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OPENNESS OF INDUCED PROJECTIONS

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Abstract. For continua X and Y it is shown that if the projection $f : X \times Y \to X$ has its induced mapping $C(f)$ open, then X is C^*-smooth. As a corollary, a characterization of dendrites in these terms is obtained.

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. To exclude some trivial statements we assume that all considered mappings are not constant. A *continuum* means a compact connected space. Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\}$$

(see e.g. [6, (0.1), p. 1, and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. The reader is referred to Nadler's book [6] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping $f : X \to Y$ between spaces X and Y is said to be *open* provided the image of an open subset of the domain is open in the range. The following results concerning induced mappings for the class of open mappings are known (see [4, Theorem 4.3]; compare also [3, Theorem 3.2]).

1. Statement. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions:*

(1.1) *$f : X \to Y$ is open;*
(1.2) *$C(f) : C(X) \to C(Y)$ is open;*
(1.3) *$2^f : 2^X \to 2^Y$ is open.*

Then (1.1) *and* (1.3) *are equivalent, and each of them is implied by* (1.2).

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An example is known [4, Section 4] of open surjective mappings $f : X \to Y$ between locally connected continua X and Y such that the induced mapping $C(f) : C(X) \to C(Y)$ is not open.

A continuum, the intersection of every two subcontinua of which is connected, is said to be *hereditarily unicoherent*. A continuum is called a *dendroid* provided that it is hereditarily unicoherent and arcwise connected. Given points a and b in a dendroid X, we denote by ab the (unique) arc in X joining these points.

The following result has been proved in [1, Theorem 21].

2. Theorem. *Let X and Y be nondegenerate continua, and let $f : X \times Y \to X$ denote the natural projection. If $C(f)$ is open, then X is hereditarily unicoherent.*

It is known that the opposite implication is not true (see [1, Example 22]). The aim of the paper is to present further results in this direction.

Given a (metric) space X we denote by d_X the metric on X, and by $B_X(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε. Given a subset $A \subset X$, we define $N_X(A, \varepsilon) = \bigcup\{B_X(a, \varepsilon) : a \in A\}$, and we use the symbol $\mathrm{cl}_X(A)$ to denote the closure of A in X. The symbol $\mathbb{N}$ stands for the set of all positive integers.

Let X be a continuum. Define $C^* : C(X) \to C(C(X))$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semicontinuous (see [6, Theorem 15.2, p. 514]), and it is continuous on a dense G_δ subset of $C(X)$ (see [6, Corollary 15.3, p. 515]). A continuum X is said to be *C^*-smooth at* $A \in C(X)$ provided that the function C^* is continuous at A. A continuum X is said to be *C^*-smooth* provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$ (see [6, Definition 5.15, p. 517]). Each arclike continuum is C^*-smooth, ([6, Theorem 15.13, p. 525]). C^*-smoothness implies hereditary unicoherence (see [2, Corollary 3.4, p. 203] and [6, Note 1, p. 530]). Thus each arcwise connected C^*-smooth continuum is a dendroid (see [6, Theorem 15.19, p. 528]). Further, a locally connected continuum is C^*-smooth if and only if it is a dendrite (see [6, Theorem 15.11, p. 522]).

3. Lemma. *Let X be a nondegenerate continuum, and let $\varepsilon > 0$ be given. Then there is a finite sequence of subcontinua $D_0 \subset D_1 \subset \ldots \subset D_m$ of X and there is an ε-net $\{a_1, \ldots, a_m\}$ in X such that $a_i \in D_i \setminus D_{i-1}$ for each $i \in \{1, \ldots, m\}$.*

Proof. Let $\{b_1, \ldots, b_m\}$ be an $\frac{\varepsilon}{2}$-net in X. Fix a point $b \in X \setminus \{b_1, \ldots, b_m\}$. Let $\alpha : [0, 1] \to C(X)$ be an order arc from $\{b\}$ to X; that is, a mapping such that $\alpha(0) = \{b\}$, $\alpha(1) = X$ and, if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$ (for the existence of order arcs see [6, Theorem 1.8, p. 59]). Let $t_0 > 0$ be such that $\alpha(t_0) \cap \{b_1, \ldots, b_m\} = \emptyset$. Define $D_0 = \alpha(t_0)$.

Let $s_1 = \min\{t \in [0, 1] : \alpha(t) \cap \{b_1, \ldots, b_m\} \neq \emptyset\}$. We may assume that $b_1 \in \alpha(s_1)$. Note that $t_0 < s_1$. Consider the set $E = (X \setminus B_X(b_1, \frac{\varepsilon}{2})) \cup D_0$. Then E is a closed subset of X and $b_1 \in X \setminus E$. Thus there exists $t_1 \in (t_0, s_1)$ such that $\alpha(t_1)$ is not contained in E. Choose a point $a_1 \in \alpha(t_1) \setminus E$. Observe that $a_1 \in B_X(b_1, \frac{\varepsilon}{2}) \setminus D_0$. Define $D_1 = \alpha(t_1)$.

Let $s_2 = \min\{t \in [0, 1] : \alpha(t) \cap \{b_2, \ldots, b_m\} \neq \emptyset\}$. We may assume that $b_2 \in \alpha(s_2)$. Note that $t_1 < s_2$. Proceeding as in the paragraph above it is possible to find a number $t_2 \in (t_1, s_2)$ and a point $a_2 \in \alpha(t_2) \cap B_X(b_2, \frac{\varepsilon}{2}) \setminus D_1$. Define $D_2 = \alpha(t_2)$.

Following this procedure we can find points $a_1, a_2, \ldots, a_m$ in X and numbers $0 < t_0 < t_1 < t_2 < \cdots < t_m < 1$ such that for each $i \in \{1, \ldots, m\}$ we have

$a_i \in \alpha(t_i) \setminus \alpha(t_{i-1})$ and $d(a_i, b_i) < \frac{\varepsilon}{2}$. Defining $D_i = \alpha(t_i)$ for each $i \in \{0, 1, \dots, m\}$ we see that the continua $D_0, D_1, \dots, D_m$ and the points $a_1, \dots, a_m$ satisfy the required conditions. The proof is complete.

Let $n \in \mathbb{N}$. A finite sequence of n sets $L_1, \dots, L_n$ is called a *chain* provided that $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Elements L_i of the chain are called its *links*.

4. Theorem. *Let X and Y be nondegenerate continua, and let $f : X \times Y \to X$ denote the natural projection. If $C(f)$ is open, then X is C^*-smooth.*

Proof. Assume the contrary. Let $\mathcal{A} = \operatorname{Lim} C(A_n) \subsetneq C(A)$ for a sequence of subcontinua A_n of X converging to a continuum A, and take $K \in C(A) \setminus \mathcal{A}$. Let $\varepsilon > 0$ be such that $B_{C(X)}(K, 2\varepsilon) \cap C(A_i) = \emptyset$ for each $i \in \mathbb{N}$.

Let $D_0, D_1, \dots, D_m$ and $\{a_1, \dots, a_m\}$ be as in Lemma 3 for the continuum K. Choose subcontinua $E_0 \subset E_1 \subset \dots \subset E_m$ of Y and points $b_i \in E_i \setminus E_{i-1}$. Fix points $a_0 \in D_0$ and $b_0 \in E_0$.

Note that the sequence $\{a_0\} \times E_0,\ D_1 \times \{b_0\},\ \{a_1\} \times E_1,\ D_2 \times \{b_1\},\ \{a_2\} \times E_2,\ \dots,\ D_m \times \{b_{m-1}\},\ \{a_m\} \times E_m,\ A \times \{b_m\}$ is a chain. Let P be the union of the chain, i.e.,

$$P = (A \times \{b_m\}) \cup (D_1 \times \{b_0\}) \cup (D_2 \times \{b_1\}) \cup \dots \cup (D_m \times \{b_{m-1}\}) \\ \cup (\{a_0\} \times E_0) \cup (\{a_1\} \times E_1) \cup \dots \cup (\{a_m\} \times E_m),$$

and note that P is a subcontinuum of $X \times Y$ and that $C(f)(P) = A$.

Choose a number η with $0 < \eta < \varepsilon$ satisfying the two conditions

$$N_X(D_i, \eta) \cap B_X(a_j, \eta) = \emptyset \quad \text{for} \quad 0 \leq i < j,$$
$$N_Y(E_i, \eta) \cap B_Y(b_j, \eta) = \emptyset \quad \text{for} \quad 0 \leq i < j.$$

It follows that the sequence

(4.1)
$$N_{X\times Y}(\{a_0\}\times E_0, \eta),\ N_{X\times Y}(D_1 \times \{b_0\}, \eta),\ N_{X\times Y}(\{a_1\} \times E_1, \eta), \\ N_{X\times Y}(D_2 \times \{b_1\}, \eta),\ N_{X\times Y}(\{a_2\} \times E_2, \eta),\ \dots, \\ N_{X\times Y}(D_m \times \{b_{m-1}\}, \eta),\ N_{X\times Y}(\{a_m\} \times E_m, \eta),\ N_{X\times Y}(A \times \{b_m\}, \eta)$$

is a chain. By interiority of $C(f)$ at P there is a $\delta > 0$ such that $B_{C(X)}(f(P), \delta) \subset C(f)(B_{C(X\times Y)}(P, \eta))$. Let $k \in \mathbb{N}$ be such that $H(A_k, A) < \delta$. Then there is a continuum $Q \subset X \times Y$ such that $H(P, Q) < \eta$ and $f(Q) = A_k$. Take a point $q \in Q$ such that $d_{X\times Y}(q, (a_0, b_0)) < \eta$. Let L be the component of $Q \setminus N_{X\times Y}(A \times \{b_m\}, \eta)$ containing q. By the Janiszewski theorem (known also as the boundary bumping theorem; see e.g. [5, §47, III, Theorems 1 and 2, p. 172] and compare [6, 20.1-20.3, p. 625-626]) there is a point $r \in L \cap \operatorname{cl} N_{X\times Y}(A \times \{b_m\}, \eta)$. Thus L intersects the first and the closure of the last link of the chain (4.1), and it is contained in the union of all links of this chain. Consequently, L intersects each intermediate link of the chain.

Let $q_i \in L \cap N_{X\times Y}(\{a_i\} \times E_i, \eta)$. Note that $d_X(f(q_i), a_i) < \eta$. Thus we have $f(L) \subset N_X(K, \eta) \subset N_X(K, 2\varepsilon)$ and

$$K \subset N_X(\{a_1, \dots, a_m\}, \varepsilon) \subset N_X(\{f(q_1), \dots, f(q_m)\}, \varepsilon + \eta) \\ \subset N_X(f(L), \varepsilon + \eta) \subset N_X(f(L), 2\varepsilon).$$

It follows that $H(K, f(L)) < 2\varepsilon$ and $f(L) \in C(A_k)$, contrary to the definition of ε. The proof is then complete. □

5. Corollary. *For a fixed continuum X let $f_Y : X \times Y \to X$ denote the natural projection. The following conditions are equivalent for a locally connected continuum X:*

(5.1) *X is a dendrite;*
(5.2) *for each continuum Y the induced mapping $C(f_Y)$ is open;*
(5.3) *the induced mapping $C(f_{[0,1]})$ is open;*
(5.4) *there exists a continuum Y such that the induced mapping $C(f_Y)$ is open.*

Proof. The implication from (5.1) to (5.2) is known from [1, Corollary 35]. The implications (5.2) $\implies$ (5.3) $\implies$ (5.4) are obvious. Finally (5.4) implies that X is C^*-smooth according to Theorem 4, which for locally connected continua is equivalent to be a dendrite (see [6, Theorem 15.11, p. 522]). □

The following problem seems to be interesting.

6. Problem. Characterize the continua X for which the converse implication to that of Theorem 4 is true, i.e., the continua X such that the induced mapping $C(f_Y) : C(X \times Y) \to C(X)$ is open for each continuum Y. In particular, is C^*-smoothness of X sufficient?

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