is locally given by an immersion $f:\mathbb{R}^n\to\mathbb{R}^{n+k}$ and consider the distance squared functions family,

$$\begin{array}{llll} d: & \mathbb{R}^n\times\mathbb{R}^{n+k} & \longrightarrow & \mathbb{R} \\ & (x,a) & \longmapsto & d_a(x)=\|f(x)-a\|^2. \end{array}$$

It is easy to see that x is a critical point of d_a if and only if $a-f(x)\in N_{f(x)}M$. Moreover, given a unit normal field ν, defined in a neighbourhood of $p=f(x_0)$, it can be shown that $p\in\mathcal{U}_\nu(k_i,k_j)$, for some $i\neq j$, if and only if x_0 is a singularity of corank at least 2, either of the function $d_{p+\frac{1}{\lambda}\nu(p)}$ with $\lambda=k_i(p)=k_j(p)$ when $\lambda\neq 0$, or of the height function in the direction $\nu(p)$, $f_{\nu(p)}(x)=\langle x,\nu(p)\rangle$, when $\lambda=0$. Here, by the corank of a function at a point x_0 we mean the corank of its Hessian matrix at x_0. It follows from Looijenga's genericity theorem ([6], see also [10] for an alternative version of this result) that there exists a residual subset $\mathcal{I}(M)$ of the space of immersions of M in $\mathbb{R}^{n+k}$ (provided with the Whitney C^∞-topology) such that the family d associated to any immersion of this subset is structurally stable. In this case, it follows from standard methods of Singularity Theory developed in [6] (see also [15]) that the focal subset of M:

$$\mathcal{F}(M)=\{(x,a)\in NM : x \text{ is a degenerate singularity of } d_a\},$$

is a stratified subset, whose local structure at a point (x,a) is equivalent to the one induced by the Looijenga's stratification of the space, $C^\infty(M)$, of smooth functions on M on a transversal slice to the orbit of the function d_a under the action of the group, $Dif(M)\times Dif(\mathbb{R})$, of smooth diffeomorphisms pairs on $C^\infty(M)$. In particular, this implies that the subset

$$\Sigma_2=\{(x,a)\in NM : x \text{ is a singularity of } d_a \text{ with corank } \geq 2\}$$

is a stratified subset of codimension 3 in NM. Consider now the cone over Σ_2:

$$C(\Sigma_2)=\{(x,\beta a)\in NM,\ \text{for } a\in\Sigma_2 \text{ and } \beta\in\mathbb{R}-\{0\}\}.$$

This is a stratified subset of codimension 2 in NM. Clearly, we have that $p=f(x)\in\mathcal{U}_\nu$ if and only if $(x,\nu(x))\in C(\Sigma_2)$. Then the Elementary Transversality Theorem ([4]) implies that, given any immersion of M lying in $\mathcal{I}(M)$, there is a residual subset of the space, $\mathcal{N}(M)$, of C^∞-sections of the normal bundle $NM\to M$, with the Whitney C^∞-topology, such that any normal field in this subset meets transversally each stratum of $C(\Sigma_2)$. Clearly, given such a normal field, we have that $\mathcal{U}_\nu=\nu^{-1}(C(\Sigma_2))$ is a stratified subset of codimension 2 in M. Consequently, $\mathcal{U}_\nu$ has measure zero in M and hence $M-\mathcal{U}_\nu$ is an open and dense submanifold.

The differential equation of ν-lines of curvature is

$$S_\nu(X(p))=\lambda(p)X(p). \tag{1}$$

Suppose that $(\phi,\ U)$ is a local chart on M with corresponding local coordinates $(u_1,\dots,u_n)$. Let $g_{ij}=<\phi_{u_i},\phi_{u_j}>$, where $\phi_{u_k}=d\phi(\frac{\partial}{\partial u_k})$, $k=i,j$, are the coefficients of the first fundamental form in this coordinate chart. The coefficients of the second fundamental form are

$$\begin{aligned} \bar{G}^{\nu}_{ij} &= -<\alpha(\phi_{u_i},\phi_{u_j}),\nu>, \\ &= -<\phi_{u_i},\nu_{u_j}>=-<\phi_{u_iu_j},\nu>=\bar{G}^{\nu}_{ji}. \end{aligned} \tag{2}$$

Assume that $(g^{ij})=(g_{ij})^{-1}$ is the inverse of the metric. Thus, define

$$G^{\nu}_{ij}=\sum_{k=1}^{n}\bar{G}^{\nu}_{ik}g^{kj}. \tag{3}$$

We write the tangent component of ν_{u_j} as $\nu^T_{u_j}=a_{1j}\phi_{u_1}+\cdots+a_{nj}\phi_{u_n}$. Then the shape operator in this basis has the expression

$$S_\nu=(a_{ij}).$$

To obtain the expression of the coefficients a_{ij}, we consider

$$-\bar{G}^{\nu}_{ji}=<\phi_{u_i},\nu_{u_j}>=a_{1j}g_{i1}+\cdots+a_{nj}g_{in}=a_{1j}g_{1i}+\cdots+a_{nj}g_{ni}.$$

Therefore we get the following equation:

$$-(\bar{G}^{\nu}_{ij})=(S_\nu)^T(g_{ij}).$$

Thus, we have

$$(S_\nu)^T=-(\bar{G}^{\nu}_{ij})(g^{ij})=-(G^{\nu}_{ij}). \tag{4}$$

In order to obtain the differential equation of ν-lines of curvature in local coordinates and considering that S_ν is symmetric, we write the system $S_\nu(u)=\lambda(u)$ as

$$\begin{aligned} G^{\nu}_{11}\dot{u}_1+\cdots+G^{\nu}_{1n}\dot{u}_n &= \lambda\dot{u}_1, \\ G^{\nu}_{21}\dot{u}_1+\cdots+G^{\nu}_{2n}\dot{u}_n &= \lambda\dot{u}_2, \\ &\vdots \\ G^{\nu}_{n1}\dot{u}_1+\cdots+G^{\nu}_{nn}\dot{u}_n &= \lambda\dot{u}_n. \end{aligned} \tag{5}$$

The matrix $(G^{\nu}_{ij}(u))$ is a multiple of the identity, if and only if u is a ν-umbilic.

3 ν- umbilicity and lines of curvature

Given a normal field ν on a manifold M, we say that M is ν-umbilical if each point of M is ν-umbilic. We next analyze sufficient conditions on M to ensure that it is ν-umbilical. More precisely, we see that the fact that a certain number of linearly independent normal fields share a certain number of principal lines implies the existence of some normal field ν such that M is ν-umbilic.

Suppose that M is an n-manifold immersed in $\mathbb{R}^{n+k}$ through an immersion ϕ, and that $\nu_1, ..., \nu_k$ are k linearly independent normal fields on M. For any linear combination, $\nu = \sum_{l=1}^{k} \lambda_l \nu_l$, we have that

$$\begin{aligned} G_{ij}^{\nu} &= G_{ij}^{\sum_{l=1}^{k} \lambda_l \nu_l} = \sum_{m=1}^{n} (\bar{G}_{im}^{\sum_{l=1}^{k} \lambda_l \nu_l} g^{mj}), \\ &= \sum_{m=1}^{n} < \alpha(\phi_{u_i}, \phi_{u_m}), \sum_{l=1}^{k} \lambda_l \nu_l > g^{mj}, \\ &= \sum_{l=1}^{k} \lambda_l (\sum_{m=1}^{n} \bar{G}_{im}^{\nu_l} g^{mj}) = \sum_{l=1}^{k} \lambda_l G_{ij}^{\nu_l}. \end{aligned}$$

The matrix (G_{ij}^{ν}) is a multiple of the identity, or equivalently, M is ν-umbilic provided $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ is a non-trivial solution of the following linear system of $\frac{1}{2}(n+2)(n-1)$ equations:

$$\begin{aligned} \sum_{l=1}^{k} \lambda_l (G_{11}^{\nu_l} - G_{22}^{\nu_l}) &= 0, \\ &\vdots \\ \sum_{l=1}^{k} \lambda_l (G_{11}^{\nu_l} - G_{nn}^{\nu_l}) &= 0, \\ \sum_{l=1}^{k} \lambda_l G_{ij}^{\nu_l} &= 0, \ i < j. \end{aligned}$$

Notice that the first $(n-1)$-equations are determined by the fact that the diagonal coefficients must be equal, while the other $\frac{n(n-1)}{2}$ equations correspond to the fact that the upper triangle coefficients must vanish. Since S_ν is symmetric these equations guarantee that M is ν-umbilic. In the following argument, in order to study the existence of the solution of this system, we shall consider the dependence of the coefficients of this system on the immersion ϕ.

Denote by Δ^ϕ the matrix of this system, and by $\{\Delta_i^\phi\}_{i=1}^m$ the different minors of maximal order (=min $\{\frac{1}{2}(n+2)(n-1), k\}$) of Δ^ϕ. It follows from expression (4) that the coefficients of Δ^ϕ are given in terms of the first and the second derivatives of the immersion ϕ. The rank of Δ^ϕ is not maximum at a given point p provided all the determinants of all these minors vanish at p. This condition is thus determined by polynomial equations, in the first and second derivatives of ϕ, given by $det\ (\Delta_i^\phi) = 0, i = 1, ..., m$. Such equations determine a closed algebraic variety, Γ, in the space, $J^2(M, \mathbb{R}^{n+k})$, of 2-jets of maps from M to $\mathbb{R}^{n+k}$. A consequence of the Thom Transversality Theorem ([4]) is that the subset of immersions of M (locally given as $\phi(\mathbb{R}^n)$) in $\mathbb{R}^{n+k}$, for which the 2-jet extension, $j^2\phi : \mathbb{R}^n \to J^2(M, \mathbb{R}^{n+k})$, is transversal to Γ, is open and dense in the set of all immersions of $\mathbb{R}^n$ in $\mathbb{R}^{n+k}$ with the Whitney C^∞- topology. We shall call *generic immersions* those having this property. Clearly, if ϕ is a generic immersion, or in other words, M is a generic n-manifold in $\mathbb{R}^{n+k}$, the subset $M' = \{p \in M : rank\Delta < min\{\frac{1}{2}(n+2)(n-1), k\}\}$ is a closed stratified subset whose codimension in M is equal to that of Γ in $J^2(M, \mathbb{R}^{n+k})$.

Since Γ is given by at least one polynomial equation, we have that it has at least codimension one and thus, $M - M'$ is an open and dense submanifold of M.

Moreover, the above is a homogeneous linear system of $\frac{1}{2}(n+2)(n-1)$ equations in k variables. So, we can find non-trivial solutions provided the number of linearly independent equations is smaller than that of variables. In particular, this happens in case that

$$(n+2)(n-1) < 2k.$$

We observe that the coefficients of this system are smooth functions and so are their solutions all over the submanifold $M - M'$.

Lemma 3.1. *Suppose that M is a generic n-manifold immersed in $\mathbb{R}^{n+k}, n > 3$ and that $k > \frac{1}{2}(n+2)(n-1)$. Then M' is empty. In the case of a generic surface in $\mathbb{R}^5$, we have that M' is made of isolated points in M.*

Proof: If $n = 2, k = 3$, we have that Δ is a (2×3)-matrix. So, Γ has codimension 2 in $J^2(M, \mathbb{R}^5)$ and, since M is generic, so has M' in M. But this means that M' must be made of isolated points. It follows analogously that, for $n = 2, k \geq 4$, M' has codimension at least 3 in M and hence must be empty. On the other hand, when $n \geq 3$, we have that $\frac{1}{2}(n+2)(n-1) \geq n+2$ and thus, $k > n+2$. Therefore, codim $\Gamma \geq n+1$. And then it follows from the genericity of M, that M' must have codimension $n+1$ in M and thus, be empty. ∎

As a consequence of all these considerations, we can state:

Theorem 3.2. *Given a generic n-manifold M immersed in $\mathbb{R}^{\frac{1}{2}n(n+3)}$, $n \geq 3$, it is possible to find some locally defined smooth normal field ν at every point of M, such that M is ν-umbilical.*

Remark 3.3. In the case of a generic surface immersed in $\mathbb{R}^{n \geq 5}$ we can find such a field ν, locally defined at each point of M except by a subset M' of isolated points. We analyze this fact, from the geometrical viewpoint, in Section 5.

Remark 3.4. When the vector field ν is parallel and globally defined, Chen and Yano's result ([1]) implies that M is hyperspherical.

Moreover, we consider the following:

Definition 3.5. *Given a subbundle U of $\mathcal{N}M$, we say that M is U-umbilical provided that it is ν-umbilical for any section ν of the subbundle U.*

Now, if ϕ is a generic immersion of the n-manifold M into $\frac{1}{2}n(n+3)+l$-dimensional euclidean space, we have that for $k = l+1$, there exist k solutions of the system defined by Δ^{ϕ}. So we can assert:

Theorem 3.6. *Given any n-dimensional generic manifold M in $\mathbb{R}^{\frac{1}{2}n(n+3)+l}$ we can always find some rank (l+1) subbundle U (locally defined at each point of M)of $\mathcal{N}M$, such that M is U- umbilical.*

We observe that, from the global viewpoint, we may not have as many linearly independent normal fields as the codimension of M allows. For instance, for an orientable compact surface embedded in $\mathbb{R}^4$, we may just ensure the existence of one global non-vanishing section of the normal bundle. On the other hand, it is possible to find non-orientable surfaces in $\mathbb{R}^4$ without globally defined normal fields ([13]).

Suppose now that the k fields $\nu_1, ..., \nu_k$ share $r < n$ linearly independent principal directions at a given point $p \in M$. We can choose a coordinate system centered at p having these r directions as the tangents to the first coordinate lines. Clearly, the first r rows of the matrix S_{ν_s} diagonalize in this system, for $s = 1, ..., k$. Therefore, $G_{ij}^{\nu_s} = 0$, for $s = 1, ..., k$, $i = 1, ..., r$ and $i < j \leq n$. In other words, we have that $(n-1)+(n-2)+...+(n-r) = \frac{1}{2}r(2n-r+1)$ of the equations of the above system are identically zero on M. So we are left with at most $\frac{1}{2}(n+2)(n-1)-\frac{1}{2}r(2n-r+1)$ linearly independent equations in the system. Now, in order to ensure the existence of a non-trivial solution, we must ask that the number, k, of unknowns be strictly bigger than $\frac{1}{2}(n+2)(n-1) - \frac{1}{2}r(2n-r+1)$. So we can assert:

Lemma 3.7. *Suppose that M is a generic n-manifold immersed in $\mathbb{R}^{n+k}$ and that we can find k linearly independent normal fields, $\nu_1, ..., \nu_k$, sharing $r < n$ linearly independent principal directions all over M. Let r be a natural number satisfying*

$$k + \frac{1}{2}r(2n-r+1) > \frac{1}{2}(n+2)(n-1).$$

Then it is possible to find some locally defined smooth normal field ν at every point of M, such that M is ν-umbilical.

Theorem 3.8. *Suppose that there are k linearly independent normal fields over an n-dimensional manifold M, sharing $\delta(n,k)$ linearly independent principal directions at each point of M, where δ ranges as in the table below. Then there exists some smooth vector field ν normal to M for which M is ν-umbilical, locally defined at each point of M.*

$n\backslash k$	2	3-5	6-9	10-14	15-20	21-27	...
2	1	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
3	2	1	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$
4	3	2	1	0	$\cdots$	$\cdots$	$\cdots$
5	4	3	2	1	0	$\cdots$	$\cdots$

Proof: It follows from calculating the adequate values of r for the different choices of n and k according to the above lemma. In fact, for a given n, we have that $\delta(n,k) = 0$ if and only if $k \geq \frac{1}{2}n(n+1)$. Moreover, it can be seen that for a fixed k, $\delta(n+1,k) = \delta(n,k) + 1$. ∎

Remark 3.9. Notice that $\delta(n,k)$ is the minimal number of lines of curvature that must be shared by the k (maximun number of linearly independent) generic normal fields over M. By appliyng the same arguments, we see that in some cases it is possible to replace the number k by a smaller one, l, provided we substitute the number $\delta(n,k)$ by an appropriate higher one $\delta(n,k,l)$.

Remark 3.10. Suppose that the k normal fields $\nu_1, ..., \nu_k$ are generic, in the sense that the subsets $\mathcal{U}_{\nu_i}, i = 1, ..., k$, have measure zero in M. In this case, we can substitute the requirement that they share $\delta(n, k)$ linearly independent principal directions at each point of M by the condition of sharing $\delta(n, k)$ curvature lines over the open dense submanifold $M - \bigcup_{i=1}^{k} \mathcal{U}_{\nu_i}$. It can be shown that there is an open and dense subset of immersions of an n-manifold M in $\mathbb{R}^{n+k}$ (with the Whitney C^∞-topology), for which the immersed submanifold admits k linearly independent generic normal fields.

4 Submanifolds of codimension 2

Suppose that M is a generic n-manifold in $\mathbb{R}^{n+2}$, then the ν-umbilicity result above can be expressed as follows:

Theorem 4.1. *M is ν-umbilical if and only if all the normal fields $\eta \neq \nu$ on M have the same principal directions all over M.*

Proof: That M is ν-umbilical provided any other normal fields $\eta \neq \nu$ on M have the same principal directions follows from the Theorem 3.8 above, for in this case M must admit two linearly independent normal fields with $n - 1$ common linearly independent principal directions. To see the converse suppose that M is ν-umbilical and let $\eta \neq \nu$. Consider the orthogonal frame for the normal bundle of M given by ν and its orthogonal $\nu^\perp$. Then we can write, $\eta = \lambda_1\nu + \lambda_2\nu^\perp$. The corresponding shape operator is given by,

$$S_\eta = S_{\lambda_1\nu+\lambda_2\nu^\perp} = \lambda_1 S_\nu + \lambda_2 S_{\nu^\perp} = \lambda_1\lambda_\nu Id + \lambda_2 S_{\nu^\perp},$$

where λ_ν is the curvature function associated to the field ν. This means that S_η and $S_{\nu^\perp}$ have the same eigenvectors and the result is proven. ■

We shall now concentrate our attention on a special kind of normal fields, known as binormal fields, on submanifolds of codimension 2 in $I\!R^n$. These were introduced in [9] and are defined as follows:

Given an embedding, $f : M \to \mathbb{R}^{n+2}$, of an n- manifold M, the family of height functions on M associated to f is defined as

$$\begin{array}{rcccl} \lambda(f) & : & M \times S^{n+1} & \longrightarrow & \mathbb{R} \\ & & (x, v) & \longmapsto & \langle f(x), v\rangle = f_v(x). \end{array}$$

For any fixed $v \in S^{n+1}$, we have a height function, f_v, on M, that satisfies: f_v has a singularity at $x \in M$ if and only if v is a normal vector to $f(M)$ at $f(x)$. The singularity type of the function f_v at a point x characterizes the geometric contact of the submanifold M at the point $f(x)$ with the tangent hyperplane orthogonal to the vector v (see [10]). We shall say that a hyperplane H_v with orthogonal direction v has *higher order contact with* M at a point x if H_v is tangent to $f(M)$ at x and f_v has a degenerate (i.e., non-Morse) singularity at x. By analogy with the case of curves in $\mathbb{R}^3$, where the binormal direction is precisely the unit vector whose corresponding height function has a degenerate singularity at the considered point, each direction v corresponding to some height function f_v having a degenerate singularity at x is called *binormal.* The hyperplane that, being orthogonal to such direction, passes through the point x is called *osculating hyperplane* of M at x. So, the osculating hyperplanes of M are those whose contact with M is stronger in some sense. Submanifolds of codimension two do not necessarily have binormals at every point. Moreover, in case that they have, these do not need to be unique at every point. Interesting examples of (non-compact) surfaces having no binormal directions are given by minimal surfaces in 4-space ([8]). On the other hand, locally convex surfaces in $\mathbb{R}^4$, in the sense that they admit a locally support hyperplane at each point, illustrate the case of manifolds having two binormal fields globally defined ([7]). In general, for a closed (compact without boundary) n-manifold M embedded in $\mathbb{R}^{n+2}$, it can be seen that (see [9]) :

a) if n is odd, all the points of M admit at least one binormal direction, and at most n of them;

b) if n is even, there is always an open region of M all whose points admit at least one binormal direction and at most n of them.

Suppose that b is a binormal vector for M at x, so x is a degenerate critical point of the height function f_b. In this case, the Hessian quadratic form $\mathcal{H}(f_b)(x)$ is degenerate over T_xM and hence there exists some vector $u \in T_xM - \{0\}$ such that $\mathcal{H}(f_b)(x)(u) = 0$. The vector u defines an *asymptotic* direction for M at x.

Each binormal direction has associated a unique asymptotic direction at each point, except at the *inflection points.* These points are characterized as singularities of corank two of appropriate height functions, which leads to the fact that the asymptotic directions at them fill a tangent plane. Under certain conditions on the k-jet extension of the family of height functions, $j^k\lambda(f) : M \times S^{n+1} \to J^k(M, \mathbb{R})$, for a big enough k, we can guarantee that the inflection points form closed stratified subsets of codimension 2 in M. Consequently, a binormal field defined by such an immersion of M provides a unique asymptotic field on M whose critical points lie in the set of inflection points. It follows from Looijenga's theorem ([6]) that there is a residual subset for the Whitney C^∞-topology over the set of immersions of M in $\mathbb{R}^{n+2}$, for which the k-jet extension of the family of height functions satisfies the above referred transversality conditions. On the other hand, any asymptotic field comes from some binormal field and although in principle, asymptotic fields corresponding to different binormal fields might coincide at some points, the fact that two different binormals have coincident arcs of asymptotic lines is non-generic, in the sense that it can be avoided by a small enough perturbation of the immersion. Moreover, it can also be shown that, generically, two different binormal direction fields may only coincide over zero measure sets. In what follows we shall understand by generic manifolds those that are generic in the sense of Section 3 and also verify appropriate transversality conditions, in the above sense, such that: i) their inflection points lie in closed subsets of codimension at least 2, and ii) their binormal fields determine different asymptotic directions at each non-inflection point over which they are defined.

Given a binormal b, the principal curvature associated to one of the b-principal directions is null. We have the following relation between the asymptotic direction associated to b and the b-principal direction with vanishing principal curvature:

Lemma 4.2. *If b is a binormal field on M, then the asymptotic direction field associated to b determines a b-principal direction field with vanishing principal b-curvature in the following way: If the vector $\bar{a}_b$ determines the asymptotic direction associated to b and (g_{ij}) is the matrix of the metric, then $a_b = (g_{ij})\bar{a}_b$ determines the b-principal direction with vanishing b-principal curvature.*

Proof: Given a normal field η on M, it is not difficult to check that, in local coordinates given by the Monge form at any point of M, the Hessian matrix of the height function f_η at the given point coincides with the matrix $(\bar{G}^\eta_{ij})$ whose coefficients are described by equation (2). In the case of a binormal field b, we have that the corresponding height function is degenerate and thus its Hessian has some vanishing eigenvalue. Let $\bar{a}_b$ be a non-null vector in the kernel of the Hessian of f_b. Then, equation (4) implies that $a_b = (g_{ij})\bar{a}_b$ is in the kernel of S_b, namely, it is an eigenvector of this operator. ∎

Remark 4.3. When the matrix of the metric at a point is a multiple of the identity, an asymptotic direction with binormal b is a b-principal direction associated to a null b-principal curvature. This is always the case for the isothermic coordinate charts of M.

Assume that two binormal fields b_1 and b_2 on M have the same principal configurations and consider their corresponding principal directions associated to vanishing principal curvatures, a_{b_i}, $i = 1, 2$, as in the previous lemma. Clearly, a_{b_j}, $i \neq j$ defines another principal direction for b_i. Denote by k_i, $i, j \in \{1, 2\}$, the b_i-principal curvature associated to the principal direction a_{b_j}, $i \neq j$. We observe that a point $p \in M$ at which $k_1(p) = k_2(p) = 0$ is an inflection point of M. So, we have that the normal field

$$\nu = \frac{k_2 b_1 + k_1 b_2}{\| k_2 b_1 + k_1 b_2 \|},$$

is defined over the open dense submanifold, $\bar{M}$, determined by the complementary of the inflection points in a generic manifold M. Theorem 4.1 above leads us to the following:

Theorem 4.4. *Let M be a generic n-manifold. Suppose that M has (at least) two globally defined binormal fields, b_1 and b_2, sharing all their principal directions. Then $\bar{M}$ is ν-umbilical. Moreover, all the normal fields differing from ν at least over some dense subset of M, have the same principal directions on M and thus, determine a unique principal configuration.*

On the other hand, if M has n different globally defined asymptotic fields, then they are mutually orthogonal and coincide with this unique principal configuration.

Proof: We can assume that the curvature directions $a_{b_i}, i = 1, 2$, are different on $\bar{M}$ because otherwise $k_1(p) = k_2(p) = 0$. At each point of $\bar{M}$ consider a coordinate chart (ϕ, U) with a tangent basic frame $\{\phi_{u_i}, 1 \leq i \leq n\}$, such that ϕ_{u_1} is tangent to the principal direction a_{b_1} of b_1, and ϕ_{u_2} is tangent to the principal direction a_{b_2} of b_2. Let G^1_{ij} denote the coefficients of the shape operator S_{b_1} and G^2_{ij} those of the shape operator S_{b_2} in this basis. Since ϕ_{u_1} and ϕ_{u_2} are eigenvectors of the shape operators S_{b_k}, $k = 1, 2$, at each point of $\bar{M}$, we have that $G^1_{22} = k_1$ and $G^2_{11} = k_2$.

On the other hand, as b_1 and b_2 have the same lines of curvature, Theorem 3.6 implies that there is a smooth vector field ν, locally defined at each point, such that $\bar{M}$ is ν-umbilic. Let $\nu = \frac{1}{B}\{\lambda b_1 + \mu b_2\}$, where $B = \|\lambda b_1 + \mu b_2\|$. Therefore,

$$S_\nu = \frac{1}{B}\{\lambda S_{b_1} + \mu S_{b_2}\} = k_\nu Id,$$

where k_ν is the ν-umbilic curvature.

The following system of equations holds at each point of the coordinate chart:

$$\lambda G^1_{22} - \mu G^2_{11} = 0, \tag{6}$$

$$\begin{aligned} \lambda(G^1_{kk} - G^1_{22}) + \mu G^2_{kk} &= 0, \\ \lambda G^1_{kk} + \mu(G^2_{kk} - G^2_{11}) &= 0, \ k = 3, \ldots, n, \end{aligned} \tag{7}$$

$$\lambda G^1_{ij} + \mu G^2_{ij} = 0, \ i > j. \tag{8}$$

Equation (6) is obtained by observing that the b_1-principal curvature vanishes along the first coordinate curve and the b_2-principal curvature vanishes along the second coordinate line. In fact, we have that

$$\lambda G^1_{22} = \mu G^2_{11} = k_\nu.$$

By substituting these two equations in

$$\lambda G^1_{kk} + \mu G^2_{kk} = k_\nu,$$

we obtain system (7). Finally, system (8) holds because S_ν has null entrance functions off its diagonal.

So the pair (λ, μ) defined by ν satisfies systems (6), (7) and (8). Furthermore, since these systems are linear, they hold for any multiple of this pair too. Thus, the line of solutions of system (6) is the set of solutions of systems (7) and (8). In particular, the pair defined by

$$\lambda = G^2_{11} = k_2, \ \mu = G^1_{22} = k_1,$$

satisfies the three systems and determines the coefficients of the normal field ν. And hence we get that $\bar{M}$ is ν-umbilic. The second statement follows as a consequence of Theorem 4.1. ∎

We say that a submanifold M of codimension 2 in $\mathbb{R}^{n+2}$ is *hyperspherical* provided that it lies in some $(n+1)$-sphere. B. Y. Chen and K. Yano proved ([1]) that this is equivalent to being ν-umbilical for some parallel normal field ν, where parallel means that the normal component of the covariant derivative of ν vanishes at every point. As a consequence we obtain the following:

Corollary 4.5. *Suppose that M is an n-manifold with two globally defined binormal fields b_1 and b_2 having the same principal directions. Under the conditions of the previous Theorem, assume also that the field*

$$\nu = \frac{k_2 b_1 + k_1 b_2}{|| \; k_2 b_1 + k_1 b_2 \; ||}$$

is parallel. Then M is hyperspherical. Moreover, in this case M admits n globally defined binormal fields whose corresponding asymptotic fields are mutually orthogonal.

Proof: The hypersphericity follows from the previous theorem by applying the result of B. Y. Chen and K. Yano to the field ν on $\bar{M}$. Since $\bar{M}$ is dense in M, we get that M must lie in a hypersphere too. On the other hand, it was proven in [9] that hyperspherical submanifolds of codimension 2 always have the maximum number of asymptotic fields and these are mutually orthogonal. ∎

Remark 4.6. Surfaces in $\mathbb{R}^4$ with the property of having everywhere defined asymptotic lines are locally convex [8]. It was proven in [16] that the two asymptotic fields of a locally convex surface in $\mathbb{R}^4$ are orthogonal if and only if M is ν-umbilical, with $\nu = k_2 b_1 + k_1 b_2$, where k_1 and k_2 are the non-vanishing principal curvatures associated to the fields b_1 and b_2, respectively. Notice that since we can define on M isothermic coordinates, the expression of the metric implies that asymptotic directios are b_i-principal directions, $i = 1, 2$. In this case, the condition of parallelism on ν is equivalent to asking that $(\frac{k_1}{k_2} + \frac{k_2}{k_1} + 2cos\theta)E = constant$, where θ is the angle between the two binormals at each point and E is the coefficient of the first fundamental form in isothermic coordinates on M.

5 ν-umbilicity of surfaces

It follows from Theorem 3.2, as pointed out in Remark 3.3, that generic surfaces in $\mathbb{R}^N, N > 4$, always admit normal fields for which they are umbilic. In this section, we analyze this fact from the geometrical viewpoint.

Definition 5.1. *Let M be a surface embedded in $\mathbb{R}^N, N \geq 4$. Given $p \in M$, consider the unit circle in T_pM parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ the curve obtained by intersecting M with the hyperplane at p composed by the direct sum of the normal subspace N_pM and the straight line in the tangent direction represented by θ. Such curve is called the normal section of $\phi(M)$ in the direction θ. The curvature vector $\eta(\theta)$ of γ_θ in p lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the curvature ellipse of M at p.*

In fact, M can be locally given by an embedding $f : \mathbb{R}^2 \to \mathbb{R}^N$. And if we take isothermic coordinates $\{x, y\}$ and an orthonormal frame $\{e_1, e_2, ..., e_N\}$ in a neighbourhood of a point $p = f(0, 0) \in M$, in such a way that $\{e_1, e_2\}$ is the tangent frame determined by these coordinates and $\{e_3, ..., e_N\}$ is a normal frame, the matrix of the second fundamental form of M at p is given by

$$\alpha_f(p) = \begin{bmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_{N-2} & b_{N-2} & c_{N-2} \end{bmatrix},$$

where

$$\begin{aligned} a_i &= \frac{\partial^2 f}{\partial x^2}(0,0)\cdot e_{i+2} = \bar{G}_{11}^{\nu_{i+2}}, \quad b_i = \frac{\partial^2 f}{\partial x \partial y}(0,0)\cdot e_{i+2} = \bar{G}_{12}^{\nu_{i+2}}, \\ c_i &= \frac{\partial^2 f}{\partial y^2}(0,0)\cdot e_{i+2} = \bar{G}_{22}^{\nu_{i+2}}, \end{aligned}$$

for $i = \{1, \cdots, N-2\}$.

Then the curvature ellipse is the image of the affine map

$$\eta : S^1 \subset T_pM \longrightarrow N_pM,$$

given by

$$\theta \longmapsto \eta(\theta) = \sum_{i=1}^{N-2} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot e_{i+2},$$

that is,

$$\eta(\theta) = H + B\cos 2\theta + C\sin 2\theta,$$

where $H = 1/2\sum_{i=1}^{N-2}(a_i + c_i)e_{i+2}$, $B = 1/2\sum_{i=1}^{N-2}(a_i - c_i)e_{i+2}$ and $C = \sum_{i=1}^{N-2} b_i e_{i+2}$.

Example: Consider the surface M defined by the following immersion

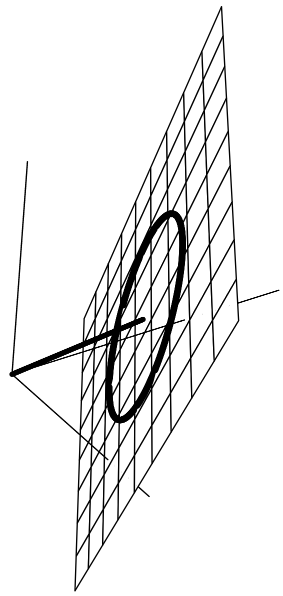
$$\begin{array}{cccc} \psi : & \big(\mathbb{R}^2, (0,0)\big) & \longrightarrow & \big(\mathbb{R}^5, (0, \cdots, 0)\big) \\ & (x,y) & \longmapsto & (x, y, x^2, y^2, xy) \end{array}.$$

Straightforward computations show that the curvature ellipse of M at $p = \psi(0,0)$ is given by

$$\eta(\theta) = (0,0,1,1,0) + (0,0,1,-1,0)cos2\theta + (0,0,0,0,1)sin2\theta.$$

This ellipse is contained in a plane with parametric equation $(x,y,z,u,w) = (0,0,1,1,0) + (0,0,1,-1,0)\lambda + (0,0,0,0,1)\nu$. The figure below illustrates the position of both, the ellipse and the plane with respect to the origin in the normal space of M at p.

The curvature ellipse may degenerate into a segment or even a point at certain points of M. We denote by E_p the vector subspace parallel to the affine hull of the ellipse in N_pM. Provided that this ellipse is non-degenerate at p the subspace E_p is a plane. The next result tells us that the degenerate situation may be generically avoided.



Theorem 5.2. *Given a surface* M*, there is an open and dense subset of immersions of* M *in* $\mathbb{R}^N$ *(with the Whitney* C^∞*-topology) such that,*
a) If $N \geq 6$*, the curvature ellipse is non-degenerate at every point of* M*.*
b) If $N = 5$*, the curvature ellipse is non-degenerate at every point except at most at isolated points.*

Proof: We observe that the curvature ellipse at p degenerates into a segment if and only if the vectors $B = (a_1 - c_1, ..., a_{N-2} - c_{N-2})$ and $C = (b_1, ..., b_{N-2})$ are linearly dependent. But this is equivalent to the vanishing of $N-3$ quadratic equations in the variables a_i, b_i and c_i, which can be taken as coordinates in the jet space $J^2(M, \mathbb{R}^N)$. The zeroes of these equations determine a stratified subset V of codimension $n-3$ in $J^2(M, \mathbb{R}^N)$. And saying that the ellipse at a point p is degenerate, is equivalent to ask that the image of the 2-jet map $j^2 f : M \to J^2(M, \mathbb{R}^N)$ at p hits this subset. Now, it follows from the Thom Transversality Theorem ([4]) that there is an open and dense subset in the set of all the immersions of M in $\mathbb{R}R^N$, for which the map $j^2 f$ is transversal to V. Since the dimension of M is 2, transversality in this case implies that $j^2 f$ may only hit V at isolated points when $N = 5$ and at no point when $N > 5$. ■

According to the theorem above, we have that for a generic surface in $\mathbb{R}^5$ it is possible to find some normal field, locally defined at each point of an open and dense submanifold $\bar{M}$ of M, with the property that it is orthogonal to the normal plane E_p at every point. Yet in the case that the surface is immersed in a higher dimensional euclidean space $\mathbb{R}^N$, we can define such a field in a neighbourhood of every point of M. Moreover, there are at least $N-4$ locally defined linearly independent fields with this property. The (isolated) points of $M' = M - \bar{M}$, characterized by the fact that the curvature ellipse degenerates into a segment, or a point, are called *semiumbilic* ([13]).

Examples: a) The following family of immersions

$$\begin{array}{rccc} \psi: & \big(\mathbb{R}^2,(0,0)\big) & \longrightarrow & \big(\mathbb{R}^5,(0,\cdots,0)\big) \\ & (x,y) & \longmapsto & (x,x^2,x^3+y,y^2,ax^3+by^4), \end{array}$$

where a and b are constants, provides examples of surfaces without semiumbilics in $\mathbb{R}^5$.

b) Consider the smooth map

$$\begin{array}{rccc} \xi: & I\!\!R^3 & \longrightarrow & I\!\!R^6 \\ & (x,y,z) & \longmapsto & (x^2,y^2,z^2,\sqrt{2}xy,\sqrt{2}xz,\sqrt{2}yz). \end{array}$$

The restriction of ξ to an ellipsoid $S=\{(x,y,z)\in\mathbb{R}^3 : ax^2+by^2+cz^2=1\}$, where a,b and c are non-zero real numbers, provides an example of surface without semiumbilics substantially contained in a hyperplane $I\!\!R^5\subset\mathbb{R}^6$. In the particular case of $a=b=c$, we have an immersion of the projective plane, known as the Veronese surface. This surface is contained in a 4-sphere and the curvature ellipse at each one of its points is a circle. Now, if we slightly translate S along one of the axes, we have that its image by ξ is a surface without semiumbilics substantially contained in $I\!\!R^6$.

Theorem 5.3. *Given any surface M immersed in $\mathbb{R}^N$, $N\geq 5$, suppose that ν is a normal field which is orthogonal to E_p at every point. Then, M is ν-umbilical.*

Proof: With the same notation as above, we have that, since $\nu(p)$ is orthogonal to E_p, for all p, $\langle\nu,B\rangle=\langle\nu,C\rangle=0$ at every point. But $\langle\nu,B\rangle=\Sigma_{i=1}^{n-2}(a_i-c_i)\nu_i$. And hence, we have that $\Sigma_{i=1}^{n-2}a_i\nu_i=\Sigma_{i=1}^{n-2}c_i\nu_i=\lambda$. On the other hand, $\langle\nu,C\rangle=\Sigma_{i=1}^{n-2}b_i\nu_i$. But from the definition of the a_i,b_i and c_i it follows that $\Sigma_{i=1}^{n-2}a_i\nu_i=\frac{\partial^2 f_\nu}{\partial x^2}$, $\Sigma_{i=1}^{n-2}b_i\nu_i=\frac{\partial^2 f_\nu}{\partial x\partial y}$, $\Sigma_{i=1}^{n-2}c_i\nu_i=\frac{\partial^2 f_\nu}{\partial y^2}$, where f_ν is the height function in the direction ν. Therefore, the Hessian of f_ν is a diagonal matrix with the same function λ in the two entries of the diagonal. And since this matrix coincides with that of the shape operator associated to the normal field ν at each point, we can conclude that all the points are ν-umbilic. ∎

Suppose that the curvature ellipse is non-degenerate at each point of a surface M except perhaps at isolated (semiumbilic) points. We denote by E the rank 2 subbundle of NM whose fibre at each non-semiumbilic point p is the plane E_p. In the case $N=5$, $E^\perp$ represents the corresponding orthogonal (line) subbundle of NM. An immediate consequence of the Theorem 5.3 is the following:

Corollary 5.4. *$M-M'$ is $E^\perp$-umbilic.*

And now, from Chen's results ([1]), we can conclude:

Corollary 5.5. *If M is a surface in $\mathbb{R}^N, N\geq 5$, such that the subbundle $E^\perp$ is globally defined and there is some parallel normal field whose image lies in $E^\perp$, then M lies in a hypersphere.*

For instance, in the case of the Veronese surface, V, defined above, we have that the subbundle E is globally defined and $E^\perp$ is a line bundle whose direction at each point is given by the radial field of the 4- sphere that contains V.

Definition 5.6. *A normal field η on a surface M is called essential if $\eta(p) \in E_p$ for all non-semiumbilic points $p \in M$. The corresponding principal configuration is also called essential.*

We observe that not every surface admits globally defined essential normal fields. In fact, any globally defined normal field over the Veronese surface must be perpendicular to E_p at some points (see [13]).

Given a normal field ξ on M, we denote by M_ξ the E- support of ξ, that is, the closure of the region over which the E-component of ξ does not vanish.

Theorem 5.7. *Let ξ be a normal field such that M_ξ contains some open submanifold of M. Then the ξ-principal configuration is essential over M_ξ.*

Proof: We observe that we can write $\xi = \eta_1 + \eta_2$, where η_1 is a section of $E^\perp$ and η_2 a section of E. The hypothesis on ξ implies that η_2 is essential in the interior of M_ξ. Moreover, since all the points of M are umbilic for the field η_1, we have that ξ and η_2 induce the same differential equation for the principal lines and thus have the same principal configuration. ■

Therefore we can say that all the principal configurations of M are exhausted by the essential ones.

A surface M at which the curvature ellipse degenerates into a segment at every point is said to be *totally semiumbilic*. In this case, E is a rank 1 subbundle of NM and by using similar arguments as above, we can assert:

Corollary 5.8. *All the non-umbilic normal fields on a totally semiumbilic surface in $\mathbb{R}^N, N \geq 5$, share the same principal configuration.*

A source of examples of totally semiumbilic surfaces is given by some translation surfaces associated to curves in $\mathbb{R}^N$. Given two curves $\gamma_i : S^1 \to \mathbb{R}^N,\ i = 1, 2$, we define the *translation surface associated to* γ_1 and γ_2 as the image of the map

$$\begin{array}{rcccl} T_{\gamma_1,\gamma_2} & : & S^1 \times S^1 & \longrightarrow & \mathbb{R}^N \\ & & (s, t) & \longmapsto & \frac{1}{2}(\gamma_1(s) + \gamma_2(t)). \end{array}$$

It can be seen that, for most pairs (γ_1, γ_2), the corresponding surface is a torus immersed with isolated double points if $N = 4, 5$ or embedded if $N \geq 6$. Moreover, in the particular case that the curves γ_1 and γ_2 are contained in orthogonal subspaces of $\mathbb{R}^N$, it can be shown that T_{γ_1,γ_2} is a totally semiumbilical torus.

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