PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 10, October 2010, Pages 3707–3716
S 0002-9939(2010)10504-X
Article electronically published on May 27, 2010

PROPER ACTIONS ON TOPOLOGICAL GROUPS: APPLICATIONS TO QUOTIENT SPACES

SERGEY A. ANTONYAN

(Communicated by Alexander N. Dranishnikov)

Abstract. Let X be a Hausdorff topological group and G a locally compact subgroup of X. We show that the natural action of G on X is proper in the sense of R. Palais. This is applied to prove that there exists a closed set $F \subset X$ such that $FG = X$ and the restriction of the quotient projection $X \to X/G$ to F is a perfect map $F \to X/G$. This is a key result to prove that many topological properties (among them, paracompactness and normality) are transferred from X to X/G, and some others are transferred from X/G to X. Yet another application leads to the inequality $\dim X \le \dim X/G + \dim G$ for every paracompact topological group X and a locally compact subgroup G of X having a compact group of connected components.

1. Introduction

By a G-space we mean a completely regular Hausdorff space together with a fixed continuous action of a Hausdorff topological group G on it.

The notion of a proper G-space was introduced in 1961 by R. Palais [23] with the purpose of extending a substantial portion of the theory of compact Lie group actions to the case of noncompact ones.

A G-space X is called proper (in the sense of Palais [23, Definition 1.2.2]) if each point of X has a, so-called, *small* neighborhood, i.e., a neighborhood V such that for every point of X there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G.

Clearly, if G is compact, then every G-space is proper.

Many important problems in the theory of proper actions are conjugated (see [18], [1], [2], [8], [7]) to the following major open problem:

Conjecture 1. *Let G be a locally compact group. Then the orbit space X/G of any paracompact proper G-space X is paracompact.*

This conjecture is open even if X is metrizable; in this case it is equivalent (see [8]) to the following old problem going back to R. Palais [23]:

Conjecture 2. *Let G be a locally compact group and X a metrizable proper G-space. Then the topology of X is metrizable by a G-invariant metric.*

Received by the editors May 15, 2009.
2010 *Mathematics Subject Classification.* Primary 22A05, 22F05, 54H11, 54H15, 54F45.
Key words and phrases. Proper G-space, orbit space, locally compact group, dimension.
The author was supported in part by grants #IN102608 from PAPIIT (UNAM) and #79536 from CONACYT (Mexico).

Due to Palais [23], it is known that Conjecture 2 is true for a separable metrizable proper G-space X provided the acting group G is Lie. Other special cases are discussed in [20] and [8]. In particular, in [8], Palais' result is extended to the case of a locally compact separable G and a metrizable locally separable X.

In this paper we prove Conjecture 1 in an important special case, namely, when X is a topological group endowed with the natural action of its locally compact subgroup G (see Corollary 1.5). We first show that X is a proper G-space and then we establish a more general result (Theorem 1.2) which has many interesting applications in the theory of topological groups.

Below all topological groups are assumed to satisfy the Hausdorff separation axiom.

Theorem 1.1. *Let X be a topological group and G a locally compact subgroup of X. Then the action of G on X given by the formula $g * x = xg^{-1}$, $g \in G$, $x \in X$, is proper.*

Recall that a subset S of a proper G-space X is called *G-fundamental*, or just *fundamental*, if S is a small set and the saturation $G(S) = \{gs \mid g \in G, s \in S\}$ coincides with X.

Here is the key result of the paper:

Theorem 1.2. *Let X be a topological group and G a locally compact subgroup of X. Then there exists a closed G-fundamental set in X.*

It is easy to prove (see [2, Proposition 1.4]) that in every proper G-space X, the restriction of the orbit map $p : X \to X/G$ to any closed small set is perfect (i.e., is closed and has compact fibers). In combination with Theorem 1.2 this yields the following:

Corollary 1.3. *Let X be a topological group, G a locally compact subgroup of X, and X/G the quotient space of all right cosets $xG = \{xg \mid g \in G\}$, $x \in X$. Then there exists a closed subset $F \subset X$ such that the restriction $p|_F : F \to X/G$ is a perfect surjective map.*

This fact has the following immediate corollary about transfer of properties from X to X/G:

Corollary 1.4. *Let $\mathcal{P}$ be a topological property stable under perfect maps and also inherited by closed subsets. Assume that X is a topological group with the property $\mathcal{P}$ and let G be a locally compact subgroup of X. Then the quotient space X/G also has the property $\mathcal{P}$.*

Among such properties $\mathcal{P}$ we single out just some of those which provide new results in Corollary 1.4; these are: paracompactness, countable paracompactness, weak paracompactness, normality, perfect normality, Čech-completeness, being a k-space (see [16, §5.1, §5.2, §5.3, §1.5, §3.9, §3.3]) and stratifiability (see [14]). Thus, we get the following positive solution of Conjecture 1 in an important special case:

Corollary 1.5. *Let X be a paracompact topological group and G a locally compact subgroup of X. Then the quotient space X/G is paracompact.*

In this connection it is in order to recall the following remarkable result of A. V. Arhangel'skii [9]: every topological group is the quotient of a paracompact zero-dimensional group. Hence, the local compactness of G is essential in Corollary 1.5.

We recall that a locally compact group is called *almost connected* if its space of connected components endowed with the quotient topology is compact.

Corollary 1.6. *Let $\mathcal{P}$ be a topological property stable under open perfect maps and also inherited by closed subsets. Assume that X is a paracompact group with the property $\mathcal{P}$ and let G be an almost connected subgroup of X. Then the quotient space X/G also has the property $\mathcal{P}$.*

Among properties stable under open perfect maps and also inherited by closed subsets we highlight strong paracompactness and realcompactness (see [16, Exercises 5.3.C(c), 5.3H(d), and Theorem 3.11.4 and Exercises 3.11.G]). Thus, we get the following:

Corollary 1.7. *Let X be a strongly paracompact (resp., paracompact and realcompact) topological group and let G be an almost connected subgroup of X. Then the quotient space X/G is strongly paracompact (resp., paracompact and realcompact).*

Remark 1.8. We note that the converse of the first statement in this corollary is not true. Namely, it is known that the Baire space $B(\aleph_1)$ of weight $\aleph_1$ (which is, in fact, homeomorphic to a commutative metrizable topological group) is strongly paracompact and its product with the additive group $\mathbb{R}$ of the reals is not (see [16, Exercises 5.3F(a) and 5.3F(b)]). Hence, the direct product $X = \mathbb{R} \times B(\aleph_1)$ and its subgroup $G = \mathbb{R} \times \{0\}$ provide the desired counterexample, answering negatively a question from [11]. Furthermore, [12, Open Problem 3.2.1, p. 151] asks whether every locally strongly paracompact group is strongly paracompact? The same group $\mathbb{R}\times B(\aleph_1)$ provides a negative answer to this question too. Indeed, since the product of a compact space and a strongly paracompact space is strongly paracompact (see [16, Exercise 5.3H(a)]), we infer that $\mathbb{R} \times B(\aleph_1)$ is locally strongly paracompact.

Combining our Corollary 1.5 with a result of Abels [1, Main Theorem], we obtain the following:

Corollary 1.9. *Let X be a paracompact group, G an almost connected subgroup of X, and K a maximal compact subgroup of G. Then there exists a K-invariant subset $S \subset X$ such that X is K-homeomorphic to the product $G/K \times S$. In particular, X is homeomorphic to $\mathbb{R}^n \times S$ for some $n \geq 1$.*

In [10] A. V. Arhangel'skii has studied properties which are transferred in the opposite direction, i.e., from X/G to X. The next corollary is a unified result of this sort which implies many of those in [10] as well as provides some new ones.

Corollary 1.10. *Let $\mathcal{P}$ be a topological property invariant and inverse invariant of perfect maps, and also stable under multiplication by a locally compact group. Assume that X is a topological group and let G be a locally compact subgroup of X such that the quotient space X/G has the property $\mathcal{P}$. Then the group X also has the property $\mathcal{P}$.*

Among such properties we mention just some: paracompactness, being a k-space, Čech-completeness (see [16, §5.1, §3.3, §3.9]). That paracompactness is stable under multiplication by a locally compact group follows from a result of Morita [21] since every locally compact group is paracompact (even, strongly paracompact [12, Theorem 3.1.1]).

Corollary 1.11. *Let $\mathcal{P}$ be a topological property invariant and inverse invariant of open perfect maps, and also stable under multiplication by a locally compact group. Assume that X is a paracompact group and let G be an almost connected subgroup of X such that the quotient space X/G has the property $\mathcal{P}$. Then the group X also has the property $\mathcal{P}$.*

Among such properties we highlight realcompactness (see [16, Theorem 3.11.14 and Exercise 3.11.G], and we also take into account that every locally compact group is realcompact).

Corollary 1.5 is further applied to prove the following Hurewicz type formula:

Theorem 1.12. *Let X be a paracompact topological group and G an almost connected subgroup of X. Then*

$$\dim X \leq \dim X/G + \dim G.$$

Remark 1.13 ([24]). If in this theorem X is a locally compact group, then, in fact, the equality holds:

$$\dim X = \dim X/G + \dim G.$$

All the proofs are given in section 3.

2. Preliminaries

Throughout the paper, unless otherwise stated, by a *group* we shall mean a topological group G satisfying the Hausdorff separation axiom; by e we shall denote the unity of G.

All topological spaces are assumed to be Tychonoff (= completely regular and Hausdorff). The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in G. Bredon [15] and in R. Palais [22]. Our basic reference on proper group actions is Palais' article [23]. Other good sources are [20], [1] and [2].

For the convenience of the reader we recall, however, some more special definitions and facts below.

By a G-space we mean a topological space X together with a fixed continuous action $G \times X \to X$ of a topological group G on X. By gx we shall denote the image of the pair $(g, x) \in G \times X$ under the action.

If Y is another G-space, a continuous map $f : X \to Y$ is called a G-map or an equivariant map if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$.

If X is a G-space, then for a subset $S \subset X$ and for a subgroup $H \subset G$, the H-hull (or H-saturation) of S is defined as follows: $H(S) = \{hs \mid h \in H,\ s \in S\}$. If S is the one-point set $\{x\}$, then the G-hull $G(\{x\})$ is usually denoted by $G(x)$ and called the orbit of x. The orbit space X/G is always considered in its quotient topology.

A subset $S \subset X$ is called H-invariant if it coincides with its H-hull, i.e., $S = H(S)$. By an invariant set we shall mean a G-invariant set.

For any $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the stabilizer (or stationary subgroup) at x.

A compatible metric ρ on a metrizable G-space X is called invariant or G-invariant if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$. If ρ is a G-invariant metric on any G-space X, then it is easy to verify that the formula

$$\widetilde{\rho}\big(G(x), G(y)\big) = \inf\{\rho(x', y') \mid \ x' \in G(x),\ y' \in G(y)\}$$

defines a pseudometric $\widetilde{\rho}$, compatible with the quotient topology of X/G. If, in addition, X is a proper G-space, then $\widetilde{\rho}$ is, in fact, a compatible metric on X/G [23, Theorem 4.3.4].

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{gH |\ g \in G\}$ under the action induced by left translations.

A locally compact group G is called *almost connected* if the quotient group G/G_0 of G modulo the connected component G_0 of the identity is compact.

Such a group has a maximal compact subgroup K; i.e., every compact subgroup of G is conjugate to a subgroup of K [1, Theorem A.5]. The corresponding classical theorem on Lie groups can be found in [19, Ch. XV, Theorem 3.1].

In 1961 Palais [23] introduced the very important concept of a *proper action* of an arbitrary locally compact group G and extended a substantial part of the theory of compact Lie transformation groups to noncompact ones.

Let X be a G-space. Two subsets U and V in X are called thin relative to each other [23, Definition 1.1.1] if the set $\langle U, V\rangle = \{g \in G|\ gU \cap V \neq \emptyset\}$, called *the transporter* from U to V, has a compact closure in G. A subset U of a G-space X is called *G-small*, or just *small*, if every point in X has a neighborhood thin relative to U. A G-space X is called *proper* (in the sense of Palais) if every point in X has a small neighborhood.

Clearly, if G is compact, then every G-space is proper. Furthermore, if G acts properly on a compact space, then G has to be compact as well. If G is discrete and X is locally compact, the notion of a proper action is the same as the classical notion of a *properly discontinuous* action. When $G = \mathbb{R}$, the additive group of the reals, proper G-spaces are precisely the *dispersive* dynamical systems with regular orbit space (see [13, Ch. IV]).

Important examples of proper G-spaces are the coset spaces G/H with H a compact subgroup of a locally compact group G. The reader can find other interesting examples in [1], [2], [5], [6] and [20].

In what follows we shall also need the definition of a twisted product $G \times_K S$, where K is a closed subgroup of G, and S a K-space. $G \times_K S$ is the orbit space of the K-space $G \times S$ on which K acts by the rule: $k(g, s) = (gk^{-1}, ks)$. Furthermore, there is a natural action of G on $G \times_K S$ given by $g'[g, s] = [g'g, s]$, where $g' \in G$ and $[g, s]$ denotes the K-orbit of the point (g, s) in $G \times S$. We shall identify S by means of the K-equivariant embedding $s \mapsto [e, s]$, $s \in S$, with the K-invariant subset $\{[e, s] \mid s \in S\}$ of $G \times_K S$. This K-equivariant embedding $S \hookrightarrow G \times_K S$ induces a homeomorphism of the K-orbit space S/K onto the G-orbit space $(G \times_K S)/G$ (see [15, Ch. II, Proposition 3.3]).

The twisted products are of particular interest in the theory of transformation groups (see [15, Ch. II, § 2]). It turns out that every G-space locally is a twisted product. For a more precise formulation we need to recall the following well known notion of a slice (see [23, p. 305]).

Definition 2.1. Let X be a G-space and K a closed subgroup of G. A K-invariant subset $S \subset X$ is called a K-kernel if there is a G-equivariant map $f : G(S) \to G/K$, called the slicing map, such that $S{=}f^{-1}(eK)$. The saturation $G(S)$ is called a *tubular* set, and the subgroup K will be referred to as the slicing subgroup.

If in addition $G(S)$ is open in X, then we shall call S a K-slice in X.

If $G(S) = X$, then S is called *a global* K-slice of X.

It turns out that each tubular set with a compact slicing subgroup is a twisted product. The tubular neighborhood $G(S)$ is G-homeomorphic to the twisted product $G \times_K S$; namely, the map $\xi : G \times_K S \to G(S)$ defined by $\xi([g, s]) = gs$ is a G-homeomorphism (see [15, Ch. II, Theorem 4.2]). In what follows we will use this fact without a specific reference.

One of the most powerful results in the theory of topological transformation groups is Palais' slice theorem [23, Proposition 2.3.1], which states that if X is a proper G-space with G a Lie group, then for any point $x \in X$, there exists a G_x-slice S in X such that $x \in S$. In general, when G is not a Lie group, it is no longer true that a G_x-slice exists at each point of X (see [4]). Generalizing the case of Lie group actions, in [2] and [7] (see also [4] for the case of compact non-Lie group actions), approximate versions of Palais' slice theorem for non-Lie group actions were proved. Below, in the proof of Theorem 1.12, we shall need the following global slice theorem established by H. Abels [1, Main Theorem]:

Theorem 2.2 (Global Slice Theorem)**.** *Let G be an almost connected group, K a maximal compact subgroup of G, and X a proper G-space with a paracompact orbit space. Then X admits a global K-slice.*

On any group G one can define two natural (but equivalent) actions of G given by the formulas

$$g \cdot x = gx \qquad \text{and} \quad g * x = xg^{-1},$$

respectively, where in the right-hand sides the group operations are used with $g, x \in G$.

Throughout we shall consider the second action only.

By $U(G)$ we shall denote the Banach space of all right uniformly continuous bounded functions $f : G \to \mathbb{R}$ endowed with the supremum norm. Recall that f is called right uniformly continuous if for every $\varepsilon > 0$ there exists a neighborhood O of the unity in G such that $|f(y) - f(x)| < \varepsilon$ whenever $yx^{-1} \in O$.

We shall consider the induced action of G on $U(G)$, i.e.,

$$(gf)(x) = f(xg), \quad \text{for all} \quad g, x \in G.$$

It is easy to check that this action is continuous, linear and isometric (see, e.g., [3, Proposition 7]).

Proposition 2.3. *Let G be a group. Then for every $f \in U(G)$, the map*

$$f_* : G \to U(G)$$

defined by $f_*(x)(g) = f(xg^{-1})$, $x, g \in G$, is a right uniformly continuous G-map.

Proof. A simple verification. □

3. Proofs

Proof of Theorem 1.1. Choose a neighborhood U of the identity in X such that $U = U^{-1}$ and $U^3 \cap G$ has a compact closure in G. We claim that for every $x \in X$, the neighborhood xU is G-small. Indeed, let $y \in X$ be any point. Two cases are possible.

Case 1. Assume that $y \in xU^2G$. Then $y = xu_1u_2h$ with $u_1, u_2 \in U$ and $h \in G$. We claim that xU^2h is a neighborhood of y thin relative to xU. Indeed, if $g \in \langle xU, xU^2h \rangle$, then $g^{-1}h^{-1} \in U^3 \cap G$. Since $U^3 \cap G$ has a compact closure, we see that so does the set $(U^3 \cap G)h$ which contains g^{-1}. This yields that $\langle xU, xU^2h\rangle$ is contained in $h^{-1}(U^3 \cap G)^{-1}$, which also has a compact closure. Hence the transporter $\langle xU, xU^2h\rangle$ has a compact closure, as required.

Case 2. Assume that $y \notin xU^2G$. In this case $y \notin \overline{xUG}$. Indeed, if $y \in \overline{xUG}$, then the neighborhood $xUx^{-1}y$ of y should meet the set xUG. Then $xux^{-1}y = xvh$ for suitable elements $u, v \in U$ and $h \in G$. Then $y = xu^{-1}vh \in xU^2G$, a contradiction.

Hence the open set $V = X \setminus \overline{xUG}$ is a G-invariant neighborhood of y, and V is thin relative to xU because the transporter $\langle xU, V\rangle$ is empty in this case. □

Proposition 3.1. *Let X be a group and G a locally compact subgroup of X. Then there exists a locally finite covering of X consisting of G-invariant open sets of the form S_iG, where each S_i is an open G-small subset of X.*

Proof. By Theorem 1.1, X is a proper G-space, and hence, one can choose a G-small neighborhood U of the unity in X. By virtue of Markov's theorem [12, Theorem 3.3.9], there exists a right uniformly continuous function $f : X \to [0, 1]$ such that

$$f(e) = 0 \quad \text{and} \quad f^{-1}\big([0, 1)\big) \subset U. \tag{3.1}$$

Then, by Proposition 2.3, f induces an X-equivariant map $f_* : X \to U(X)$ defined by the rule:

$$f_*(x)(g) = f(xg^{-1}), \quad x, g \in X.$$

Denote by Z the image $f_*(X)$. Clearly, Z is the X-orbit of the point $f_*(e)$ in the X-space $U(X)$, and the metric of $U(X)$ induces an X-invariant metric on Z. We claim that

$$f_*^{-1}(\Gamma_{x,V}) \subset x^{-1}U, \quad \text{for every} \quad x \in X, \tag{3.2}$$

where $V = [0, 1)$ and $\Gamma_{x,V}$ is the open subset $\{\varphi \in U(X) \mid \varphi(x) \in V\}$ of U.

First we observe that $\Gamma_{x,V} = x^{-1}\Gamma_{e,V}$ and

$$f_*^{-1}(\Gamma_{e,V}) \subset f^{-1}(V).$$

Then (3.2) follows from (3.1) and the X-equivariance of f_*.

Besides, since $f_*(x) \in \Gamma_{x,V}$ for every $x \in X$, we see that the sets $\Gamma_{x,V}$, $x \in X$, constitute a covering of Z.

From now on we restrict ourselves only by the induced actions of the subgroup $G \subset X$; i.e., we will consider X and Z just as G-spaces.

Now, it follows from (3.2) and from the G-equivariance of f_* that

$$f_*^{-1}\big(G(\Gamma_{x,V})\big) \subset x^{-1}UG, \quad \text{for every} \quad x \in X. \tag{3.3}$$

Since $f_* : X \to Z$ is G-equivariant, it induces a continuous map $\widetilde{f_*}$ of the G-orbit spaces; i.e., we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f_*} & Z \\ \downarrow p & & \downarrow q \\ X/G & \xrightarrow{\widetilde{f_*}} & Z/G \end{array}$$

where p and q are the G-orbit maps.

It follows from (3.3) that

$$\widetilde{f}_*^{\,-1}\big(q(\Gamma_{x,V})\big) \subset p(x^{-1}U), \quad \text{for every} \quad x \in X. \tag{3.4}$$

Thus, the open covering $\{p(xU) \mid x \in X\}$ of the G-orbit space X/G is refined by the open covering $\{\widetilde{f}_*^{\,-1}\big(q(\Gamma_{x,V})\big) \mid x \in X\}$.

Since the metric of Z is G-invariant, the orbit space Z/G is pseudometrizable (see Preliminaries). Hence the open covering $\{q(\Gamma_{x,V}) \mid x \in X\}$ of Z/G admits an open locally finite refinement, say $\{W_i \mid i \in \mathcal{I}\}$ (see [16, Theorem 4.4.1 and Remark 4.4.2]). Then, clearly, $\{p^{-1}\big(\widetilde{f}_*^{\,-1}(W_i)\big) \mid i \in \mathcal{I}\}$ is an open locally finite refinement of $\{xUG \mid x \in X\}$ consisting of G-invariant sets. It then follows that each set $p^{-1}\big(\widetilde{f}_*^{\,-1}(W_i)\big)$ is contained in some set xUG, $x \in X$, which yields that

$$p^{-1}\big(\widetilde{f}_*^{\,-1}(W_i)\big) = S_iG,$$

where $S_i = p^{-1}\big(\widetilde{f}_*^{\,-1}(W_i)\big) \cap xU$.

Now, S_i, being a subset of the G-small set xU, is itself G-small. Thus $\{S_iG \mid i \in \mathcal{I}\}$ is the desired covering. □

Proof of Theorem 1.2. Let $\{S_iG \mid i \in \mathcal{I}\}$ be the locally finite open covering of X from Proposition 3.1. Then the union $S = \bigcup_{i\in\mathcal{I}} S_i$ is a G-small set (see, e.g., [2, Proposition 1.2(d)]). On the other hand,

$$SG = \Big(\bigcup_{i\in\mathcal{I}} S_i\Big)G = \bigcup_{i\in\mathcal{I}} S_iG = X,$$

yielding that S is a G-fundamental subset of X. Since the closure of a G-small set is G-small (see, e.g., [2, Proposition 1.2(b)]), the closure $\overline{S}$ is the desired closed G-fundamental set. □

Proof of Corollary 1.6. Since G is almost connected, it has a maximal compact subgroup K (see Preliminaries). Since by Corollary 1.5 the quotient X/G is paracompact, due to a result of Abels [1, Main Theorem], X admits a global K-slice S, and hence it is G-homeomorphic to the twisted product $G \times_K S$ (see Preliminaries). Since the group K is compact, it then follows that the K-orbit map $G \times S \to G \times_K S \cong_G X$ is open and perfect. This yields immediately that the restriction $p|_S : S \to X/G$ of the G-orbit map $p : X \to X/G$ is an open and perfect surjection. Now the result follows. □

Proof of Corollary 1.10. By virtue of Corollary 1.3, X admits a closed subset $F \subset X$ such that the restriction of the quotient projection $p : X \to X/G$ to F is a perfect surjection $p|_F : F \to X/G$. It then follows from the hypothesis that F has the property $\mathcal{P}$. Since a locally compact group is paracompact (even, strongly paracompact [12, Theorem 3.1.1]), then again by the hypothesis, the product $G\times F$ has the property $\mathcal{P}$. Since F is a closed G-small set, the action map $G \times F \to X$ is perfect (see Abels [2, Proposition 1.4]), and since $FG = X$ we see that the map $G \times F \to X$ is surjective. Then X has the property $\mathcal{P}$ by the hypothesis. □

Proof of Corollary 1.11. It is quite similar to the proof of Corollary 1.6. □

Proof of Theorem 1.12. Since G is almost connected, it has a maximal compact subgroup, say, K (see Preliminaries). Since, by Corollary 1.5, the quotient X/G is paracompact, one can apply the Global Slice Theorem (see Theorem 2.2), according to which X admits a global K-slice, say S. Then X is G-homeomorphic to the twisted product $G\times_K S$ (see Preliminaries). In turn, due to a result of H. Abels [1, Theorem 2.1], the twisted product $G\times_K S$ is homeomorphic to the product $G/K\times S$. Thus, $X \cong G/K \times S$.

Since G is locally compact and paracompact (see [12, Theorem 3.1.1]) and the quotient map $G \to G/K$ is open and closed, we infer that G/K is also locally compact and paracompact. Further, since S is paracompact, according to a theorem of Morita [21] one has:

$$\dim (G/K \times S) \le \dim G/K + \dim S. \tag{3.5}$$

Since K is compact, according to a result of V. V. Filippov [17] one has the inequality:

$$\dim S \le \dim S/K + \dim q, \tag{3.6}$$

where $q : S \to S/K$ is the K-orbit projection and

$$\dim q = \sup \{\dim g^{-1}(a) \mid a \in S/K\}.$$

Further, since K acts freely on S, we see that $\dim q = \dim K$.

Consequently, combining (3.5) and (3.6) one obtains:

$$\dim (G/K \times S) \le \dim G/K + \dim K + \dim S/K. \tag{3.7}$$

Next, since $X/G \cong (G \times_K S)/G \cong S/K$ (see Preliminaries) and $\dim G/K + \dim K = \dim G$ (see Remark 1.13), it then follows from (3.7) that

$$\dim (G/K \times S) \le \dim X/G + \dim G,$$

as required. □

Acknowledgement

The author thanks the referee for useful comments.

References

1. H. Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Ann. **212** (1974), 1–19. MR0375264 (51:11460)
2. H. Abels, *A universal proper G-space*, Math. Z. **159** (1978), 143-158. MR0501039 (58:18504)
3. S. A. Antonian, *Equivariant embeddings into G-AR's*, Glasnik Matematički **22 (42)** (1987), 503–533. MR957632 (89k:54041)
4. S. A. Antonyan, *Existence of a slice for arbitrary compact transformation groups*, Matematicheskie Zametki **56:5** (1994), 3-9: English transl. in: Math. Notes **56 (5-6)** (1994), 101-1104. MR1330386 (96f:57034)
5. S. A. Antonyan, *The Banach-Mazur compacta are absolute retracts*, Bull. Acad. Polon. Sci. Ser. Math. **46, no. 2** (1998), 113-119. MR1631246 (99d:54020)
6. S. A. Antonyan, *Extensorial properties of orbit spaces of proper group actions*, Topology Appl. **98** (1999), 35-46. MR1719992
7. S. A. Antonyan, *Orbit spaces and unions of equivariant absolute neighborhood extensors*, Topology Appl. **146-147** (2005), 289-315. MR2107153 (2005i:54022)
8. S. A. Antonyan and S. de Neymet, *Invariant pseudometrics on Palais proper G-spaces*, Acta Math. Hung. **98 (1-2)** (2003), 41-51. MR1958466 (2003m:22025)
9. A. V. Arhangel'skii, *Classes of topological groups*, Russian Math. Surveys **36, no. 3** (1981), 151-174. MR622722 (83d:54082)

10. A. V. Arhangel'skii, *Quotients with respect to locally compact subgroups*, Houston J. Math. **31, no. 1** (2005), 215-226. MR2123011 (2006a:54028)
11. A. V. Arhangel'skii and V. V Uspenskij, *Topological groups: local versus global*, Applied General Topology **7, no. 1** (2006), 67-72. MR2284936 (2007k:54042)
12. A. Arhangel'skii and M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Press/World Scientific, Amsterdam-Paris, 2008. MR2433295
13. N. P. Bhatia, G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer Verlag, Berlin, 1970. MR0289890 (44:7077)
14. C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17, no. 1** (1966), 1-16. MR0188982 (32:6409)
15. G. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972. MR0413144 (54:1265)
16. R. Engelking, *General Topology*, PWN-Pol. Sci. Publ., Warsaw, 1977. MR0500780 (58:18316b)
17. V. V. Filippov, *Dimensionality of spaces with the action of a bicompact group*, Math. Notes, **25, no. 3** (1979), 171-174. MR534089 (80f:54031)
18. O. Hájek, *Parallelizability revisited*, Proc. Amer. Math. Soc., **27, no. 1** (1971), 77-84. MR0271925 (42:6806)
19. G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965. MR0207883 (34:7696)
20. J. L. Koszul, *Lectures on groups of transformations*, Tata Institute of Fundamental Research, Bombay, 1965. MR0218485 (36:1571)
21. K. Morita, *On the dimension of product spaces*, American Journal of Mathematics, **75, no. 2** (1953), pp. 205-223. MR0054240 (14:893c)
22. R. Palais, *The classification of G-spaces*, Memoirs of the AMS, **36** (1960). MR0177401 (31:1664)
23. R. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295-323. MR0126506 (23:A3802)
24. E. G. Skljarenko, *On the topological structure of locally bicompact groups and their quotient spaces*, Amer. Math. Soc. Transl., Ser. 2, **39** (1964), 57-82. MR0147575 (26:5090)

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 México Distrito Federal, México

E-mail address: `antonyan@unam.mx`