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Kernels in pretransitive digraphs

Hortensia Galeana-Sánchez[a], Rocío Rojas-Monroy[b]

[a] Instituto de Matemáticas, UNAM, Universidad Nacional Autónoma de México, Ciudad Universitaria, México, D.F. 04510, Mexico

[b] Facultad de Ciencias, Universidad Autónoma del Estado de México, Instituto Literario No. 100, Centro 50000, Toluca, Edo. de Méx., Mexico

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Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. A kernel N of D is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from w to N. A digraph D is called *right-pretransitive* (resp. *left-pretransitive*) when $(u,v) \in A(D)$ and $(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(w,v) \in A(D)$ (resp. $(u,v) \in A(D)$ and $(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(v,u) \in A(D)$). This concepts were introduced by P. Duchet in 1980. In this paper is proved the following result: Let D be a digraph. If $D = D_1 \cup D_2$ where D_1 is a right-pretransitive digraph, D_2 is a left-pretransitive digraph and D_i contains no infinite outward path for $i \in \{1,2\}$, then D has a kernel.

MSC: 05C20

Keywords: Kernel; Kernel-perfect digraph; Right-pretransitive digraph; Left-pretransitive digraph

1. Introduction

For general concepts we refer the reader to [1]. In the paper we write digraph to mean 1-digraph in the sense of Berge [1]. In this paper D will denote a possibly infinite digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. Often we shall write u_1u_2 instead of (u_1u_2). An arc $u_1u_2 \in A(D)$ is called asymmetrical (resp. symmetrical) if $u_2u_1 \notin A(D)$ (resp. $u_2u_1 \in A(D)$). The asymmetrical part of D

E-mail address: hgalaena@matem.unam.mx (H. Galeana-Sánchez).

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(resp. symmetrical part of D), which is denoted by Asym(D) (resp. Sym(D)), is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. We recall that a subdigraph D_1 of D is a spanning subdigraph if $V(D_1)=V(D)$. If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph with vertex set S and whose arcs are those arcs of D which join vertices of S.

A directed path is a finite or infinite sequence $(x_1,x_2,\ldots)$ of distinct vertices of D such that $(x_i,x_{i+1})\in A(D)$ for each i. When D is infinite and the sequence is infinite we call the directed path an *infinite outward path*. Let S_1 and S_2 be subsets of $V(D)$, a finite directed path $(x_1,x_2,\ldots,x_n)$ will be called and S_1S_2-directed path whenever $x_1\in S_1$ and $x_n\in S_2$ in particular when the directed path is an arc.

Definition 1.1. A set $I\subseteq V(D)$ is independent if $A(D[I])=\emptyset$. A kernel N of D is an independent set of vertices such that for each $z\in V(D)-N$ there exists a zN-arc in D.

A digraph D is called kernel-perfect digraph when every induced subdigraph of D has a kernel.

The concept of kernel was introduced by Von Neumann and Morgenstern [7] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [8,9], Duchet and Meyniel [4], Duchet [2,3], Galeana-Sánchez and Neumann-Lara [6].

It is well known that a finite transitive digraph is kernel-perfect and a finite symmetrical digraph is kernel perfect. (We recall that a digraph D is transitive whenever $(u,v)\in A(D)$ and $(v,w)\in A(D)$ implies $(u,w)\in A(D)$.)

Definition 1.2 (Duchet [2]). A digraph D is called right- (resp. left-) pretransitive if every nonempty subset B of $V(D)$ possesses a vertex $t(B)=b$ such that: $(x,b)\in A(D)$ and $(b,y)\in A(D)$ implies $(x,y)\in A(D)$ or $(y,b)\in A(D)$ (resp. $(x,b)\in A(D)$ and $(b,y)\in A(D)$ implies $(x,y)\in A(D)$ or $(b,x)\in A(D)$), for any two vertices $x,y\in V(D)$.

Clearly taking $B=\{b\}$ for each $b\in V(D)$ (taking all the possible singletons of $V(D)$) in Definition 1.2, we obtain that Definition 1.2 is equivalent to those given in the Abstract, which for technical reasons will be used in this paper.

Theorem 1.1 (P. Duchet [2]). *A finite right-pretransitive (resp. left-pretransitive) digraph is kernel-perfect.*

The result proved in this paper generalize Theorem 1.1 and the following result of Sands et al. [10].

Theorem 1.2 (Sands et al. [10]). *Let D be a digraph whose arcs are coloured with two colors. If D contains no monochromatic infinite outward path, then there exists a set S of vertices of D such that: no two vertices of S are connected by a monochromatic directed path and, for every vertex x not in S there is a monochromatic directed path from x to a vertex in S.*

In order to understand Theorem 1.2 in terms of kernels we include the following definitions:

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path is called monochromatic if all of its arcs are coloured alike.

Definition 1.3 (Galeana-Sánchez [5]). Let D be an m-coloured digraph. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

(i) For every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and,
(ii) For every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

Definition 1.4. If D is an m-coloured digraph then the closure of D, denoted $\mathscr{C}(D)$ is the m-coloured multidigraph defined as follows:

$$V(\mathscr{C}(D)) = V(D);$$

$$A(\mathscr{C}(D)) = A(D) \cup \{uv \text{ with colour } i \mid \text{there exits an } uv\text{-monochromatic directed path of colour } i \text{ contained in } D\}.$$

Note that for any digraph D, $\mathscr{C}(\mathscr{C}(D)) \cong \mathscr{C}(D)$ and D has a kernel by monochromatic paths if and only if $\mathscr{C}(D)$ has a kernel. (Although the concept of kernel was defined in [1] for 1-digraphs, the same concept is valid and can be considered in multidigraphs).

In this terminology Theorem 1.2 asserts that if D is a 2-coloured digraph, which contains no monochromatic infinite outward path, then $\mathscr{C}(D)$ has a kernel (in fact $\mathscr{C}(D)$ is a kernel-perfect digraph).

Now it is clear that Theorem 1.2 is equivalent to the following assertion: Let D be a digraph; D_1 and D_2 transitive subdigraphs of D such that $D = D_1 \cup D_2$. If D has no infinite outward path contained in D_i; $(i = 1, 2)$ then D has a kernel.

Finally, we will introduce some notation: Given two subdigraphs of D; D_1 and D_2 (possibly $A(D_1) \cap A(D_2) \neq \emptyset$). For distinct vertices x, y of D; $x \overset{i}{\rightarrow} y$ will mean that the arc $(x, y) \in A(D_i)$, and $x \overset{i}{\rightarrow} S$ will mean that there exists an arc in D_i from x to a vertex in S, $S \subseteq V(D)$, where $i = 1, 2$. When we do not know if the arc is in D_1 or in D_2 we write simply $x \rightarrow y$. The negation of $x \overset{i}{\rightarrow} y$ (resp. $x \overset{i}{\rightarrow} S$) will be denoted $x \overset{i}{\nrightarrow} y$ (resp. $x \overset{i}{\nrightarrow} S$) for $i = 1, 2$.

2. Kernels in pretransitive digraphs

The main result of this section is Theorem 2.1, to prove this result we use a method closely related to the one of Sands et al. [10].

Lemma 2.1. *Let D be a right-pretransitive or left-pretransitive digraph. If $(x_1,x_2,\dots,x_n)$ is a sequence of vertices such that $(x_i,x_{i+1})\in A(D)$ and $(x_{i+1},x_i)\notin A(D)$, then the sequence is a directed path and for each $i\in\{1,\dots,n-1\}$, $(x_i,x_j)\in A(D)$ and $(x_j,x_i)\notin A(D)$, for every $j\in\{i+1,\dots,n\}$.*

Proof. We proceed by induction on n. The result is obvious for $n\leqslant 2$. Assume the result is true for a sequence $(x_1,x_2,\dots,x_n)$, which satisfies the hypothesis of Lemma 2.1. Consider a sequence $T=(x_1,x_2,\dots,x_n,x_{n+1})$ such that for each $i\in\{1,\dots,n\}$, $(x_i,x_{i+1})\in A(D)$ and $(x_{i+1},x_i)\notin A(D)$. Since $(x_1,\dots,x_n)$ and $(x_2,\dots,x_{n+1})$ satisfy the inductive hypothesis we only need to prove $x_1\neq x_{n+1}$, $(x_1,x_{n+1})\in A(D)$ and $(x_{n+1},x_1)\notin A(D)$.

First assume by contradiction that $x_{n+1}=x_1$. It follows from the inductive hypothesis on $(x_1,\dots,x_n)$ that $(x_1,x_n)\in A(D)$, and so $(x_{n+1},x_n)\in A(D)$, contradicting our assumption on T; so T is a directed path. Now consider the arcs (x_1,x_n) and (x_n,x_{n+1}); since D is a right-pretransitive or left-pretransitive digraph, $(x_n,x_1)\notin A(D)$ and $(x_{n+1},x_n)\notin A(D)$, we conclude $(x_1,x_{n+1})\in A(D)$. Finally suppose $(x_{n+1},x_1)\in A(D)$; when D is a right-pretransitive digraph considering the arcs (x_{n+1},x_1) and (x_1,x_n), and when D is a left-pretransitive considering the arcs (x_n,x_{n+1}) and (x_{n+1},x_1), we conclude that $(x_{n+1},x_n)\in A(D)$ or $(x_n,x_1)\in A(D)$, which is impossible. □

Lemma 2.2. *Let D be a right-pretransitive or left-pretransitive digraph. If D has no infinite outward paths, and $\emptyset\neq U\subseteq V(D)$, then there exists $x\in U$ such that $(x,y)\in A(D)$ with $y\in U$ implies $(y,x)\in A(D)$.*

Proof. Suppose by contradiction that for each $x\in U$, there exists $y\in U$ such that $(x,y)\in A(D)$ and $(y,x)\notin A(D)$. Consider some $x_1\in U$, then there exists $x_2\in U$ such that $(x_1,x_2)\in A(D)$ and $(x_2,x_1)\notin A(D)$. So for each $n\in\mathbb{N}$; given $x_n\in U$, there exists $x_{n+1}\in U$ such that $(x_n,x_{n+1})\in A(D)$ and $(x_{n+1},x_n)\notin A(D)$. It follows from Lemma 2.1 that $T_{n+1}=(x_1,\dots,x_n,x_{n+1})$ is a directed path. Consider the sequence $T=(x_n)_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, $(x_n,x_{n+1})\in A(T_{n+1})\subseteq A(D)$; for $n<m$ we have $\{x_n,x_m\}\subseteq V(T_m)$, and since T_m is a directed path we obtain $x_n\neq x_m$; hence T is an infinite outward path, a contradiction. □

Theorem 2.1. *Let D be a digraph. If there exists two subdigraphs of D say D_1 and D_2 such that $D=D_1\cup D_2$ (possibly $A(D_1)\cap A(D_2)\neq\emptyset$), where D_1 is a right-pretransitive digraph, D_2 is a left-pretransitive digraph, and D_i contains no infinite outward path for $i\in\{1,2\}$. Then D is a kernel-perfect digraph.*

Proof. It suffices to prove that D has a kernel, as any induced subdigraph of D satisfies the hypothesis of Theorem 2.1.

For independent sets of vertices of D; S, T; we write $S\leqslant T$ if and only if, for each $s\in S$ there exists $t\in T$, such that either $s=t$ or ($s\overset{1}{\rightarrow}t$ and $t\overset{1}{\nrightarrow}$ s). Notice that in particular $S\subseteq T$ implies $S\leqslant T$.

(1) The collection of all independent sets of vertices of D is partially ordered by $\leqslant$.

(1.1) $\leqslant$ is reflexive.

This follows from the fact $S \subseteq S$.

(1.2) $\leqslant$ is transitive.

Let S, T and R be independent sets of vertices of D, such that $S \leqslant T$ and $T \leqslant R$, and let $s \in S$. Since $S \leqslant T$ there exists $t \in T$ such that either, $s = t$ or ($s \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} s$) and; $T \leqslant R$ implies there exists $r \in R$ such that either, $t = r$ or ($t \xrightarrow{1} r$ and $r \overset{1}{\nrightarrow} t$). If $s = t$ or $t = r$, then $s = r$ or ($s \xrightarrow{1} r$ and $r \overset{1}{\nrightarrow} s$) with $r \in R$. So we can assume $s \neq t$, $t \neq r$, ($s \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} s$) and ($t \xrightarrow{1} r$ and $r \overset{1}{\nrightarrow} t$). And since D_1 is a right-pretransitive digraph it follows from Lemma 2.1 on the sequence (s, t, r) that $s \xrightarrow{1} r$ and $r \overset{1}{\nrightarrow} s$.

(1.3) $\leqslant$ is antisymmetrical.

Let S and T be independent sets of vertices of D such that $S \leqslant T$ and $T \leqslant S$, and let $s \in S$. Since $S \leqslant T$ there exists $t \in T$ such that either, $s = t$ or ($s \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} s$). Suppose $s \neq t$; the fact $T \leqslant S$ implies that there exists $s' \in S$ such that either, $t = s'$ or ($t \xrightarrow{1} s'$ and $s' \overset{1}{\nrightarrow} t$). When $t = s'$ we obtain $s \xrightarrow{1} s'$ contradicting that S is an independent set; so $t \neq s'$ and ($t \xrightarrow{1} s'$ and $s' \overset{1}{\nrightarrow} t$). Now applying Lemma 2.1 on the sequence (s, t, s'), we have $s \xrightarrow{1} s'$ contradicting that S is an independent set. We conclude $t = s$ and consequently $s \in T$ and $S \subseteq T$. Analogously it can be proved $T \subseteq S$.

Let $\mathscr{F}$ be the family of all nonempty independent sets S of vertices of D such that, $S \xrightarrow{2} y$ implies $y \to S$ for all vertices y of D_2.

(2) $(\mathscr{F}, \leqslant)$ has maximal elements.

(2.1) $\mathscr{F} \neq \emptyset$.

Since D_2 is a left-pretransitive digraph, which has no infinite outward paths; it follows from Lemma 2.2 (taking $D = D_2$ and $U = V(D_2)$), that there exists a vertex $x \in V(D_2)$ such that $x \xrightarrow{2} y$ implies $y \to x$, for all vertices y of D_2, so $\{x\} \in \mathscr{F}$.

(2.2) Every chain in $(\mathscr{F}, \leqslant)$ is upper bounded.

Let $\mathscr{C}$ be a chain in $(\mathscr{F}, \leqslant)$, and define $S^\infty = \{s \in \bigcup_{S \in \mathscr{C}} S \mid$ there exists $S \in \mathscr{C}$ such that $s \in T$ whenever $T \in \mathscr{C}$ and $T \geqslant S\}$ (S^∞ consists of all vertices of D that belong to every member of $\mathscr{C}$ from some point on).

We will prove that S^∞ is an upper bound of $\mathscr{C}$.

(2.2.1) $S^\infty \neq \emptyset$, and for each $S \in \mathscr{C}$, $S^\infty \geqslant S$.

Let $S \in \mathscr{C}$ and $t_0 \in S$, we will prove that there exists $t \in S^\infty$ such that either, $t_0 = t$ or ($t_0 \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} t_0$). If $t_0 \in S^\infty$ we are done, so assume $t_0 \notin S^\infty$. We proceed by contradiction; suppose that if $t \in V(D)$ with ($t_0 \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} t_0$), then $t \notin S^\infty$. Take $T_0 = S$; since $t_0 \notin S^\infty$ we have that there exists $T_1 \in \mathscr{C}$, $T_1 \geqslant T_0$ such that $t_0 \notin T_1$. Hence there exists $t_1 \in T_1$ such that $t_0 \xrightarrow{1} t_1$ and $t_1 \overset{1}{\nrightarrow} t_0$. And our assumption implies $t_1 \notin S^\infty$. The fact $t_1 \notin S^\infty$ implies $t_1 \notin T_2$ for some $T_2 \in \mathscr{C}$, $T_2 \geqslant T_1$, and there exists $t_2 \in T_2$ such that $t_1 \xrightarrow{1} t_2$ and $t_2 \overset{1}{\nrightarrow} t_1$. Since D_1 is a right-pretransitive digraph, it follows from Lemma 2.1 on the sequence $\tau_2 = (t_0, t_1, t_2)$, that τ_2 is a directed path, $t_0 \xrightarrow{1} t_2$ and $t_2 \overset{1}{\nrightarrow} t_0$, and $t_2 \notin S^\infty$. We may continue that way and we obtain, for each $n \in \mathbb{N}$; $T_n \in \mathscr{C}, t_n \in T_n, (t_0 \xrightarrow{1} t_n$ and $t_n \overset{1}{\nrightarrow} t_0)$ and $t_n \notin S^\infty$, hence there exists $T_{n+1} \in \mathscr{C}$ such that $T_{n+1} \geqslant T_n$ and $t_n \notin T_{n+1}$; so there exists $t_{n+1} \in T_{n+1}$ with $t_n \xrightarrow{1} t_{n+1}$ and $t_{n+1} \overset{1}{\nrightarrow} t_n$.

Since D_1 is a right-pretransitive digraph, and $(t_n \xrightarrow{1} t_{n+1}$ and $t_{n+1} \overset{1}{\nrightarrow} t_n)$ for each $n \in \mathbb{N}$; it follows from Lemma 2.1 (on the sequence) $\tau_{n+1} = (t_0, t_1, \ldots, t_{n+1})$, that τ_{n+1} is a directed path in D_1 and $(t_0 \xrightarrow{1} t_{n+1}$ and $t_{n+1} \overset{1}{\nrightarrow} t_0)$. And our assumption implies $t_{n+1} \notin S^\infty$. Now consider the sequence $\tau = (t_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$ we have $t_n \xrightarrow{1} t_{n+1}$, and for $n < m, \{t_n, t_m\} \subseteq V(\tau_m)$; and since τ_m is a directed path we have $t_n \neq t_m$. Hence τ is an infinite outward path contained in D_1. We conclude that there exists $t \in S^\infty$ such that $(t_0 \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} t_0)$.

(2.2.2) S^∞ is an independent set.

Let $s_1, s_2 \in S^\infty$ and suppose without loss of generality that $S_1, S_2 \in \mathscr{C}$ are such that: $s_1 \in S_1, s_1 \in S$ whenever $S \in \mathscr{C}$ and $S \geqslant S_1, s_2 \in S_2$ and $S_1 \leqslant S_2$. Then $s_1 \in S_2$ and since S_2 is independent there is no arc between s_1 and s_2 in D.

(2.2.3) $S^\infty \in \mathscr{F}$.

Suppose $S^\infty \xrightarrow{2} y$ with $y \in V(D_2)$, so there exists $s \in S^\infty$ with $s \xrightarrow{2} y$. Let $S \in \mathscr{C}$ such that $s \in T$ for all $T \in \mathscr{C}, T \geqslant S$.

Since $S \in \mathscr{F}$ we have $y \to S$, so there exists $s' \in S$ with $y \to s'$. When $s' \in S^\infty$ we are done. When $s' \notin S^\infty$ we analyze the two possibilities, $y \xrightarrow{1} s'$ or $y \xrightarrow{2} s'$. First suppose $y \xrightarrow{2} s'$; since $s \xrightarrow{2} y$, and D_2 is a left-pretransitive digraph it follows $s \xrightarrow{2} s'$ or $y \xrightarrow{2} s$, now the fact S is an independent set and $\{s, s'\} \subseteq S$ implies $s \overset{2}{\nrightarrow} s'$, so $y \xrightarrow{2} s$ and consequently $y \to S^\infty$. Now suppose $y \xrightarrow{1} s'$; since $s' \in S$ and since $S \leqslant S^\infty$ by (2.2.1), and $s' \notin S^\infty$. There exists $t \in S^\infty$ such that $s' \xrightarrow{1} t$ and $t \overset{1}{\nrightarrow} s'$, finally the fact that D_1 is a right-pretransitive digraph implies $y \xrightarrow{1} t$.

We have proven that any chain in $\mathscr{F}$ has an upper bound in $\mathscr{F}$, and so by Zorn's Lemma, $(\mathscr{F}, \leqslant)$ contains maximal elements. Let S be a maximal element of $(\mathscr{F}, \leqslant)$.

(3) S is a kernel of D.

Since $S \in \mathscr{F}$, S is an independent set of vertices of D.

(3.1) For each $x \in (V(D) - S)$ there exists an xS-arc.

Suppose by contradiction there exists $x \in (V(D) - S)$ such that $x \nrightarrow S$.

(3.1.1) There exists a vertex $x_0 \in V(D)$ such that $x_0 \nrightarrow S$, and x_0 satisfies: $x_0 \xrightarrow{2} y$ and $y \nrightarrow S$ imply $y \to x_0$ for all vertices $y \in V(D_2)$

Let $U = \{z \in V(D_2) - S \mid z \nrightarrow S\}$. When $U \neq \emptyset$ it follows from Lemma 2.2 (applied on D_2 and U) that there exists x_0 with the required properties. When $U = \emptyset$ it follows from our assumption that $z \nrightarrow S$, for some vertex z in $V(D_1) - (S \cup V(D_2))$. And we take x_0 any such a vertex.

Notice that the choice of x_0 implies $x_0 \nrightarrow S$ and since $S \in \mathscr{F}$ also we have $S \overset{2}{\nrightarrow} x_0$.

Let $T = \{s \in S \mid s \overset{1}{\nrightarrow} x_0\}$, it follows from above that $T \cup \{x_0\}$ is an independent set of vertices of D.

(3.1.2) $T \cup \{x_0\} \in \mathscr{F}$.

Suppose $T \cup \{x_0\} \overset{2}{\rightarrow} y$ and $y \nrightarrow T$; we will prove $y \rightarrow x_0$. Before to start the proof of (3.1.2) we make the following observation.

(3.1.2.1) If $y \overset{1}{\rightarrow} (S - T)$ then $y \overset{1}{\rightarrow} x_0$.

Assume $y \overset{1}{\rightarrow} (S - T)$; since $s \overset{1}{\rightarrow} x_0$ for any $s \in (S - T)$, and D_1 is a right-pretransitive digraph, we have $y \overset{1}{\rightarrow} x_0$ or $x_0 \overset{1}{\rightarrow} (S - T)$. Now, we know $x_0 \nrightarrow S$, so, we conclude $y \overset{1}{\rightarrow} x_0$. We proceed to prove (3.1.2) by considering the two following cases:

Case a: $T \overset{2}{\rightarrow} y$.

Since $T \subset S$ we have $S \overset{2}{\rightarrow} y$ and the fact $S \in \mathscr{F}$ implies $y \rightarrow S$. So $y \rightarrow (S - T)$ (as we are assuming $y \nrightarrow T$).

When $y \overset{1}{\rightarrow} (S - T)$ it follows from (3.1.2.1) that $y \overset{1}{\rightarrow} x_0$.

When $y \overset{2}{\rightarrow} (S - T)$; since we have $T \overset{2}{\rightarrow} y$ and D_2 is a left-pretransitive digraph, it follows $y \overset{2}{\rightarrow} T$ or $T \overset{2}{\rightarrow} (S - T)$; now $T \overset{2}{\rightarrow} (S - T)$ is impossible as $T \subset S$ and S is an independent set, we conclude $y \overset{2}{\rightarrow} T$, a contradiction.

Case b: $x_0 \overset{2}{\rightarrow} y$.

We consider two possible subcases:

Case b.1: $y \nrightarrow S$.

Since $x_0 \overset{2}{\rightarrow} y$ we have $y \in V(D_2)$ and the choice of x_0 (see (3.1.1)) implies $y \rightarrow x_0$.

Case b.2: $y \rightarrow S$.

In this case we have $y \rightarrow (S - T)$ (as we are assuming $y \nrightarrow T$). When $y \overset{1}{\rightarrow} (S - T)$ it follows from (3.1.2.1) that $y \overset{1}{\rightarrow} x_0$.

When $y \overset{2}{\rightarrow} (S - T)$, since $x_0 \overset{2}{\rightarrow} y$ and D_2 is a left-pretransitive digraph, we obtain $x_0 \overset{2}{\rightarrow} (S - T)$ or $y \overset{2}{\rightarrow} x_0$; now recalling that $x_0 \nrightarrow S$, so, we conclude $y \overset{2}{\rightarrow} x_0$.

(3.1.3) $S < T \cup \{x_0\}$.

For $s \in (S - T)$ we have $s \overset{1}{\rightarrow} x_0$ and we have noted $x_0 \nrightarrow S$; hence $S \leqslant T \cup \{x_0\}$, and it follows from the fact $x_0 \notin S$ (by the construction in (3.1.1)) that $S < T \cup \{x_0\}$.

Clearly propositions (3.1.2) and (3.1.3) contradict that S is a maximal element of $(\mathscr{F}, \leqslant)$. □

Remark 2.1. The hypothesis D_i has no infinite outward paths in Theorem 2.1 is necessary.

Consider the following digraph D; $V(D)=\{u_n \mid n \in \mathbb{N}\}$ and $A(D)=\{(u_n, u_m) \mid n, m \in \mathbb{N}$ and $n < m\}$, $D_1 = D$ and $D_2 = D$.

It is easy to see that if H is any right-pretransitive digraph, and we consider D_1 and D_2 such that: $V(D_1)=V(D)\cup V(H)$, $A(D_1)=A(H)\cup\{(u,v) \mid u \in V(H), v \in V(D)\}$ and $D_2 = D$ then $D = D_1 \cup D_2$ has no kernel.

Remark 2.2. The following digraph D is the union of two right-pretransitive digraphs, D_1 and D_2, and it has no kernel.

$$V(D_1) = V(D_2) = \{u, v, w, x\}, \quad A(D_1) = \{(x,u),(u,w),(w,u),(v,w)\},$$

$$A(D_2) = \{(u,v),(x,v),(v,x),(w,x)\} \quad \text{and} \quad D = D_1 \cup D_2.$$

Remark 2.3. There exists a digraph D which is the union of two left-pretransitive digraphs and has no kernel.

$$V(D_1) = V(D_2) = \{u, v, w, x\}, \quad A(D_1) = \{(u,v),(u,w),(w,u),(w,x)\},$$

$$A(D_2) = \{(x,u),(x,v),(v,x),(v,w)\} \quad \text{and} \quad D = D_1 \cup D_2.$$

It is easy to see that by adding vertices to this digraphs one can construct arbitrarily large finite examples as those given in Remarks 2.2 and 2.3.

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References

[1] C. Berge, Graphs, in: North-Holland Mathematical Library, Vol. 6, North-Holland, Amsterdam, 1985.
[2] P. Duchet, Graphes Noyau-Parfaits, Ann. Discrete Math. 9 (1980) 93–101.
[3] P. Duchet, A sufficient condition for a digraph to be kernel-perfect, J. Graph Theory 11 (1) (1987) 81–85.
[4] P. Duchet, H. Meyniel, A note on kernel-critical graphs, Discrete Math. 33 (1981) 103–105.
[5] H. Galeana-Sánchez, Kernels in edge-coloured digraphs, Discrete Math. 184 (1998) 87–99.
[6] H. Galeana-Sánchez, V. Neumann-Lara, On kernels and semikernels of digraphs, Discrete Math. 48 (1984) 67–76.
[7] J. Von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1944.
[8] M. Richardson, Solutions of irreflexive relations, Ann. Math. 58 (2) (1953) 573–580.
[9] M. Richardson, Extensions theorems for solutions of irreflexive relations, Proc. Nat. Acad. Sci. USA 39 (1953) 649–651.
[10] B. Sands, N. Sauer, R. Woodrow, On monochromatic paths in edge-coloured digraphs, J. Combin. Theory Ser. B 33 (1982) 271–275.