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A SHORT PROOF THAT HYPERSPACES OF PEANO CONTINUA ARE ABSOLUTE RETRACTS

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ABSTRACT. We give a short proof of Wojdyslawski's famous theorem.

Theorem (Wojdyslawski [6]). *Let X be a Peano continuum. Then the hyperspace 2^X of all nonempty compact subsets of X is an absolute retract for metric spaces.*

This result is an essential step in the proof of the Curtis-Schori-West Hyperspace Theorem to the effect that 2^X is a Hilbert cube for any Peano continuum X (see, e.g., the book of van Mill [5, §8.4]). Wojdyslawski's original proof is rather complicated [6]. A simpler proof was suggested later on by Kelley [4], which is, however, based on a difficult Lefschetz-Dugundji characterization of metric *ANR*'s (see [5, Theorem 5.2.1]). Yet another proof, also based on the Lefschetz-Dugundji characterization, can be found in [5, §5.3]. Our proof is elementary and it does not rely on the Lefschetz-Dugundji criterion.

Proof. Let d be any compatible metric on X and let d_H be the Hausdorff metric on 2^X. Assume that (Y, ρ) is a metric space, A is a closed subset of Y and $f : A \to 2^X$ is a continuous map. Following [3], choose a canonical cover ω of $Y \setminus A$ in Y, that is to say: (1) ω is an open cover of $Y \setminus A$, locally finite in $Y \setminus A$; (2) for each neighborhood V of a point $a \in A$ in Y there exists a neighborhood S of a in Y contained in V, such that every element $U \in \omega$ which meets S is contained in V. We note that the second condition implies that every neighborhood of any boundary point of A in Y contains infinitely many open sets in ω (see [2, Ch. III, §1]).

Let $\mathcal{N}(\omega)$ denote the nerve of ω endowed with the CW topology. We will denote by p_U the vertex of $\mathcal{N}(\omega)$ corresponding to $U \in \omega$. Then according to [3], there exist a Hausdorff space Z and a continuous map $\mu : Y \to Z$ with the following properties:

(a) Z as a set coincides with the disjoint union $A \cup \mathcal{N}(\omega)$;

(b) A is closed in Z and the restriction $\mu|_A$ is the identical homeomorphism;

(c) $Z \setminus A = \mathcal{N}(\omega)$ is taken with its CW topology and $\mu(Y \setminus A) \subset Z \setminus \mu(A)$;

(d) a base of neighborhoods of $a \in A$ in Z is determined by selecting a neighborhood W of a in Y and taking in Z the set $W \cap A$ together with the closed

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star of every vertex p_U of $\mathcal{N}(\omega)$ corresponding to a set $U \in \omega$ with $U \subset W$. This neighborhood is denoted by $\widetilde{W}$.

It is sufficient to prove that f extends to a continuous map $F : Z \to 2^X$; then the map $\Phi = F\mu : Y \to 2^X$ will be the desired extension of f.

Let $\mathcal{N}_k(\omega)$ denote the k-skeleton of $\mathcal{N}(\omega)$. First we extend f to a map $f_0 : A \cup \mathcal{N}_0(\omega) \to 2^X$ as follows: in every set $U \in \omega$ we select a point x_U and then choose a point $a_U \in A$ such that $\rho(x_U, a_U) < 2\rho(x_U, A)$. Set $f_0(p_U) = f(a_U)$ and $f_0(a) = f(a)$ for $a \in A$. It is readily seen that f_0 is continuous. Now we will extend f_0 over each simplex of $\mathcal{N}(\omega)$ and thus we obtain the desired map F. Since 2^X is a Peano continuum [5, Proposition 5.3.10], it is path-connected and locally path-connected by a well-known result of Mazurkiewicz (see [5, Theorem 5.3.13]). For any two points $B, C \in 2^X$ we select a path $l_{B,C} : [0,1] \to 2^X$ such that $l_{B,C}(0) = B$, $l_{B,C}(1) = C$ and

$$\operatorname{diam} l_{B,C}([0,1]) < 2\inf\{\operatorname{diam} \gamma([0,1]) : \ \gamma \text{ is a path from } B \text{ to } C\}.$$

We now extend f_0 to a map $f_1 : A \cup \mathcal{N}_1(\omega) \to 2^X$ by the rule: $f_1(a) = f_0(a)$ for $a \in A$ and $f_1\big(tp_U + (1-t)p_V\big) = l_{f_0(p_U), f_0(p_V)}(t)$, $\ 0 \le t \le 1$. One needs to prove f_1 continuous only at points of A. Let $a \in A$, $\varepsilon > 0$ and $O(f(a), \delta)$ be the δ-neighborhood of $f(a)$ in 2^X. By the local path-connectedness of 2^X, there is a path-connected neighborhood Q of $f_0(a) = f(a)$ contained in $O(f(a), \varepsilon/8)$. By continuity of f_0, there exists a neighborhood of a in Z of the form $\widetilde{W}$ such that $f_0\big(\widetilde{W} \cap \big(A \cup \mathcal{N}_0(\omega)\big)\big) \subset Q$. Then $f_1\big(\widetilde{W} \cap \big(A \cup \mathcal{N}_1(\omega)\big)\big) \subset O\big(f(a), \varepsilon\big)$. Indeed, if $z = tp_U + (1-t)p_V \in \widetilde{W} \cap \mathcal{N}_1(\omega)$, then $f_0(p_U), f_0(p_V) \in Q$; so Q contains a path γ, connecting $f_0(p_U)$ and $f_0(p_V)$. Hence $\operatorname{diam} \gamma\big([0,1]\big) < \varepsilon/4$, which implies that $\operatorname{diam} l_{f_0(p_U), f_0(p_V)}\big([0,1]\big) < \varepsilon/2$. Then $d_H\big(f_1(z), f_1(a)\big) < \varepsilon$ because $f_1(z) \in l_{f_0(p_U), f_0(p_V)}\big([0,1]\big)$.

Now suppose that a continuous extension $f_k : A \cup \mathcal{N}_k(\omega) \to 2^X$ of f_{k-1}, $k \ge 1$ has already been constructed. We shall construct an extension $f_{k+1} : A \cup \mathcal{N}_{k+1}(\omega) \to 2^X$ of f_k. Let σ be any $(k+1)$-dimensional simplex in $\mathcal{N}(\omega)$. Let $\mathbb{B}^{k+1}$ be the $(k+1)$-dimensional Euclidean closed unit ball and $\mathbb{S}^k$ be its boundary sphere. We aim at applying the following well-known easy fact: for every $k \ge 1$ there exists a continuous function $r : \mathbb{B}^{k+1} \to 2^{\mathbb{S}^k}$ such that $r(y) = \{y\}$ for all $y \in \mathbb{S}^k$ (see, e.g., [5, Proposition 5.3.11]). To this end, it is convenient to identify the pair $(\sigma, \partial\sigma)$ with $(\mathbb{B}^{k+1}, \mathbb{S}^k)$. Then the preceding fact insures the existence of a continuous map $r_\sigma : \sigma \to 2^{\partial\sigma}$ such that $r_\sigma(z) = \{z\}$ for every $z \in \partial\sigma$. The map $g_\sigma : 2^{\partial\sigma} \to 2^X$ defined by $g_\sigma(C) = \bigcup_{c \in C} f_k(c)$ is continuous [5, Corollary 5.3.7]. Then $f_\sigma = g_\sigma r_\sigma : \sigma \to 2^X$ is a continuous extension of $f_k|_{\partial\sigma}$. Now we set $f_{k+1}(z) = f_\sigma(z)$ if $z \in \sigma$, and $f_{k+1}(a) = f_k(a)$ if $a \in A$. Then f_{k+1} extends f_k and is continuous on $\mathcal{N}_{k+1}(\omega)$. We define the map $F : Z \to 2^X$ as follows: $F(z) = f_k(z)$ whenever $z \in A \cup \mathcal{N}_k(\omega)$. Clearly, F is continuous on $\mathcal{N}(\omega)$. Let us check its continuity at points of A. Let $a \in A$ and $\varepsilon > 0$. By continuity of f_1, there is a neighborhood of a in Z of the form $\widetilde{W}$ such that $f_1\big(\widetilde{W} \cap \big(A \cup \mathcal{N}_1(\omega)\big)\big) \subset O\big(f(a), \varepsilon\big)$. We claim that $F\big(\widetilde{W}\big) \subset O\big(f(a), \varepsilon\big)$. We shall prove by induction on the dimension of σ that $F(\sigma) \subset O\big(f(a), \varepsilon\big)$ for every simplex $\sigma \subset \widetilde{W}$. If $\dim \sigma = 1$, then $F(\sigma) = f_1(\sigma) \subset O\big(f(a), \varepsilon\big)$. Assume that the claim is true for all simplices $s \subset \widetilde{W}$ with $\dim s \le k$. Let $\sigma \subset \widetilde{W}$, $\dim \sigma = k+1$ and $z \in \sigma$. As $F(z) = f_{k+1}(z) = g_\sigma\big(r_\sigma(z)\big)$, we have $F(z) = \bigcup_{c \in r_\sigma(z)} f_k(c)$. But $d_H\big(f_k(c), f(a)\big) < \varepsilon$ for all $c \in \partial\sigma$, and in particular, for all $c \in r_\sigma(z)$. This yields that $d_H\Big(\bigcup_{c\in r_\sigma(z)} f_k(c), f(a)\Big) < \varepsilon$, i.e., $d_H\big(F(z), f(a)\big) < \varepsilon$, completing the inductive step. □

The reader can easily observe that the same proof serves also for Curtis' theorem [1, Theorem 1.6] on *growth* hyperspaces $\mathcal{G} \subset 2^X$, where X is any connected and locally continuum-connected metrizable space.

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