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PRODUCTS OF QUASI-p-PSEUDOCOMPACT SPACES

M. SANCHIS (Castelló) and A. TAMARIZ-MASCARÚA (México)

Abstract. Given $p \in \beta(\omega) \setminus \omega$, we determine when a product of quasi-p-pseudocompact spaces preserves this property. In particular, we analyze the product of quasi-p-pseudocompact subspaces of $\beta(\omega)$ containing ω. We give examples of spaces X, Y, X_s, Y_s which are quasi-p-pseudocompact for every $p \in \omega^*$, but $X \times Y$ is not pseudocompact, and $X_s \times Y_s$ is pseudocompact and it is not quasi-s-pseudocompact for each $s \in \omega^*$. Besides, we prove that every pseudocompact space X of $\beta(\omega)$ with $\omega \subset X$, is quasi-p-pseudocompact for some $p \in \omega^*$. Finally, we introduce, for each $p \in \omega^*$, the class $\mathcal{P}_p$ of all spaces X such that $X \times Y$ is quasi-p-pseudocompact when so is Y; and we prove: (1) the intersection of classes $\mathcal{P}_p$ $(p \in \omega^*)$ coincides with the Frolík class; (2) every class $\mathcal{P}_p$ is closed under arbitrary products; (3) the partial ordered set $\big(\{\mathcal{P}_p : p \in \omega^*\}, \supset\big)$ is isomorphic to the set of equivalence classes of free ultrafilters on ω with the Rudin–Keisler order. A topological characterization of RK-minimal ultrafilters is also given.

1. Introduction

All spaces considered in this paper will be Tychonoff spaces. ω is the set of natural numbers, $\beta(\omega)$ is its Stone–Čech compactification and $\omega^* = \beta(\omega) \setminus \omega$, that is, the set of all free ultrafilters on ω. The Rudin–Keisler order $\leqq_{RK}$ on $\beta(\omega)$ is defined by $p \leqq_{RK} q$ if there exists a function $g : \omega \to \omega$ such that $g^\beta(q) = p$, where g^β is the continuous extension to $\beta(\omega)$ of g. If $p \leqq_{RK} q$ and $q \leqq_{RK} p$, for $p, q \in \omega^*$, then we say that p and q are RK-equivalent and we write $p \approx_{RK} q$. It is not difficult to verify that $p \approx_{RK} q$ if and only if there is a permutation σ of ω such that $\sigma^\beta(p) = q$. For $p \in \omega^*$, we set $P_{RK}(p) = \big\{ r \in \beta(\omega) : r \leqq_{RK} p \big\}$. The type of $p \in \omega^*$ is the set $T(p) = \{r \in \omega^* : p \approx_{RK} r\}$. Finally, we denote by $\Sigma(p)$ the set $T(p) \cup \omega$.

The deduction of topological properties by means of the theory of ultrafilters on ω has been widely studied in the literature. The well-known Frolík's Theorem [5, Theorem 3.6] on pseudocompactness and the techniques developed by Ginsburg and Saks in [9] are just two seminal examples. Recently, another kind of topological properties related to pseudocompactness has been introduced and studied by using the concept of free ultrafilter (see e.g, [7], [8],

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[13], [14]); namely the authors consider the notion of M-pseudocompactness for several subsets M of ω^* introduced by García-Ferreira in [7]. The starting point is the following

1.1. Definition. For $p \in \omega^\star$, a point $x \in X$ is said to be a *p-limit point* of a sequence $(U_n)_{n<\omega}$ of nonempty subsets of X (in symbols: $x = p\text{-lim}\,(U_n)_{n<\omega}$) if for each neighborhood V of x, the set $\{n < \omega : U_n \cap V \neq \emptyset\}$ belongs to p.

This notion was introduced by Ginsburg and Saks [9] by generalizing the notion of p-limit point discovered and investigated by Bernstein in [1]. It should be mentioned that Bernstein's p-limit concept was also introduced, in a different form, by Frolík [6] and Katětov [10], [11].

Now, let us agree to say that a space X is *M-pseudocompact,* where $\emptyset \neq M \subset \omega^*$, if for every sequence $(U_n)_{n<\omega}$ of nonempty open sets in X, there are $p \in M$ and $x \in X$ such that $x = p\text{-lim}\,(U_n)_{n<\omega}$. Thus, X is pseudocompact if and only if X is ω^*-pseudocompact; X is quasi-p-pseudocompact if and only if it is $(P_{RK}(p) \setminus \omega)$-pseudocompact; and X is p-pseudocompact if and only if it is $\{p\}$-pseudocompact. In this paper we are interested in analyzing M-pseudocompactness for $M = P_{RK}(p) \setminus \omega$. In particular we are going to study the product of this kind of spaces; besides, we analyze the class $\mathcal{P}_p$ of spaces X for which its product with every quasi-p-pseudocompact space preserves this property. We prove that $\bigcap_{p\in\omega^*} \mathcal{P}_p$ coincides with the class $\mathcal{P}$ of Frolík spaces studied in [5]. We also prove that every pseudocompact subspace of $\beta(\omega)$ containing ω is quasi-p-pseudocompact for some $p \in \omega^*$, and we obtain a topological characterization of RK-minimal free ultrafilters on ω.

2. Products of M-pseudocompact spaces

In this section we give some results about products of M-pseudocompact spaces for arbitrary nonempty $M \subset \omega^*$.

The proof of the next theorem follows from a standard argument.

2.1. Theorem. *Let $\emptyset \neq M \subset \omega^*$. Let $\{X_s : s \in S\}$ be a family of topological spaces. Then, the product space $X = \prod_{s\in S} X_s$ is M-pseudocompact if and only if $\prod_{s\in S_0} X_s$ is M-pseudocompact for every countable subset S_0 of S.*

So, the problem of knowing when a product of spaces is M-pseudocompact can be reduced to the case of the product of countably many factors.

For a family $\{X_s : s \in S\}$ of topological spaces, we will denote by $\mathcal{O}_s$ the set of nonempty open subsets of X_s for each $s \in S$, and π_s will be the natural projection from $\prod_{s\in S} X_s$ to X_s. The next lemma will be useful.

2.2. LEMMA. *Let $\{X_s : s \in S\}$ be a family of topological spaces. Let $x = (x_s)_{s\in S} \in X = \prod_{s\in S} X_s$ be an r-limit of a sequence $(V_n)_{n<\omega}$ of subsets of X, with $r \in \omega^*$. Then $x_s = r\text{-}\lim\big(\pi_s(V_n)\big)_{n<\omega}$ for every $s \in S$.*

PROOF. Let s be an arbitrary element of S. Let W_s be a neighborhood of x_s. Then $Y = \prod_{g\in S} Y_g$, where $Y_s = W_s$ and $Y_g = X_g$ whenever $g \neq s$, is a neighborhood of x. So,

$$\{n < \omega : V_n \cap Y \neq \emptyset\} \in r.$$

It happens that $V_n \cap Y \neq \emptyset$ if and only if $\pi_s(V_n) \cap W_s \neq \emptyset$. Therefore

$$\big\{n < \omega : \pi_s(V_n) \cap W_s \neq \emptyset\big\} \in r.$$

This means that $x_s = r\text{-}\lim\big(\pi_s(V_n)\big)_{n<\omega}$. $\square$

2.3. THEOREM. *Let $\emptyset \neq M \subset \omega^*$, $0 < \mathfrak{t} \leqq \omega$ and $\{X_s : s < \mathfrak{t}\}$ be a family of topological spaces. Then the product space $X = \prod_{s<\mathfrak{t}} X_s$ is M-pseudocompact if and only if for every sequence $\big((U_s^n)_{s<\mathfrak{t}}\big)_{n<\omega}$ of elements in $\prod_{s<\mathfrak{t}} \mathcal{O}_s$, there exist $r \in M$ and $(x_s)_{s<\mathfrak{t}} \in X$ such that $x_s = r\text{-}\lim\,(U_s^n)_{n<\omega}$ for every $s < \mathfrak{t}$.*

PROOF. Assume that X is M-pseudocompact and let $(U_s^n)_{n<\omega}$ be a sequence of open sets in X_s for each $s < \mathfrak{t}$. For each $n < \omega$, let V_n be the open set of X defined as follows:

$$V_n = \begin{cases} \prod_{s\leqq n} U_s^n \times \prod_{s>n} X_s & \text{if } \ \mathfrak{t} = \omega, \\ \prod_{s<\mathfrak{t}} U_s^n & \text{if } \ \mathfrak{t} < \omega. \end{cases}$$

As X is M-pseudocompact, we can find $x = (x_s)_{s<\mathfrak{t}} \in X$ and $r \in M$ such that $x = r\text{-}\lim\,(V_n)_{n<\omega}$. Applying Lemma 2.2 it is an easy matter to see that $x_s = r\text{-}\lim\,(U_s^n)_{n<\omega}$ for each $s < \mathfrak{t}$.

Now, we are going to prove the converse. For each $n < \omega$, let $U_n = \prod_{s<\mathfrak{t}} V_s^n$ be a standard open set in X for each $n < \omega$. We shall prove that the sequence $(U_n)_{n<\omega}$ has an r-limit point for some $r \in M$.

For this in turn, we take for each $s < \mathfrak{t}$ the sequence $(V_s^n)_{n<\omega}$. By assumption, there exist $r \in M$ and $(x_s)_{s<\mathfrak{t}} \in X$ such that $x_s = r\text{-}\lim\,(V_s^n)_{n<\omega}$ for every $s < \mathfrak{t}$. We shall finish the proof by showing that $x = (x_s)_{s<\mathfrak{t}} = r\text{-}\lim(U_n)_{n<\omega}$. In fact, let $W_{i_1} \times \cdots \times W_{i_k} \times \prod_{j\in\mathfrak{t}\setminus\{i_1,\ldots,i_k\}} X_j$ be a standard neighborhood of $(x_s)_{s<\mathfrak{t}}$ in X. Then

$$E = \bigcap_{j=1}^{k} \{n < \omega : W_{i_j} \cap V_{i_j}^n \neq \emptyset\} \in r.$$

Since

$$E \subset \Big\{ n < \omega :\ \big(W_{i_1} \times \cdots \times W_{i_k} \times \prod_{j \notin \{i_1, \ldots, i_k\}} X_j \big) \cap U_n \neq \emptyset \Big\},$$

the proof is complete. □

In [7] the following concept was introduced. A space X is said to be (α, M)*-pseudocompact* if for every set $\big\{ (V_n^\xi)_{n<\omega} : \xi < \gamma \big\}$ of γ-many sequences, for $\gamma \leqq \alpha$, of nonempty open subsets of X, there are $p \in M$ and $x_\xi \in X$, for each $\xi < \gamma$, such that $x_\xi = p\text{-}\lim\, (V_n^\xi)_{n<\omega}$ for all $\xi < \gamma$.

As a consequence of Theorems 2.1 and 2.3 we obtain the following generalization of Theorems 2.2 and 2.3 in [7].

2.4. Corollary. *Let* $\mathfrak{t}$ *be a cardinal number,* X *a topological space and* $M \subset \omega^*$. *Then the following assertions are equivalent:*

(1) $X^{\mathfrak{t}}$ *is* M*-pseudocompact.*

(2) X *is* $(\mathfrak{t}, M)$*-pseudocompact.*

Moreover, if $\mathfrak{t}$ *is an infinite cardinal,* (1) *and* (2) *are equivalent to*

(3) X *is* (ω, M)*-pseudocompact.*

M-pseudocompactness for the product of subspaces of $\beta(\omega)$ which contain ω can be determined by sequences of natural numbers as we are going to see in Theorem 2.6. First we present a well-known lemma. We include its proof for the sake of completeness.

2.5. Lemma. *Let* $r, p \in \omega^*$ *and let* $(k_n)_{n<\omega}$ *be a sequence in* ω. *Then* $r = p\text{-}\lim\, (k_n)_{n<\omega}$ *if and only if* $f^\beta(p) = r$ *where* $f(n) = k_n$.

Proof. Assume that $r = p\text{-}\lim\, (k_n)_{n<\omega}$. Then, for each $B \in r$, we have that $f^{-1}(B) = \{ n < \omega : k_n \in B \} \in p$. On the other hand, $f^\beta(p) = \big\{ A \subset \omega : f^{-1}(A) \in p \big\}$. Thus $r = f^\beta(p)$.

Now, assume that $f(n) = k_n$ for every $n < \omega$ and $f^\beta(p) = r$. Let $B \in r$. We have that $\{ n < \omega : k_n \in B \} = f^{-1}(B)$. Since $r = f^\beta(p) = \big\{ A \subset \omega : f^{-1}(A) \in p \big\}$, then $f^{-1}(B) \in p$. So, $r = p\text{-}\lim\, (k_n)_{n<\omega}$. □

2.6. Theorem. *Let* $\emptyset \neq M \subset \omega^*$, $\mathfrak{t} \leqq \omega$ *and* $\{ X_s : s < \mathfrak{t} \}$ *be a family of topological spaces such that* $\omega \subset X_s \subset \beta(\omega)$ *for every* $s < \mathfrak{t}$. *Then the following assertions are equivalent:*

(1) *The product space* $X = \prod_{s<\mathfrak{t}} X_s$ *is* M*-pseudocompact.*

(2) *For every* $(f_s)_{s<\mathfrak{t}} \in (\omega^\omega)^{\mathfrak{t}}$, *there exist* $r \in M$ *and* $(x_s)_{s<\mathfrak{t}} \in X$ *such that* $f_s^\beta(r) = x_s$ *for every* $s < \mathfrak{t}$.

(3) *For every* $(f_s)_{s<\mathfrak{t}} \in (\omega^\omega)^{\mathfrak{t}}$, $M \cap \bigcap_{s<\mathfrak{t}} (f_s^\beta)^{-1}(X_s) \neq \emptyset$.

PROOF. (1) ⇒ (2). Let $(f_s)_{s<\mathfrak{t}} \in (\omega^\omega)^{\mathfrak{t}}$. For each $n < \omega$ we take the open set

$$U_n = \begin{cases} \prod_{s \leqq n} \{ f_s(n) \} \times \prod_{s>n} X_s & \text{if } \mathfrak{t} = \omega, \\ \prod_{s<\mathfrak{t}} \{ f_s(n) \} & \text{if } \mathfrak{t} < \omega. \end{cases}$$

Since X is M-pseudocompact, we can find $x = (x_s)_{s<\mathfrak{t}}$ and $r \in M$ such that $x = r\text{-}\lim\, (U_n)_{n<\omega}$. By Lemma 2.2, $x_s = r\text{-}\lim\, \big(f_s(n)\big)_{n<\omega}$ for every $s < \mathfrak{t}$. By Lemma 2.5, these equalities imply that $f_s^\beta(r) = x_s$ for all $s < \mathfrak{t}$.

(2) ⇒ (3). It is trivial.

(3) ⇒ (1). Let $(U_n)_{n<\omega}$ be a sequence of open sets in X. The set $\omega^{\mathfrak{t}}$ is dense in X; thus, for every $n < \omega$, there exists $(k_s^n)_{s<\mathfrak{t}} \in U_n \cap \omega^{\mathfrak{t}}$. Let $f_s : \omega \to \omega$ be defined by $f_s(n) = k_s^n$. By assumption, there exist $r \in M$ and $(x_s)_{s<\mathfrak{t}} \in X$ such that $f_s^\beta(r) = x_s$ for every $s < \mathfrak{t}$. By Lemma 2.5, we have that $x_s = r\text{-}\lim\, (k_s^n)_{n<\omega}$. This means that $(x_s)_{s<\mathfrak{t}} = r\text{-}\lim\, (U_n)_{n<\omega}$. □

3. Products of quasi-p-pseudocompact spaces

Now, we are going to reproduce explicitly some corollaries of Theorems 2.1 and 2.3 when $M = P_{RK}(p) \setminus \omega$ and when, for every $s \in S$, X_s is equal to a space X.

Let $(U_n)_{n<\omega}$ be a sequence of subsets of a space X, and let $0 < \mathfrak{t} \leqq \omega$. A family $(\mathcal{V}_k)_{k<\mathfrak{t}}$ of pairwise disjoint subsequences of $(U_n)_{n<\omega}$ is called *a* $\mathfrak{t}$-*partition* of $(U_n)_{n<\omega}$ if every element of $(U_n)_{n<\omega}$ belongs to $\mathcal{V}_k$ for some $k < \mathfrak{t}$.

3.1. THEOREM. *Let $p \in \omega^\star$ and let X be a topological space. Then the following assertions are equivalent:*

(1) *$X^{\mathfrak{t}}$ is quasi-p-pseudocompact for every cardinal number* $\mathfrak{t}$.

(2) *$X^{\mathfrak{t}}$ is quasi-p-pseudocompact for an infinite cardinal number* $\mathfrak{t}$.

(3) *X^ω is quasi-p-pseudocompact.*

3.2. THEOREM. *Let $p \in \omega^\star$, $0 < \mathfrak{t} \leqq \omega$ and let X be a topological space. Then the following assertions are equivalent:*

(1) *$X^{\mathfrak{t}}$ is quasi-p-pseudocompact.*

(2) *For each sequence $(U_n)_{n<\omega}$ of open sets in X and each $\mathfrak{t}$-partition $(\mathcal{V}_s)_{s<\mathfrak{t}}$ of $(U_n)_{n<\omega}$, there exist $r \leqq_{RK} p$ and $(x_s)_{s<\mathfrak{t}} \subset X$ such that x_s is an r-limit of $\mathcal{V}_s$ for each $s < \mathfrak{t}$.*

(3) *For each sequence $(\mathcal{V}_s)_{s<\mathfrak{t}}$ of sequences of open sets in X, there exist $r \leqq_{RK} p$ and $(x_s)_{s<\mathfrak{t}} \subset X$ such that x_s is an r-limit of $\mathcal{V}_s$ for each $s < \mathfrak{t}$.*

PROOF. By virtue of Theorem 2.3 we only have to prove (1) ⇒ (2) ⇒ (3).

(1) ⇒ (2). Let $(U_n)_{n<\omega}$ be a sequence of open sets of X. For each $s < \mathfrak{t}$, let $\mathcal{V}_s = \big(U^s_{s(n)}\big)_{n<\omega}$ be a subsequence of $(U_n)_{n<\omega}$ such that the family $(\mathcal{V}_s)_{s<\mathfrak{t}}$

is a $\mathfrak{t}$-partition of $(U_n)_{n<\omega}$. For each $n<\omega$, let V_n be the open set of $X^{\mathfrak{t}}$ defined as follows:

$$V_n = \begin{cases} \prod_{s\leqq n} U^s_{s(n)} \times \prod_{s>n} X_s & \text{if } \mathfrak{t}=\omega, \\ \prod_{s<\mathfrak{t}} U^s_{s(n)} & \text{if } \mathfrak{t}<\omega, \end{cases}$$

where $X_s = X$ for every $s>n$.

As $X^{\mathfrak{t}}$ is quasi-p-pseudocompact, we can find $x=(x_s)_{s<\mathfrak{t}} \in X^{\mathfrak{t}}$ and $r \leqq_{RK} p$ such that $x = r\text{-lim}\,(V_n)_{n<\omega}$. By Lemma 2.2 this means that $x_s = r\text{-lim}\,\big(U^s_{s(n)}\big)_{n<\omega}$ for each $s<\mathfrak{t}$.

(2) $\Rightarrow$ (3). For each $s<\mathfrak{t}$, let $\mathcal{V}_s = (V^n_s)_{n<\omega}$ be a sequence of open sets in X. Let $(A_s)_{s<\mathfrak{t}}$ be a $\mathfrak{t}$-partition of ω, and consider a faithful enumeration $\{s(n) : n<\omega\}$ of A_s for each $s<\mathfrak{t}$. Define the sequence $(W_n)_{n<\omega}$ in X as $W_{s(n)} = V^n_s$. Then, the family $\Big(\big(W_{s(n)}\big)_{n<\omega}\Big)_{s<\mathfrak{t}}$ is a $\mathfrak{t}$-partition of $(W_n)_{n<\omega}$. By assumption there exist $(x_s)_{s<\mathfrak{t}}$ and $r\leqq_{RK} p$ such that $x_s = r\text{-lim}\big(W_{s(n)}\big)_{n<\omega} = r\text{-lim}\,(V^n_s)_{n<\omega} = r\text{-lim}\,\mathcal{V}_s$ for each $s<\mathfrak{t}$. □

3.3. Corollary. *Let $\omega \subset X \subset \beta(\omega)$ and $0<\mathfrak{t}\leqq\omega$. If X is quasi-p-pseudocompact for some $p\in\omega^\star$, then the following conditions are equivalent:*

(1) *$X^{\mathfrak{t}}$ is quasi-p-pseudocompact.*

(2) *For each sequence $(m_n)_{n<\omega}$ in ω and each ω-partition $(\mathcal{V}_k)_{k<\omega}$ of $(m_n)_{n<\omega}$, there exist $r\leqq_{RK} p$ and $(x_n)_{n<\omega}\subset X$ such that x_k is an r-limit of $\mathcal{V}_k$ for each $k<\omega$.*

(3) *For each sequence $(s_k)_{k<\omega}$ of sequences in ω, there exist $r\leqq_{RK} p$ and $(x_n)_{n<\omega}\subset X$ such that x_k is an r-limit of s_k for each $k<\omega$.*

It is easy to prove the following lemma.

3.4. Lemma. *Let $\emptyset \neq M \subset \omega^*$, and let Y be a dense subspace of X. If Y is M-pseudocompact, then X is M-pseudocompact.*

So, we obtain:

3.5. Corollary. *Let $\omega\subset X\subset Y\subset\beta(\omega)$. If $X^{\mathfrak{t}}$ is quasi-p-pseudocompact for some $p\in\omega^\star$ and some cardinal number $\mathfrak{t}$, then so is $Y^{\mathfrak{t}}$.*

In [13] the authors analyzed the quasi-p-pseudocompact spaces. In particular, they proved the following result.

3.6. Theorem. *Let $\omega\subset X\subset\beta(\omega)$ and $p\in\omega^*$. Then the following assertions are equivalent:*

(1) *X is quasi-p-pseudocompact.*

(2) *$X\cap P_{RK}(p)$ is quasi-p-pseudocompact.*

(3) *$\big(X\cap P_{RK}(p)\big)\setminus\omega$ is dense in ω^*.*

Thus, the space $\Sigma(p)$ is a "small" enough quasi-p-pseudocompact subspace of $\beta(\omega)$. So it is interesting to know if the powers of $\Sigma(p)$ are quasi-

p-pseudocompact. In this way, the previous results permit us to obtain the following theorem.

3.7. THEOREM. *Let $p \in \omega^*$, $0 < \mathfrak{t} \leqq \omega$, and for each $s \leqq \mathfrak{t}$, let X_s be a subspace of $\beta(\omega)$ containing $\Sigma(p)$. Then $X = \prod_{s \leqq \mathfrak{t}} X_s$ is quasi-p-pseudocompact.*

PROOF. This theorem is a consequence of Lemma 3.4 and Corollary 2.4 above, and Theorem 2.6 in [7] which, in particular, establishes that the space $\Sigma(p)$ is $(\omega, P_{RK}(p) \setminus \omega)$-pseudocompact. □

3.8. COROLLARY. *For every cardinal number $\mathfrak{t} > 0$, $\Sigma(p)^{\mathfrak{t}}$ is a quasi-p-pseudocompact space.*

Related to the previous results, the following example is in order.

3.9. EXAMPLE. There exist $p \in \omega^*$ and countably many ultrafilters $(p_n)_{n<\omega}$ in ω^* such that each $\Sigma(p_n)$ is quasi-p-pseudocompact and the product space $\prod_{n<\omega} \Sigma(p_n)$ is not pseudocompact.

PROOF. Choose an increasing (in the Rudin–Keisler order) sequence $(p_n)_{n<\omega}$ in ω^*. Let p be an upper bound of our sequence. Then, for each $n < \omega$, $\Sigma(p_n)$ is quasi-p-pseudocompact but, by a theorem of Comfort (see [3]), the product space $\prod_{n<\omega} \Sigma(p_n)$ is not pseudocompact. □

4. Products of subspaces of $\beta(\omega)$

In this section we construct two spaces X' and Y' such that they are quasi-p-pseudocompact for every $p \in \omega^*$ and $X' \times Y'$ is not pseudocompact. Besides, for each $s \in \omega^*$, we obtain spaces X_s and Y_s which are quasi-p-pseudocompact for every $p \in \omega^*$, and $X_s \times Y_s$ is a pseudocompact non-quasi-s-pseudocompact space. On the other hand, we will prove that if X and Y are subspaces of $\beta(\omega)$ containing ω, and $X \times Y$ is pseudocompact, then $X \times Y$ is quasi-p-pseudocompact for some $p \in \omega^*$.

For a subset A of ω, we denote by $\widehat{A}$ the set $\{p \in \omega^* : A \in p\}$. The following results are well known.

4.1. LEMMA. (1) *The family $\mathcal{B}' = \{\widehat{A} : A \subset \omega\}$ is a base for the topology of $\beta(\omega)$, and $|\mathcal{B}'| = 2^\omega$.*

(2) *For each infinite subset A of ω and each $p \in \omega^*$, we have $|\widehat{A} \cap T(p)| = 2^\omega$.*

(3) *For each $p \in \omega^*$, $T(p)$ is dense in ω^*.*

(4) *If $p, q \in \omega^*$ with $p \not\approx_{RK} q$, then $T(p) \cap T(q) = \emptyset$.*

4.2. EXAMPLE. There exist spaces X' and Y' which are quasi-p-pseudocompact for every $p \in \omega^*$ but $X' \times Y'$ is not pseudocompact.

PROOF. Let $\mathcal{B}'$ be as in Lemma 4.1.(1). Let $\mathcal{B} = \{ A \subset \omega : \widehat{A} \in \mathcal{B}' \text{ and } |A| = \aleph_0 \}$. Enumerate faithfully the set $\mathcal{B}$ as $\{A_\lambda : \lambda < 2^\omega\}$.

By Lemma 4.1 we can choose, by induction, points a^p_λ and b^p_λ for each $p \in \omega^*$ and $\lambda < 2^\omega$ such that

(1) $a^p_\lambda, b^p_\lambda \in \widehat{A}_\lambda \cap T(p)$;

(2) $a^p_\lambda \neq a^p_\xi$ if $\lambda \neq \xi$ and $b^p_\lambda \neq b^p_\xi$ if $\lambda \neq \xi$;

(3) $\{a^p_\lambda : \lambda < 2^\omega\} \cap \{b^p_\lambda : \lambda < 2^\omega\} = \emptyset$.

We set $X' = \{a^p_\lambda : p \in \omega^*,\ \lambda < 2^\omega\}$ and $Y' = \{b^p_\lambda : p \in \omega^*,\ \lambda < 2^\omega\}$. It happens that for every $p \in \omega^*$, both $X' \cap T(p)$ and $Y' \cap T(p)$ are dense in ω^*, so they are quasi-p-pseudocompact for every $p \in \omega^*$ (Theorem 3.6). Moreover, $X' \times Y'$ is not pseudocompact because the sequence $\big((n,n)\big)_{n<\omega}$ of open sets in $X' \times Y'$ does not have a limit point in $X' \times Y'$. □

4.3. EXAMPLE. For each $s \in \omega^*$, there exist spaces X_s and Y_s which are quasi-p-pseudocompact for every $p \in \omega^*$ and $X_s \times Y_s$ is a pseudocompact non-quasi-s-pseudocompact space.

PROOF. Let X' and Y' be the spaces defined in the previous example. Let p be an element in ω^* such that p is not less or equal to s in the Rudin–Keisler order. For each $L = (f,g) \in (\omega^\omega)^2$ we are going to take a point $(x_L, y_L) \in \beta(\omega)$ as follows:

If $\mathcal{F} = \{ f^{-1}(n) : n \in \omega \}$ and $\mathcal{G} = \{ g^{-1}(n) : n \in \omega \}$ are finite sets, then we take $(x_L, y_L) = \big(f^\beta(p), g^\beta(p) \big)$. Observe that in this case $x_L, y_L \in \omega$.

If $\mathcal{F}$ is infinite and there is an infinite subset A of ω such that $f|_A$ and $g \mid A$ are one-to-one functions, then we take $q \in T(p) \cap \widehat{A}$ and $(x_L, y_L) = \big(f^\beta(q), g^\beta(q) \big)$. In this case $x_L, y_L \in T(p)$.

If $\mathcal{F}$ is infinite and there is no infinite subset A of ω in which both f and g are one-to-one, then there exist $k_1, k_2 \in \omega$ such that either

(1) for all $n > k_2$, we have $g^{-1}(n) \subset f^{-1}(k_1)$, or

(2) for all $n > k_2$, $f^{-1}(n) \subset g^{-1}(k_1)$.

If (1) happens, we take $A \subset \omega$ with $|A| = \aleph_0$ and $\big| A \cap f^{-1}(m) \big| = 1$ for all $m > k_1$. Then we take $q \in T(p) \cap \widehat{A}$, and $(x_L, y_L) = \big(f^\beta(q), g^\beta(q) \big)$. In this case $x_L \in T(p)$ and $y_L \in \omega$.

If (2) happens, we take an infinite subset A of ω such that $\big| A \cap f^{-1}(m) \big| = 1$ for all $m > k_2$. Then we take $q \in T(p) \cap \widehat{A}$, and $(x_L, y_L) = \big(f^\beta(q), g^\beta(q) \big)$. Again, in this case $x_L \in T(p)$ and $y_L \in \omega$.

The last possible case is when $\mathcal{F}$ is finite and $\mathcal{G}$ is infinite. In this case we take an infinite subset A of ω such that $g\,|_A$ is an one-to-one function. Then, we choose $q \in T(p) \cap \widehat{A}$, and we take $(x_L, y_L) = \big(f^\beta(q), g^\beta(q) \big)$. In this case $x_L \in \omega$ and $y_L \in T(p)$.

Let $N = \{ (x_L, y_L) : L \in (\omega^\omega)^2 \}$, $X_s = X' \cup \pi_1(N)$ and $Y_s = Y' \cup \pi_2(N)$ where, for $i = 1, 2$, π_i is the i-th-projection map.

Since X' and Y' are dense subspaces of X_s and Y_s, respectively, then X_s and Y_s are quasi-p-pseudocompact for every $p \in \omega^*$. Notice that $X_s \times Y_s$ is not quasi-s-pseudocompact, because the sequence $\big((n,n)\big)_{n<\omega}$ of open sets in $X_s \times Y_s$ does not have an s-limit point. Moreover, due to Theorem 2.6, $X_s \times Y_s$ is quasi-p-pseudocompact (so, pseudocompact). $\square$

It is not possible to construct a pseudocompact product of subspaces of $\beta(\omega)$ containing ω which is not quasi-p-pseudocompact for any $p \in \omega^*$. Indeed, we have:

4.4. THEOREM. *Let $\mathfrak{t}$ be a cardinal number satisfying $0 < \mathfrak{t} \leqq \omega$. For each $s < \mathfrak{t}$, let X_s be a subspace of $\beta(\omega)$ such that $\omega \subset X_s$. If $X = \prod_{s<\mathfrak{t}} X_s$ is pseudocompact, then there is $p \in \omega^*$ such that X is quasi-p-pseudocompact.*

PROOF. Since X is pseudocompact, Theorem 2.6 proclaims that for every $L = (f_s)_{s<\mathfrak{t}} \in (\omega^\omega)^\mathfrak{t}$, there exist $r_L \in \omega^*$ and $(x_s)_{s<\mathfrak{t}} \in X$ such that $f_s^\beta(r_L) = x_s$ for every $s < \mathfrak{t}$. The set $\{r_L : L \in (\omega^\omega)^\mathfrak{t}\}$ has cardinality 2^ω. Thus, there exists $p \in \omega^*$ such that $r_L \leqq p$ for every $L \in (\omega^\omega)^\mathfrak{t}$ (see Proposition 2.6 in [1]). Then, by Theorem 2.6, we conclude that X is quasi-p-pseudocompact. $\square$

Again, using the fact that every collection of free ultrafilters on ω having cardinality $\leqq 2^\omega$ has an upper bound in the $\leqq_{RK}$-order (see Proposition 2.6 in [4]), and using Theorem 4.4, Theorem 2.1 and Theorem 2.6, we obtain:

4.5. THEOREM. *Let $\mathfrak{t}$ be a cardinal number with $0 < \mathfrak{t} \leqq 2^\omega$. For each $s < \mathfrak{t}$, let X_s be a subspace of $\beta(\omega)$ such that $\omega \subset X_s$. If $X = \prod_{s<\mathfrak{t}} X_s$ is pseudocompact, then there is $p \in \omega^*$ such that X is quasi-p-pseudocompact.*

4.6. COROLLARY. *Let $\omega \subset X \subset \beta(\omega)$. If X is pseudocompact, then X is quasi-p-pseudocompact for some $p \in \omega^*$.*

5. The classes $\mathcal{P}_p$ and $\mathcal{P}$

A *Frolík sequence* in a space X is a sequence $(U_n)_{n<\omega}$ of subsets of X such that for each filter $\mathcal{G}$ of infinite subsets of ω,

$$\bigcap_{F\in\mathcal{G}} \mathrm{cl}_X \left(\bigcup_{n\in F} U_n \right) \neq \emptyset.$$

In the following, we say that a space X is *Frolík* if $X \times Y$ is pseudocompact for every pseudocompact space Y. The Frolík class $\mathcal{P}$ is the class consisting of exactly all Frolík spaces. In Theorem 3.6 in [5] the following result was proved:

5.1. Theorem. *A pseudocompact space belongs to the Frolík class $\mathcal{P}$ if and only if every sequence of disjoint open sets contains a subsequence which is a Frolík sequence.*

For $p \in \omega^*$, let $\mathcal{P}_p$ be the class of all spaces X satisfying that $X \times Y$ is quasi-p-pseudocompact whenever Y has this property. Since quasi-p-pseudocompactness is a property preserved under continuous functions, then every space in $\mathcal{P}_p$ is quasi-p-pseudocompact and $f(X) \in \mathcal{P}_p$ if f is a continuous function and $X \in \mathcal{P}_p$. Besides, every regular closed subspace of a space that belongs to $\mathcal{P}_p$, is an element of this class too. Also, it is easy to see that $\mathcal{P}_p$ is finitely multiplicative. We say that a sequence $(U_n)_{n<\omega}$ of subsets of X is a *Frolík sequence for* p if

$$\bigcap_{F \in p} \mathrm{cl}_X \left(\bigcup_{n \in F} U_n \right) \neq \emptyset.$$

Notice that, by the basic properties of ultrafilters, each point in $\bigcap_{F \in p} \mathrm{cl}_X \left(\bigcup_{n \in F} U_n \right)$ is a p-limit point of the sequence $(U_n)_{n<\omega}$. The following theorem characterizes the class $\mathcal{P}_p$. Following the pattern given in [2, Theorem 2.1], the starting point of the proof is to construct appropriate pseudocompact subspaces of $\beta(\omega)$ associated with special kinds of sequences of open sets in a space X.

5.2. Theorem. *Let X be a space. Then the following assertions are equivalent:*

(1) $X \in \mathcal{P}_p$.

(2) *For every sequence $(U_n)_{n<\omega}$ of pairwise disjoint open sets of X, there exists a subsequence $(U_{n_k})_{k<\omega}$ which is a Frolík sequence for every $q \leqq_{RK} p$.*

(3) *For every sequence $(U_n)_{n<\omega}$ of pairwise disjoint open sets of X, there exists a subsequence $(U_{n_k})_{k<\omega}$ such that, for each $q \leqq_{RK} p$ there exists $x_q \in X$ for which $x_q = q\text{-}\lim\, (U_{n_k})_{k<\omega}$.*

(4) *For each quasi-p-pseudocompact space Y, the product $X \times Y$ is pseudocompact.*

(5) *For each quasi-p-pseudocompact subspace Y of $\beta(\omega)$ containing ω, the product $X \times Y$ is pseudocompact.*

Proof. (1) $\Rightarrow$ (2). Suppose that there exists a sequence $(U_n)_{n<\omega}$ of pairwise disjoint open sets of X such that, for every infinite subset $N_0 = \{n_1, n_2, \ldots, n_k, \ldots\}$ of natural numbers with $n_k < n_{k+1}$, we can find $q(N_0) \leqq_{RK} p$ satisfying

$$\bigcap_{F \in q(N_0)} \mathrm{cl}_X \left(\bigcup_{k \in F} U_{n_k} \right) = \emptyset.$$

Consider now the function $f:\omega\to\omega$ defined by $f(k)=n_k$ for each $k<\omega$, and let f^β be the continuous extension of f to $\beta(\omega)$. Let q_{N_0} be such that $f^\beta\big(q(N_0)\big)=q_{N_0}$. It is clear that $q_{N_0}\leqq_{RK} p$. We prove that

$$\bigcap_{G\in q_{N_0}} \mathrm{cl}_X\left(\bigcup_{n\in G} U_n\right)=\emptyset.$$

In fact, since $f(F)\in q_{N_0}$ whenever $F\in q(N_0)$, we have that

$$\bigcap_{F\in q(N_0)} \mathrm{cl}_X\left(\bigcup_{k\in F} U_{n_k}\right)=\bigcap_{f(F)\in q_{N_0}} \mathrm{cl}_X\left(\bigcup_{n_k\in f(F)} U_{n_k}\right)\supset\bigcap_{G\in q_{N_0}} \mathrm{cl}_X\left(\bigcup_{n\in G} U_n\right).$$

So,

$$\bigcap_{G\in q_{N_0}} \mathrm{cl}_X\left(\bigcup_{n\in G} U_n\right)=\emptyset.$$

Let Y be the subspace of $\beta(\omega)$ defined as:

$$Y=\omega\cup\left\{q_{N_0}: N_0\subset\omega,\ |N_0|=\omega\right\}.$$

We prove that the space Y is quasi-p-pseudocompact. To see this, let $(n_k)_{k<\omega}$ be a subsequence of ω. Consider $N_0=\{n_1,n_2,\dots,n_k,\dots\}$. It is clear that $q_{N_0}=q(N_0)\text{-}\lim\,(n_k)_{k<\omega}$. The result follows from the fact that $q(N_0)\leqq_{RK} p$.

Now, we finish the proof by showing that $X\times Y$ is not pseudocompact. For this in turn, we prove that the sequence $\big(U_n\times\{n\}\big)_{n<\omega}$ is locally finite in $X\times Y$. Let $q_{N_0}\in Y$ be a cluster point of $(n)_{n<\omega}$ and let x be a cluster point of $(U_n)_{n<\omega}$. Since $\bigcap_{G\in q_{N_0}}\mathrm{cl}_X\left(\bigcup_{n\in G}U_n\right)=\emptyset$, there exists $G\in q_{N_0}$ such that $x\notin\mathrm{cl}_X\left(\bigcup_{n\in G}U_n\right)$; that is, there is a neighborhood V of the point x with $V\cap(\cup_{n\in G}U_n)=\emptyset$. Then, $\widehat{G}\cap Y$ is an open neighborhood of q_{N_0} such that $V\times\widehat{G}$ does not meet the sequence $\big(U_n\times\{n\}\big)_{n<\omega}$.

(2) $\Rightarrow$ (3). Let $(U_n)_{n<\omega}$ be a sequence of pairwise disjoint open sets of X. Then, there exists a subsequence $(U_{n_k})_{k<\omega}$ such that

$$\bigcap_{F\in q} \mathrm{cl}_X\left(\bigcup_{k\in F} U_{n_k}\right)\neq\emptyset$$

for each $q\leqq_{RK} p$. Take $x_q\in\bigcap_{F\in q}\mathrm{cl}_X\left(\bigcup_{k\in F}U_{n_k}\right)$. We are going to prove that $x_q=q\text{-}\lim\,(U_{n_k})_{k<\omega}$. In fact, let V be a neighborhood of x_q in X, and assume that $G = \{k < \omega : V \cap U_{n_k} \neq \emptyset\}$ does not belong to q. So, $H = \omega \setminus G \in q$. By assumption, there is $y \in V \cap U_m$ where $m \in H$. But, by definition $V \cap U_m = \emptyset$, a contradiction.

(3) ⇒ (1). Let $(U_n \times V_n)_{n<\omega}$ be a sequence of pairwise disjoint open sets of $X \times Y$ where Y is quasi-p-pseudocompact. Let $(U_{n_k})_{k<\omega}$ be a subsequence of $(U_n)_{n<\omega}$ satisfying the requirements in (3). Now, according to the fact that Y is quasi-p-pseudocompact, the sequence $(V_{n_k})_{k<\omega}$ admits an r-limit with $r \leqq_{RK} p$. Because of the properties of $(U_{n_k})_{k<\omega}$, it is clear that $(U_{n_k} \times V_{n_k})_{k<\omega}$ admits an r-limit with $r \leqq_{RK} p$.

The implications (1) ⇒ (4) ⇒ (5) are clear. On the other hand, (5) ⇒ (2) is implicit in the proof of (1) ⇒ (2). □

As an immediate consequence of Theorem 5.2 we have the following corollaries.

5.3. Corollary. *Let $\omega \subset X \subset \beta(\omega)$. Then the following assertions are equivalent:*

(1) $X \in \mathcal{P}_p$.

(2) *For every sequence $(a_n)_{n<\omega}$ of natural numbers with $a_n \neq a_m$ if $n \neq m$, there exists a subsequence $(a_{n_k})_{k<\omega}$ which is a Frolík sequence for every $q \leqq_{RK} p$.*

(3) *For every sequence $(a_n)_{n<\omega}$ of natural numbers with $a_n \neq a_m$ if $n \neq m$, there exists a subsequence $(a_{n_k})_{k<\omega}$ such that, for each $q \leqq_{RK} p$, there is a q-limit point of $(a_{n_k})_{k<\omega}$ in X.*

(4) *For every function $s : \omega \to \omega$, there exists a function $f_s : \omega \to \omega$ such that, for every $q \leqq_{RK} p$, $(s \circ f_s)^{\beta}(q) \in X$.*

5.4. Corollary. *Every space in the Frolík class $\mathcal{P}$ belongs to $\mathcal{P}_p$ for every $p \in \omega^*$.*

The previous result implies, in particular, that every Frolík space is quasi-p-pseudocompact for every $p \in \omega^*$, as was already pointed out in Theorem 2.6 in [13].

5.5. Corollary. *If p, q are two elements in ω^* such that $p \leqq_{RK} q$, then $\mathcal{P}_q \subset \mathcal{P}_p$.*

Proof. Let $X \in \mathcal{P}_q$, and let $(U_n)_{n<\omega}$ be a sequence of pairwise disjoint open subsets of X. By Theorem 5.2, there is a subsequence $(U_{n_k})_{k<\omega}$ of $(U_n)_{n<\omega}$ such that, for every $r \leqq_{RK} q$, we have

$$\bigcap_{F \in r} \mathrm{cl}_X \left(\bigcup_{k \in F} U_{n_k} \right) \neq \emptyset.$$

In particular, the previous equality holds for every $r \leqq_{RK} p$. But this means that $X \in \mathcal{P}_p$. □

Observe that the spaces X' and X_s given in Examples 4.2 and 4.3, respectively, are quasi-p-pseudocompact spaces for every $p \in \omega^*$, but they do not belong to $\bigcup_{p\in\omega^*} \mathcal{P}_p$.

By applying Theorem 4.1 in [13] and Theorem 5.2 above, we obtain:

5.6. Corollary. *Every p-pseudocompact space belongs to $\mathcal{P}_p$ for all $p \in \omega^*$.*

So, the space $P_{RK}(p)$ is an example of a space belonging to $\mathcal{P}_p$ (it is p-pseudocompact, see [9]) which is not Frolík. On the other hand, the space $\prod_{p\in\omega^*}(\beta(\omega)\setminus\{p\})$ belongs to $\mathcal{P}$ but it is not p-pseudocompact for any $p \in \omega^*$ (see Example 2.9 in [13]).

Because of the properties of $\mathcal{P}_p$, we can use the space $P_{RK}(p)$ to determine the set $\mathcal{P}_p \cap \{X : \omega \subset X \subset \beta(\omega)\}$, as we will show in the following theorem.

5.7. Theorem. *Let $\omega \subset X \subset \beta(\omega)$, and let $p \in \omega^*$. Then the following assertions are equivalent:*

(1) *The space X belongs to $\mathcal{P}_p$.*
(2) *The space $X \times P_{RK}(p)$ is an element of $\mathcal{P}_p$.*
(3) *$X \cap P_{RK}(p) \in \mathcal{P}_p$.*

Proof. (1) $\Rightarrow$ (2). $P_{RK}(p)$ is an element of $\mathcal{P}_p$, and this class is finitely productive.

(2) $\Rightarrow$ (3). The space $X \cap P_{RK}(p)$ is homeomorphic to a regular closed subset of $X \times P_{RK}(p)$.

(3) $\Rightarrow$ (1). The class $\mathcal{P}_p$ is closed under continuous functions. □

5.8. Conjecture. *Let $\omega \subset X \subset \beta(\omega)$, and let $p \in \omega^*$. Then $X \in \mathcal{P}_p$ if and only if for each open subset W of $\beta(\omega)$, there exists an open subset V of W for which $V \cap P_{RK}(p) \subset X$.*

Now we are ready to prove that the partial ordered set $(\mathfrak{T}, \leqq_{RK})$, where $\mathfrak{T}$ is the set of equivalence classes of free ultrafilters on ω, is isomorphic to $(\mathfrak{P}, \supset)$, where $\mathfrak{P} = \{\mathcal{P}_p : p \in \omega^*\}$.

5.9. Theorem. *Let $p, q \in \omega^*$. Then the following assertions are equivalent:*

(1) *$q \leqq_{RK} p$.*
(2) *$\mathcal{P}_p \subset \mathcal{P}_q$.*
(3) *$P_{RK}(p) \in \mathcal{P}_q$.*
(4) *$P_{RK}(q) \subset P_{RK}(p)$.*
(5) *$P_{RK}(p)$ is q-pseudocompact.*

Proof. The implication (1) $\Rightarrow$ (2) is Corollary 5.5. The equivalence (1) $\Leftrightarrow$ (4) is trivial and the equivalence (4) $\Leftrightarrow$ (5) is a consequence of Lemma 1.9 in [9]. Since $P_{RK}(p) \in \mathcal{P}_p$ always holds, then (2) $\Rightarrow$ (3). So, we only have to prove that (3) $\Rightarrow$ (1).

Assume that $P_{RK}(p) \in \mathcal{P}_q$; so, the space $P_{RK}(p) \times \Sigma(q)$ is pseudocompact. Thus, $P_{RK}(p) \cap T(q) \neq \emptyset$. That is, $q \leqq_{RK} p$. $\square$

Observe that, even for a subspace X of $\beta(\omega)$ containing ω, the fact of being quasi-q-pseudocompact for every $q \leqq_{RK} p$, does not imply that X belongs to $\mathcal{P}_p$. Indeed, the space $X = \beta(\omega) \setminus T(p)$, where p is not RK-minimal, is quasi-p-pseudocompact for every $p \in \omega^*$ (see Example 3.2 in [13]), but it is not a member of $\mathcal{P}_p$, because $Y = \Sigma(p)$ is quasi-p-pseudocompact (Corollary 3.8) though $X \times Y$ is not pseudocompact; in fact, the sequence $\big((n,n)\big)_{n<\omega}$ of open sets, in $X \times Y$, does not have a cluster point in $X \times Y$. Nevertheless, by Theorem 5.7 and Lemma 3.4, $X \in \mathcal{P}_r$ if $r <_{RK} p$.

Theorem 5.10 produces a topological characterization of RK-minimal ultrafilters.

5.10. Theorem. *Let $p, q \in \omega^*$. The space $\Sigma(q)$ belongs to $\mathcal{P}_p$ if and only if q is RK-minimal and $q \approx_{RK} p$.*

Proof. If q is RK-minimal and $q \approx_{RK} p$, then $\Sigma(q) = \Sigma(p) = P_{RK}(p)$. As we have already seen, $P_{RK}(p) \in \mathcal{P}_p$.

Now, assume $\Sigma(q) \in \mathcal{P}_p$. Since $\Sigma(p)$ is quasi-q-pseudocompact, $\Sigma(p) \times \Sigma(q)$ is pseudocompact. Hence, the sequence $\big((n,n)\big)_{n<\omega}$ of open sets in $\Sigma(p) \times \Sigma(q)$ has an accumulation point $(s,t) \in T(p) \times T(q)$. But, s has to be equal to t. So, $p \approx_{RK} q$.

Suppose that p is not RK-minimal, and let $r <_{RK} p$. By Theorem 5.9, $\mathcal{P}_p \subset \mathcal{P}_r$, then $\Sigma(p) \in \mathcal{P}_r$. So, $\Sigma(p) \times P_{RK}(r)$ is pseudocompact. But this is not true because the sequence $\big((n,n)\big)_{n<\omega}$ of open sets in $\Sigma(p) \times P_{RK}(r)$ does not have an accumulation point in $\Sigma(p) \times P_{RK}(r)$. $\square$

Now, we are going to prove that $\bigcap_{p\in\omega^*} \mathcal{P}_p$ is precisely the class of Frolík spaces.

5.11. Theorem. $\mathcal{P} = \bigcap_{p\in\omega^*} \mathcal{P}_p$.

Proof. Corollary 5.4 establishes that $\mathcal{P} \subset \bigcap_{p\in\omega^*} \mathcal{P}_p$.

Now, assume that $X \notin \mathcal{P}$. Then there exists a pseudocompact subspace Y of $\beta(\omega)$ containing ω, such that $X \times Y$ is not pseudocompact (see [2]). By Corollary 4.6, there is $p \in \omega^*$ for which Y is quasi-p-pseudocompact. Therefore, by Theorem 5.2, $X \notin \bigcap_{p\in\omega^*} \mathcal{P}_p$. $\square$

Let $p \in \omega^*$. Let $\mathcal{P}_{F,p}$ denote the class of all spaces X such that every closed subset of X belongs to $\mathcal{P}_p$. Of course, $\mathcal{P}_p \supset \mathcal{P}_{F,p}$ for every $p \in \omega^*$, but these classes never coincide. Indeed, every compact space belongs to $\mathcal{P}_{F,p}$.

5.12. Theorem. *Let X be a space. Then the following assertions are equivalent:*

(1) $X \in \mathcal{P}_{F,p}$.

(2) *Every discrete sequence $(a_n)_{n<\omega}$ of points of X admits a subsequence $(a_{n_k})_{k<\omega}$ which is a Frolík sequence for every $q \leqq_{RK} p$.*

PROOF. (1) $\Rightarrow$ (2). Consider a discrete sequence $(a_n)_{n<\omega}$ in X. Then we can identify $(a_n)_{n<\omega}$ with ω. Let $Y = \mathrm{cl}_X\{a_n : n < \omega\}$. Then $(a_n)_{n<\omega}$ is a sequence of pairwise disjoint open sets in Y. By (1), $Y \in \mathcal{P}_p$. Now the result follows from the theorem of characterization of $\mathcal{P}_p$.

(2) $\Rightarrow$ (1). Let Y be a closed subset of X. Consider a sequence $(U_n)_{n<\omega}$ of pairwise disjoint open sets in Y. For each $n < \omega$, let a_n be a point with $a_n \in U_n$. It is clear that $(a_n)_{n<\omega}$ is a discrete sequence in X. By (2), for some subsequence $(a_{n_k})_{k<\omega}$,

$$\bigcap_{G\in q} \mathrm{cl}_X\left(\bigcup_{k\in G}\{a_{n_k}\}\right) \neq \emptyset$$

whenever $q \leqq_{RK} p$. Since $a_{n_k} \in U_{n_k}$ and Y is closed in X, we have

$$\bigcap_{G\in q} \mathrm{cl}_Y\left(\bigcup_{k\in G} U_{n_k}\right) \neq \emptyset$$

whenever $q \leqq_{RK} p$. The result follows from the characterization theorem of the class $\mathcal{P}_p$. $\square$

It is a well-known result that the Frolík class $\mathcal{P}$ is closed under arbitrary products (see [12]). In the last part of the paper we turn our attention to this question for the classes $\mathcal{P}_p$ for any $p \in \omega^*$.

5.13. LEMMA. *Let $p \in \omega^*$ and let $\{V_1, V_2, \dots, V_l\}$ be a finite family of subsets of X. If $(U_n)_{n<\omega}$ is a Frolík sequence for every $q \leqq_{RK} p$, then the sequence $(W_n)_{n<\omega}$ defined as*

$$W_i = \begin{cases} V_i & \text{if } i \leqq l, \\ U_{i-l} & \text{if } l < i, \end{cases}$$

is also a Frolík sequence for every $q \leqq_{RK} p$.

PROOF. Let $q \leqq_{RK} p$. Consider the function $f : \omega \to \omega$ defined as

$$f(t) = \begin{cases} t & \text{if } t \leqq l, \\ t - l & \text{if } t > l. \end{cases}$$

Let r denote the ultrafilter $f^\beta(q)$. Since $r \leqq_{RK} q$, there exists $x \in X$ with

$$x \in \bigcap_{F\in r} \mathrm{cl}_X\left(\bigcup_{n\in F} U_n\right).$$

We shall prove that

$$x \in \bigcap_{G \in q} \mathrm{cl}_X \left(\bigcup_{m \in G} W_m \right).$$

In fact, supposing the contrary, we claim that there exist a neighborhood V of x and $G \in q$ such that

$$V \cap \left(\bigcup_{m \in G^*} W_m \right) = \emptyset,$$

where $G^* = G \setminus \{1, \dots l\}$. Then

$$V \cap \left(\bigcup_{n \in f(G^*)} U_n \right) = \emptyset,$$

which leads us to a contradiction because $f(G^*) \in r$ and $r \leqq_{RK} p$. □

5.14. Lemma. *Let $p \in \omega^\star$. If $X \in \mathcal{P}_p$, then every sequence of open sets in X admits a subsequence which is a Frolík sequence for every $q \leqq_{RK} p$.*

Proof. Let $(U_n)_{n<\omega}$ be a sequence of open sets in X. Choose $x_n \in U_n$ for each $n < \omega$. If for some subsequence $(U_{n_k})_{k<\omega}$, there exists $x \in X$ such that every neighborhood of x meets all but finitely many elements of $(U_{n_k})_{k<\omega}$, then $(U_{n_k})_{k<\omega}$ is a Frolík sequence for every $q \leqq_{RK} p$. Otherwise, we construct by induction on n two sequences $(V_k)_{k<\omega}$ and $(F_k)_{k<\omega}$ of open sets and of infinite subsets of ω, respectively, and a subsequence $(U_{n_k})_{k<\omega}$ of $(U_n)_{n<\omega}$ satisfying:

(1) For each $k < \omega$, $V_k \subset U_{n_k}$.
(2) For each $k < \omega$, $V_k \cap U_{n_r} = \emptyset$ whenever $r \in F_k$.
(3) For each $k < \omega$, $F_k \supseteqq F_{k+1}$.
(4) $V_r \cap V_s = \emptyset$ whenever $r \neq s$.

The induction process is as follows. For $n = 1$, there exist an infinite subset G of ω and a neighborhood $W \subset U_1$ of x_1 such that $W \cap U_r = \emptyset$ whenever $r \in G$. Then we put $V_1 = W$ and $F_1 = G$. Suppose now that we have $(V_i)_{i \leqq m}$, $(F_i)_{i \leqq m}$ and $(U_{n_i})_{i \leqq m}$ enjoying the required properties. Consider the subsequence $(U_n)_{n \in F_m}$. Let r denote the minimum of F_m. Then there exist a neighborhood $W \subset U_r$ of x_r and an infinite subset G of F_m such that $W \cap U_n = \emptyset$ whenever $n \in G$. The induction step is complete by putting $V_{m+1} = W$, $F_{m+1} = G$ and $U_{n_{m+1}} = U_r$. Now the proof follows from the fact that the elements of the sequence $(V_k)_{k<\omega}$ are pairwise disjoint. □

5.15. LEMMA. *Let $p \in \omega^*$. Let $(U_n)_{n<\omega}$ be a Frolík sequence for every $q \leqq_{RK} p$. Then every subsequence of $(U_n)_{n<\omega}$ is a Frolík sequence for every $q \leqq_{RK} p$.*

PROOF. Suppose that there exists a subsequence $(U_{n_k})_{k<\omega}$ of $(U_n)_{n<\omega}$ which is not a Frolík sequence for some $q \leqq_{RK} p$; that is

$$\bigcap_{F\in q} \mathrm{cl}_X \left(\bigcup_{k\in F} U_{n_k} \right) = \emptyset.$$

Define the function f on ω by the requirement that $f(k)$ be n_k whenever $k<\omega$. Then, if $s = f^\beta(q)$, we have $s \leqq_{RK} q$ and

$$\bigcap_{F\in q} \mathrm{cl}_X \left(\bigcup_{k\in F} U_{n_k} \right) \supset \bigcap_{F\in q} \mathrm{cl}_X \left(\bigcup_{n\in f(F)} U_n \right) \supset \bigcap_{G\in s} \mathrm{cl}_X \left(\bigcup_{n\in G} U_n \right)$$

which leads us to a contradiction. □

5.16. THEOREM. *For each $p \in \omega^*$, the class $\mathcal{P}_p$ is closed under arbitrary products.*

PROOF. As a product space is quasi-p-pseudocompact if and only if each of its countable subproducts is quasi-p-pseudocompact, a product space belongs to $\mathcal{P}_p$ if and only if each of its countable subproducts does. Thus it suffices to consider a countable product, say $X = \prod_{i<\omega} X_i$, of members of $\mathcal{P}_p$.

Let $(U_n)_{n<\omega}$ be a sequence of open sets in X where each $U_n = \prod_{i<\omega} U_n^i$ is a standard open set in X.

Applying Lemma 5.14 and Lemma 5.15, we can find, for each $i<\omega$, an infinite subset $N_i = \{i_1, i_2, \dots, i_k, \dots\}$ of ω such that the sequence $(U_{i_k}^i)_{i_k\in N_i}$ is a Frolík sequence for every $r \leqq_{RK} p$ and $N_{i+1} \subsetneqq N_i$.

Now define, for each $i<\omega$, $n(i)$ as $\min N_i$. By Lemma 5.13, for each $i<\omega$, the sequence $\left(U_{n(1)}^i, U_{n(2)}^i, \dots U_{n(i)}^i, U_{i_2}^i, U_{i_3}^i, \dots\right)$ is a Frolík sequence for every $r \leqq_{RK} p$. Then, for each $i<\omega$, Lemma 5.15 says that the sequence $\left(U_{n(k)}^i\right)_{k<\omega}$ is also a Frolík sequence for every $r \leqq_{RK} p$. It is an easy matter to prove that the sequence $(V_k)_{k<\omega}$ defined as

$$V_k = \prod_{i<\omega} U_{n(k)}^i,$$

for each $k<\omega$, is a Frolík sequence (in X) for every $r \leqq_{RK} p$ which completes the proof. □

As a consequence of Theorem 5.11 and Theorem 5.16 we have

5.17. THEOREM [12]. *The Frolík class $\mathcal{P}$ is closed under arbitrary products.*

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DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT JAUME I
CAMPUS DEL RIU SEC S/N
12071, CASTELLÓ
SPAIN
E-MAIL: SANCHIS@MAT.UJI.ES

DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
U.N.A.M.
CIUDAD UNIVERSITARIA
MÉXICO 04510
MÉXICO
E-MAIL: ATAMARIZ@SERVIDOR.UNAM.MX