Differential Geometry and its Applications 16 (2002) 213–224

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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Umbilicity of surfaces with orthogonal asymptotic lines in $\mathbb{R}^4$

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Received 24 July 2000; received in revised form 15 December 2000

Abstract

We study some properties of surfaces in 4-space all whose points are umbilic with respect to some normal field. In particular, we show that this condition is equivalent to the orthogonality of the (globally defined) fields of asymptotic directions. We also analyze necessary and sufficient conditions for the hypersphericity of surfaces in 4-space. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 53A05; 58C25

Keywords: Asymptotic direction; ν-principal curvature foliation; Umbilicity; Sphericity

1. Introduction

It is possible to define asymptotic directions over the points of the surfaces in 4-space (called conjugate directions by J. Little [5]). These directions determine fields that do not need to be globally defined on the surfaces. It was shown in [6], by means of techniques relying on the analysis of the singularities of height functions on the surface, that each field of asymptotic directions is associated to some normal field of binormal directions on the surface and that a necessary and sufficient condition for existence of two globally defined fields of this type on a surface M in $\mathbb{R}^4$ is the local convexity of M (in the sense that it has a locally support hyperplane at each one of its points). It was also proven that the critical points of these fields are the inflection points of M.

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[1] Work partially supported by DGCYT grant no. BFM2000-1110.
[2] Work partially supported by DGAPA-UNAM grant no. 127498/México.

PII: S0926-2245(02)00068-2

We say that a surface in $\mathbb{R}^4$ is *hyperspherical* provided it is contained in a hypersphere. Clearly, any hyperspherical surface is locally convex. We saw in [7] that stereographic projection transforms curvature lines of surfaces in $\mathbb{R}^3$ into asymptotic lines of their images in S^3 considered as submanifolds of $\mathbb{R}^4$. Consequently, if the surface is hyperspherical, then the two fields of asymptotic directions must be orthogonal all over the surface, except at the inflection points. It was then conjectured that this orthogonality condition on the asymptotic lines is also sufficient to guarantee the hypersphericity of surfaces in $\mathbb{R}^4$.

The main feature of this paper consists in finding some geometrical conditions which are equivalent to the orthogonality of asymptotic lines, and proving that these together with a further requirement imply the hypersphericity of the surface.

Given a surface M in $\mathbb{R}^4$ and a globally defined normal field ν on M, there is a shape operator S_ν on M intrinsically attached to the second fundamental form, II_ν, associated to ν on M. The eigenvectors of S_ν determine the ν-curvature lines of M and its eigenvalues the ν-principal curvatures. We say that a point $x \in M$ is ν-umbilic provided the two ν-principal curvatures, λ_1 and λ_2 coincide at x. The typical structure of the curvature lines for a generic normal field ν on M was analized in [11]. The ν-umbilic points were characterized as the critical points of the corresponding principal direction fields.

A surface is said to be *ν-umbilic* if all its points are umbilic for the field ν. In this case we have a curvature function λ associated to the field ν defined over the whole M. A surface M is *totally umbilic* if it is ν-umbilic for any normal field ν over M. It is well known (see [12] for instance) that a surface M in 4-space is totally umbilic with the same principal curvature for any normal direction if and only if it is a 2-sphere. On the other hand, the geometric properties of the surfaces that are umbilic for some normal field have been studied by B.Y. Chen [1,2]. In this work, we relate the property of having globally defined orthogonal asymptotic lines with the ν-umbilicity for some normal field, obtaining the following result:

Theorem 3.4(a, b). *A surface M immersed in $\mathbb{R}^4$ has two globally defined orthogonal fields of asymptotic directions if and only if it is ν-umbilic for some globally defined normal field ν on M.*

Moreover, we show that surfaces with this property have univocally defined principal curvature lines, which coincide with the asymptotic lines, independently of the choice of the normal field (different from ν) on M.

On the other hand, we prove that ν-umbilicity of M is also equivalent to the vanishing of the normal curvature of M, or in other words, to the requirement that the normal bundle of M be totally flat. It follows from this that

Theorem 3.4(b, d). *M is ν-umbilic for some globally defined normal field ν if and only if M is totally made of semi-umbilic points.*

It is interesting to observe that the semi-umbilic points can be characterized as singularities of corank 2 for distance squared functions taken from some focal centers of the surface (see [10] for an introduction to the geometrical interpretation of the singularities of distance squared functions on submanifolds and [8] for the particular case of surfaces in 4-space). It follows that the surfaces all whose points are semi-umbilic have a "degenerate" family of distance squared functions (in the sense that it is not stable). In other words, these surfaces have non generic contacts with their focal hypersphere at each point, in the sense that they are "stronger" than the usual ones at most points. In the case of a surface contained in a hypersphere, this contact is completely degenerate. In fact, the distance squared function from the center of the hypersphere is constant and thus has a non finitely determined singularity at every point. We also point out that the singularities of corank 2 for the distance squared functions on surfaces in 3-space are precisely the umbilic points of these surfaces. Therefore, the surfaces in 3-space that are totally made of corank 2 singularities for distance squared functions are either pieces of a 2-sphere or a plane.

Once we have put the things in terms of ν-umbilicity we can apply the theory developed by Chen in order to obtain results on hypersphericity. In particular, we can use the following statement [1, Corollary 3.1, p. 473], a proof of which, in the case of surfaces in 4-space, is included here for the sake of completeness:

Theorem 4.3. *The surface M is hyperspherical if and only if it is ν-umbilic for some unit normal field ν over M whose associated principal curvature λ is a nonzero constant.*

We observe that in the case of a surface with isolated inflection points this amounts to say that the surface M is hyperspherical if and only if it is ν-umbilic for some normal parallel field ν over M.

Finally, we conclude

Corollary 4.7. *The surface M is hyperspherical if and only if its asymptotic lines are globally defined and orthogonal and its binormal curvatures $\{k_i\}_{i=1,2}$ satisfy the following relation*

$$\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}+2\cos\alpha\right)E=\text{constant},$$

where α is the angle between the two binormals at each point and E represents the coefficient of the first fundamental form of M in isothermic coordinates.

We would like to point out, finally, that the stereographic projection provides a bridge between the study of the properties of asymptotic lines and inflection points of surfaces in $\mathbb{R}^4$ and that of curvature lines and umbilic points of those in $\mathbb{R}^3$. In this sense, any new results concerning the first represent a generalization of similar problems relative to the later ones.

2. Curvature lines associated to a normal vector field

Let M be a smooth oriented surface immersed in $\mathbb{R}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^4$. For each $p \in M$ consider the decomposition $T_p\mathbb{R}^4 = T_pM \oplus N_pM$, where N_pM is the orthogonal complement of T_pM in $\mathbb{R}^4$. Let $\overline{\nabla}$ be the Riemannian connection of $\mathbb{R}^4$. Given local vector fields X, Y on M, let $\overline{X}, \overline{Y}$ be some local extensions to $\mathbb{R}^4$. The tangent component of the Riemannian connection in $\mathbb{R}^4$ is the Riemannian connection of M: $\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^\top$.

Let $\mathcal{X}(M)$ and $\mathcal{N}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map,

$$\alpha : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{N}M, \quad \alpha(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y.$$

This map is well defined, symmetric and bilinear.

Let $p \in M$ and $\nu \in N_pM$, $\nu \neq 0$, define the function

$$H_\nu : T_pM \times T_pM \to \mathbb{R}, \quad H_\nu(X,Y) = \langle \alpha(X,Y), \nu \rangle.$$

Then this function is as well symmetric and bilinear. The second fundamental form of M at p is the associated quadratic form,

$$II_\nu : T_pM \to \mathbb{R}, \quad II_\nu(X) = H_\nu(X,X).$$

Recall the shape operator

$$S_\nu : T_pM \to T_pM, \quad S_\nu(X) = -(\overline{\nabla}_{\overline{X}}\bar{\nu})^\top,$$

where $\bar{\nu}$ is a local extension to $\mathbb{R}^4$ of the normal vector field ν at p and $\top$ means the tangent component. This operator is bilinear, self-adjoint and for any $X, Y \in T_pM$ satisfies the following equation: $\langle S_\nu(X), Y\rangle = H_\nu(X,Y)$. So, the second fundamental form can be expressed by $II_\nu(X) = \langle S_\nu(X), X\rangle$. Thus for each $p \in M$, there exists an orthonormal basis of eigenvectors of $S_\nu \in T_pM$, for which the restriction of the second fundamental form to the unitary vectors, $II_\nu|_{S^1}$, takes its maximal and minimal values. The corresponding eigenvalues k_1, k_2 are the *maximal* and *minimal* ν*-principal curvatures*, respectively. The point p is a ν-umbilic if the ν-principal curvatures coincide. Let $\mathcal{U}_\nu$ be the set of ν-umbilics in M. For any $p \in M \backslash \mathcal{U}_\nu$ there are two ν-principal directions defined by the eigenvectors of S_ν, these fields of directions are smooth and integrable, then they define two families of orthogonal curves, its integrals, which are called the ν*-principal lines of curvature*, one maximal and the other one minimal. The two orthogonal foliations with the ν-umbilics as its singularities form the ν*-principal configuration* of M. We say that the surface M is ν*-umbilical* if each point of M is ν-umbilic. The differential equation of ν-lines of curvature is

$$S_\nu\big(X(p)\big) = \lambda(p)X(p). \tag{1}$$

Suppose that ϕ, $U \subset M$ is an open neighborhood with local coordinates (u, v). Let E, F, G be the coefficients of the first fundamental form in this coordinate chart. The coefficients of the second fundamental form are

$$e_\nu = II_\nu(\partial_u) = -\langle \alpha(\partial_u, \partial_u), \nu\rangle,$$
$$f_\nu = -\langle \alpha(\partial_u, \partial_v), \nu\rangle = -\langle \alpha(\partial_v, \partial_u), \nu\rangle,$$
$$g_\nu = II_\nu(\partial_v) = -\langle \alpha(\partial_v, \partial_v), \nu\rangle,$$

where $\partial_u = \frac{\partial}{\partial u}$ and $\partial_v = \frac{\partial}{\partial v}$.

Eq. (1) has the following expression in this coordinate chart [11].

$$(f_\nu E - e_\nu F)\,du^2 + (g_\nu E - e_\nu G)\,du\,dv + (g_\nu F - f_\nu G)\,dv^2 = 0.$$

Assume that this coordinate chart is isothermic: $E = G > 0$, $F = 0$. Then this equation has the form

$$f_\nu\,du^2 + (g_\nu - e_\nu)\,du\,dv - f_\nu\,dv^2 = 0. \tag{2}$$

Lemma 2.1. *Assume that there exist* $\nu \in \mathcal{N}M$ *such that* M *is* ν*-umbilical. The principal configuration of any normal vector field* η *linear independent of* ν*, is univocally determined.*

Proof. For any $p \in M$ consider a local isothermic chart as above. Define $\nu^\perp = \phi_u \wedge \phi_v \wedge \nu \in \mathcal{N}M$, the cross product in $\mathbb{R}^4$ of the vector fields ϕ_u, ϕ_v, ν. At each point $p \in M$ the frame $\{\phi_u(p), \phi_v(p), \nu(p), \nu^\perp(p)\}$ is an orthogonal basis of $T_p\mathbb{R}^4$, so for any normal vector field η, there are smooth functions $a, b : U \to \mathbb{R}$ such that $\eta = a\nu + b\nu^\perp$. The coefficients of the second fundamental form can be expressed by

$$e_\eta = -\langle \phi_{uu}, \eta\rangle = -\langle \phi_{uu}, a\nu + b\nu^\perp\rangle = ae_\nu + be_{\nu^\perp}.$$

Analogously, $f_\eta = af_\nu + bf_{\nu^\perp}$ and $g_\eta = ag_\nu + bg_{\nu^\perp}$. Therefore the equation of ν-lines of curvature in these coordinates is

$$a\big(f_\nu\, du^2 + (g_\nu - e_\nu)\, du\, dv - f_\nu\, dv^2\big) + b\big(f_{\nu^\perp}\, du^2 + (g_{\nu^\perp} - e_{\nu^\perp})\, du\, dv - f_{\nu^\perp}\, dv^2\big) = 0,$$

since M is ν-umbilic $f_\nu = 0$ and $g_\nu = f_\nu$ so

$$b\big(f_{\nu^\perp}\, du^2 + (g_{\nu^\perp} - e_{\nu^\perp})\, du\, dv - f_{\nu^\perp}\, dv^2\big) = 0, \quad b(p) \neq 0,$$

which implies that the principal configurations of η and $\nu^\perp$ coincide. □

Remark 2.2. It can be seen in [1] that if M is a spherical surface in $\mathbb{R}^4$, then M is ρ-umbilical with constant normal curvature.

Lemma 2.3. *Assume that $M \subset \mathbb{R}^4$ is a smooth oriented surface immersed in $\mathbb{R}^4$ and η is a vector field in $\mathcal{N}M$, then*:

(a) *The coordinate lines of the parametrization of M coincide with the η-lines of curvature if and only if $F = 0$ and $f_\eta = 0$.*
(b) *In this system of coordinates the principal curvatures have the following expression*:

$$k_1 = \frac{e_\eta}{E}, \qquad k_2 = \frac{g_\eta}{G}.$$

Proof. (a) If the coordinate lines coincide with the η-lines of curvature, they most be orthogonal, so $F = 0$ and the differential equation of the η-lines of curvature is:

$$f_\eta E\, du^2 + (g_\eta E - e_\eta G)\, du\, dv - f_\eta G\, dv^2 = 0.$$

Since ∂_u verifies this equation, then $f_\eta E = 0$ thus f_η vanishes. The converse follows from the form of the equation of η-lines of curvature:

$$(g_\eta E - e_\eta G)\, du\, dv = 0,$$

which is obviously satisfied by the coordinate vector fields ∂_u, ∂_v.

(b) Let $X = \phi_u X^1 + \phi_v X^2$, so write in coordinates the expression of the shape operator

$$S_\eta(X) = -\big(\eta_u X^1 + \eta_v X^2\big)^\top = (a_{11}\phi_u + a_{21}\phi_v)X^1 + (a_{12}\phi_u + a_{22}\phi_v)X^2.$$

On the other hand, since the tangent component of $\eta_u = \overline{\nabla}_{\partial u}\eta$ is $-S_\eta(\partial_u)$ we can compute the coefficients of the second fundamental form with respect to η in these terms, so

$$\begin{aligned} e_\eta &= -\langle \phi_{uu}, \eta\rangle = \langle \phi_u, \eta_u\rangle = a_{11}E + a_{21}F, \\ f_\eta &= -\langle \phi_{uv}, \eta\rangle = \langle \phi_u, \eta_v\rangle = a_{11}F + a_{21}G, \\ g_\eta &= -\langle \phi_{vv}, \eta\rangle = \langle \phi_v, \eta_v\rangle = a_{12}F + a_{22}G. \end{aligned} \tag{3}$$

Solving this system for a_{ij} with the conditions $F = 0 = f_\eta$ we obtain:

$$a_{11} = \frac{e_\eta}{E}, \quad \frac{g_\eta}{G}, \quad a_{12} = a_{21} = 0. \qquad \square$$

3. Binormal fields and asymptotic directions

Let M be a surface embedded by ϕ in $\mathbb{R}^4$. Given $p \in M$, consider the unit circle in T_pM parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ the curve obtained by intersecting M with the hyperplane at p composed by the direct sum of the normal plane N_pM and the straight line in the tangent direction represented by θ. Such curve is called *normal section of $\phi(M)$ in the direction θ*. The curvature vector $\eta(\theta)$ of γ_θ in p lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the *curvature ellipse* of M at p. The points in M are classified into *hyperbolic, parabolic* or *elliptic* provided they lie outside, on or inside the curvature ellipse. When this ellipse degenerates to a segment the point is said to be a *semi-umbilic center*. In the particular case that it is a radial segment of p is known as an *inflection point* of the surface. This inflection point is of real type when p belongs to the curvature ellipse, and of imaginary type when it doesn't. A direction θ in $T_{\phi(p)}\phi(M)$ for which $\frac{\partial\eta}{\partial\theta}$ and $\eta(\theta)$ are parallel is said to be an *asymptotic direction*.

Consider an orthonormal frame $\{X_1, X_2, X_3, X_4\}$ on M and take the dual 1-forms $\{w_1, w_2, w_3, w_4\}$, given by $w_i = \langle d\phi, X_i\rangle$. Let $\{w_{ij}\}_{i,j=1}^4$ be the corresponding connection forms (see [3] or [12]). These forms have the following expression in terms of the dual 1-forms [5, p. 263]:

$$\begin{aligned}
w_{13} &= e_{X_3}w_1 + f_{X_3}w_2,\\
w_{23} &= f_{X_3}w_1 + g_{X_3}w_2,\\
w_{14} &= e_{X_4}w_1 + f_{X_4}w_2,\\
w_{24} &= f_{X_4}w_1 + g_{X_4}w_2.
\end{aligned} \tag{4}$$

The *normal curvature*, N, of M is obtained from the following formula relative to the curvature form of the normal bundle of M: $dw_{34} = -Nw_1 \wedge w_2$. The function N is a multiple of the area element on M. In fact, it can be seen [5, p. 266] that

$$\frac{1}{2}\pi|N(p)| = \text{Area of curvature ellipse at } p.$$

There is an invariant function Δ on M defined as follows: Write $e = uX_1 + vX_2$ and consider $\langle de, X_3\rangle \wedge \langle de, X_4\rangle$. Now $de = u\,dX_1 + du\,X_1 + v\,dX_2 + dv\,X_2$. Therefore, $\langle de, X_3\rangle = uw_{13} + vw_{23}$ and $\langle de, X_4\rangle = uw_{14} + vw_{24}$. And taking into account that w_{13}, w_{23}, w_{14} and w_{24} can be put in terms of the basis $\{w_1, w_2\}$ of the dual of T_pM, we obtain

$$\langle de, X_3\rangle \wedge \langle de, X_4\rangle = \delta(u, v)w_1 \wedge w_2.$$

Then the function Δ is given by $\Delta(u, v) = \det \delta(u, v)$.

It can be shown that $\Delta(p)$ is > 0, $= 0$ or < 0 according to the point p is elliptic, parabolic or hyperbolic. The inflection points are also special points at which the function Δ vanishes.

Given a normal vector to M at p, η, the height function on M associated to η is defined by $h_\eta(p) = \langle\phi(p), \eta\rangle$. It is easy to see that h_η has a singularity at the point p. In the case that this is a degenerate singularity (non Morse), we shall say that η defines a binormal direction for M at p. It can be seen [7, Lemma 4] that θ is an asymptotic direction at p if and only if θ lies in the kernel of the Hessian of some height function h_η at p. In this case we say that θ is an asymptotic direction associated to the binormal direction η at p.

We observe a field of binormal directions need not be defined over the whole surface in general. Nevertheless it was shown in [6] that according to the point is hyperbolic, parabolic or elliptic we may find exactly two, one or none binormal directions respectively. A surface M is said to be *locally convex* if and only if admits a locally support hyperplane at each one of its points. It was also proven in [6] that a generic surface M is locally convex if and only if it is composed of hyperbolic and inflection points of imaginary type. In this case, we have two globally defined asymptotic fields whose singularities are the inflection points of M. The generic structure of these fields, as well as some global properties of the inflection points has been studied in [4].

Let η be a normal field on M. Then the Hessian matrix of the height function h_η at each point is given by

$$\begin{pmatrix} e_\eta & f_\eta \\ f_\eta & g_\eta \end{pmatrix},$$

where $e_\eta = -\langle \phi_{uu}, \eta\rangle$, $f_\eta = -\langle \phi_{uv}, \eta\rangle$, $g_\eta = -\langle \phi_{vv}, \eta\rangle$. We thus observe that the Hessian matrix of h_η coincides with the Jacobian matrix of the shape operator S_η.

Therefore, if b_i is one of the binormal fields on M, we have that one of the principal directions of b_i is always given by the corresponding asymptotic direction at each point. The associated principal curvature is, clearly, identically zero. The other one, k_i, shall be called the *binormal curvature* associated to b_i.

Lemma 3.1. *Suppose that* $\{b_i\}$, $i = 1, 2$, *are the two binormal vector fields on M and the corresponding asymptotic lines are mutually orthogonal. Then*:

(a) *The asymptotic lines are the curvature lines for both binormal fields.*
(b) *M is ν-umbilic for $\nu = k_2 b_1 + k_1 b_2$.*
(c) *Given any normal vector field η linear independent to ν, the η-lines of curvature coincide with the asymptotic lines of M.*

Proof. (a) Consider b_i any of the binormal vector fields and let $u \in T_pM$ be a vector which defines an asymptotic direction. Therefore u is in the kernel of the hessian of the height function h_{b_i}. Since this hessian coincides with the shape operator S_{b_i} at p, the vector u is an eigenvector of S_{b_i} corresponding to the null eigenvalue. The other eigenvector must be orthogonal to this one, but by hypothesis this is the other asymptotic direction at a non b_i-umbilic (i.e., inflection) point. Therefore the asymptotic lines associated to the binormal b_i are the curvature lines of this field.

(b) It follows from (a) that the equation of lines of curvature with respect to b_1 and b_2 coincide. Therefore there is a real valued function r defined on M for which the following equations hold, $f_{b_1} = rf_{b_2}$ and $g_{b_1} - e_{b_1} = r(g_{b_2} - e_{b_2})$, so $0 = f_{b_1} - rf_{b_2} = |b_1 - rb_2| f_\nu$ and $0 = g_{b_1} - rg_{b_2} - (e_{b_1} - re_{b_2}) = |b_1 - rb_2|(g_\nu - e_\nu)$, which implies p is ν-umbilic. Now, observe that since b_1 and b_2 are binormal we can assume that $e_{b_1} = g_{b_2} = 0$ and thus $g_{b_1} = -re_{b_2}$. Therefore $r = -\frac{g_{b_1}}{eb_2}$. Therefore M is umbilical for the field $b_1 + \frac{g_{b_1}}{e_{b_2}} b_2$. And thus it is also umbilical for the field $\nu = \frac{e_{b_2}}{E} b_1 + \frac{g_{b_1}}{E} b_2$. But it follows from Lemma 2.3 that $\nu = k_2 b_1 + k_1 b_2$.

(c) Let η be any normal field on M linearly independent to ν. Then, since M is ν-umbilic, Lemma 2.1 tells us that the η-lines of curvature coincide with the b_1-lines of curvature, $i = 1, 2$. According to (a) these are the asymptotic lines of M. □

Lemma 3.2. *With the hypothesis of Lemma* 3.1 *the curvature associated to the field ν is given by $\lambda_\nu = k_1 k_2 E$.*

Proof. By working with the isothermic coordinates determined by the asymptotic directions we have

$$\lambda_\nu = \frac{e_\nu}{E} = \frac{e_{b_2} e_{b_1} + g_{b_1} e_{b_2}}{E}.$$

Now we can assume that $e_{b_1} = 0$ for b_1 is a binormal. Thus

$$\lambda_\nu = \frac{g_{b_1} e_{b_2}}{E} = \frac{g_{b_1} e_{b_2}}{G} = k_1 e_{b_2}.$$

But $k_2 = \frac{e_{b_2}}{E}$ and the result follows. □

Remark 3.3. Given any normal field ν on M let $\bar{\nu} = \nu / \|\nu\|$. It follows from Eq. (2) that M is ν umbilic if and only if M is $\bar{\nu}$ umbilic. Therefore in what follows we shall assume that the vector field ν is unitary.

Theorem 3.4. *Let M be a surface immersed in $\mathbb{R}^4$. The following are equivalent conditions on M:*

(a) *M has two everywhere defined orthogonal fields of asymptotic lines.*
(b) *M is ν-umbilic, for some globally defined normal field ν on M.*
(c) *The normal curvature of M vanishes on every point.*
(d) *All the points of M are semi-umbilic.*

Proof. That (a) implies (b) follows from part (b) in Lemma 3.1. Let us prove that (b) implies (c). Suppose thus that M is ν-umbilic, for some normal field ν. Consider local coordinates and the orthonormal frame, $\{X_1(p), X_2(p), \nu(p), \nu^\perp(p)\}$, corresponding to the one in the proof of Lemma 2.1. Now, it can be seen that [5, p. 266]

$$|N| = \left|(e_\nu - g_\nu) f_{\nu^\perp} - (e_{\nu^\perp} - g_{\nu^\perp}) f_\nu\right|.$$

But the ν-umbilicity of M tells us that $e_\nu - g_\nu = 0$ and $f_\nu = 0$. Therefore $N(p) = 0, \forall p \in M$.

That (c) and (d) are equivalent follows from the fact, mentioned above, that $|N|$ is proportional to the area of the curvature ellipse. For we have that the curvature ellipse degenerates to a segment (or eventually to a point) if and only if its area vanishes.

It only remains to show that (c) implies (a). But this follows from the following formula (see [5, p. 268] or [13]),

$$\tan^2(\theta_1 - \theta_2) = \frac{\Delta}{N^2},$$

where θ_1 and θ_2 represent the angles in the tangent plane, corresponding to the asymptotic directions at a point p in M. □

4. ν-umbilicity and hypersphericity

Lemma 4.1. *Suppose that M is ν-umbilic and λ is the ν curvature function on M, where ν is a unitary normal field on M. In the same frame as in Theorem* 3.4 *we have*:

(a) $w_{34} \equiv 0$ *implies that λ is constant*;
(b) *if λ constant then for each point $p \in M$ we have either, $w_{34}(p) = 0$ or $w_{14}(p) = w_{24}(p) = 0$*;
(c) *if λ is a nonzero constant then $w_{34} \equiv 0$.*

Proof. In the considered frame we have that $X_3 = \nu$ and $X_4 = \nu^{\perp}$. Since M is X_3 umbilic we can write $\nabla_X X_3 = \lambda X$, for any vector X tangent to M. Then the connection forms satisfy

$$w_{j3}(X) = \langle \overline{\nabla}_X X_3, X_j\rangle = \lambda\langle X, X_j\rangle = \lambda w_j, \quad j = 1, 2. \tag{5}$$

By taking now the exterior derivative of this equation we get

$$d\lambda \wedge w_j + \lambda\, dw_j = dw_{j3}, \quad j = 1, 2. \tag{6}$$

On the other hand, we have for $j = 1$ [3]

$$dw_{13} = w_{12} \wedge w_{23} + w_{14} \wedge w_{43}.$$

By substituting in Eq. (6) we obtain

$$d\lambda \wedge w_1 + \lambda\, dw_1 = \lambda(w_{12} \wedge w_2) + w_{14} \wedge w_{43}.$$

But $dw_1 = w_{12} \wedge w_2$, therefore

$$d\lambda \wedge w_1 = w_{14} \wedge w_{43}$$

Analogous arguments for $j = 2$ lead to the expression

$$d\lambda \wedge w_2 = w_{24} \wedge w_{43}. \tag{7}$$

Therefore $d\lambda \equiv 0$ if and only if either $w_{34} = -w_{43} = 0$, or the 1 forms w_{14} and w_{24} are collinear. This proves (a). Now by writing w_{14} and w_{24} we have from Eq. (4)

$$\begin{aligned} w_{14} &= e_{X_4} w_1 + f_{X_4} w_2, \\ w_{24} &= f_{X_4} w_1 + g_{X_4} w_2. \end{aligned} \tag{8}$$

Now since ν is globally defined in M, by Theorem 3.4 we can take the isothermic coordinates determined by the asymptotic direction fields, thus we have that f_{X_4} vanishes and

$$w_{14} = e_{X_4} w_1,$$
$$w_{24} = g_{X_4} w_2.$$

Then, if w_{14} and w_{24} are collinear they must vanish. From which follows (b).

Suppose now that λ is a nonzero constant and that $w_{34} \neq 0$. Then there is some open subset U in M such that w_{14}, w_{24} are collinear on U, this happens if and only if $w_{14} = w_{24} = 0$ on U. But in this case we have that U is X_4-umbilic with vanishing associated curvature, i.e., X_4 is a binormal over U and all the points in U are inflection points of M. On the other hand U is X_3-umbilic with associated curvature

$\lambda \neq 0$. So we have

$$w_{13} = \lambda w_1,$$
$$w_{23} = \lambda w_2.$$

Then using the structure equations and the fact that $dw_{14} = 0$ and $dw_{24} = 0$, we see that $w_{34} = \beta_1 w_1$ and $w_{34} = \beta_2 w_2$, for some β_1, β_2. But w_1 and w_2 are independent so $w_{34} = 0$ on U. Consequently $w_{34} \equiv 0$ all over M. □

Let $\overline{\nabla}^{\perp}$ be the projection of $\overline{\nabla}$ in the normal bundle on M, we say that a vector field is parallel along M if $\overline{\nabla}^{\perp}_Y X \equiv 0$ for any vector Y tangent to M.

Proposition 4.2. *If M is ν-umbilic for some unitary normal field ν and has isolated inflection points, then the ν-curvature is constant if and only if ν is parallel.*

Proof. We observe that ν is parallel if and only if $w_{34} = 0$. The result follows easily by looking at the proof of the above lemma. □

Theorem 4.3. *Let M be a surface immersed in $\mathbb{R}^4$ such that it is ν-umbilic for some unitary normal field ν with constant associated curvature λ. Then if $\lambda \neq 0$ M is hyperspherical, and if $\lambda = 0$ M is hyperplanar.*

Proof. Suppose that $\lambda \neq 0$ then we know from Lemma 4.1 that $w_{34} \equiv 0$ on M and hence ν is a parallel field along M. Since ν is umbilic, for any vector X tangent to M, we have

$$\overline{\nabla}_X \nu = \nabla_X \nu + \overline{\nabla}^{\perp}_X \nu = \lambda X + \overline{\nabla}^{\perp}_X \nu.$$

But $\overline{\nabla}^{\perp} \nu = 0$ and hence

$$\overline{\nabla}_X \nu = \lambda X.$$

Now, the covariant derivation of the radial vector field ρ is the identity, i.e., $\nabla_X \rho = X$, for any vector X tangent to $\mathbb{R}^4$, and thus the following equation holds:

$$\overline{\nabla}_X (\nu - \lambda\rho) = 0,$$

for any X tangent to M. Therefore $\nu - \lambda\rho$ is parallel along M. This means that $\nu - \lambda\rho$ is a constant vector X_0, so

$$\nu(p) - \lambda(p) = X_0,$$
$$p = \frac{X_0 - \nu(p)}{\lambda},$$

for all $p \in M$. This means that M belongs to a hypersphere with center $\frac{X_0}{\lambda}$ and radius $\frac{1}{\lambda}$.

Assume now that $\lambda = 0$ and consider the frame $\{X_i\}_{i=1}^4$ as above. Then $e_{X_3} = f_{X_3} = g_{X_3} = 0$, and hence $w_{13} = w_{23} = 0$. Then from Lemma 4.1 either $w_{14} = w_{24} \equiv 0$ in which case $e_{X_4} = f_{X_4} = g_{X_4} = 0$ and we have that M is totally umbilic with vanishing curvature and therefore a plane.

Or there exists U open set on which $w_{34} = 0$ which tells us that $dX_3 = 0$ so X_3 is constant and therefore U lies in a 3-space perpendicular to X_3.

We can then conclude that if $\lambda = 0$ the surface M must lie in a hyperplane. □

Remark 4.4. We observe that in the above theorem the hypothesis $\lambda = 0$ implies that all the points of M are inflection points so we can state the following:

Corollary 4.5. *If M is a surface in $\mathbb{R}^4$ whose inflection points are isolated and it is ν-umbilic for some unitary normal field ν with constant curvature then M is hyperspherical.*

Remark 4.6. The existence of two globally defined orthogonal asymptotic fields implies automatically that the inflection points of M are isolated.

Corollary 4.7. *Suppose that M is a surface with isolated inflection points in $\mathbb{R}^4$. Then M is hyperspherical if and only if its asymptotic lines are globally defined and orthogonal and its binormal curvatures $\{k_i\}_{i=1,2}$ satisfy the following relation*

$$\left(\frac{k_1}{k_2} + \frac{k_2}{k_1} + 2\cos\alpha\right)E = \text{constant},$$

where α is the angle between the two binormals at each point.

Proof. As we have seen previously the field $\nu = k_2 b_1 + k_1 b_2$ has curvature $\lambda_\nu = k_1 k_2 E$. Now, the unit field $\nu/\|\nu\|$ is constant but this is equivalent to the above requirement. □

Remark 4.8. A submanifold M is said to be *isoparametric* provided its normal bundle is flat and the principal curvatures along any parallel normal field of M are constant. It has been shown in [9, p. 123] that an isoparametric n-manifold of $\mathbb{R}^{n+k}$ is compact if and only if it is contained in a standard hypersphere. We can then conclude from the above results that:

(a) Any isoparametric surface M in $\mathbb{R}^4$ is ν-umbilic for some globally defined normal field ν on M. Moreover, M is locally convex and has everywhere defined orthogonal asymptotic lines.
(b) If M is a compact isoparametric surface in $\mathbb{R}^4$ then there is some globally defined parallel normal field ν on M, such that M is ν-umbilic.

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