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Combinatorial derived invariants for gentle algebras

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Abstract

We define derived equivalent invariants for gentle algebras, constructed in an easy combinatorial way from the quiver with relations defining these algebras. Our invariants consist of pairs of natural numbers and contain important information about the algebra and the structure of the stable Auslander–Reiten quiver of its repetitive algebra. As a by-product we obtain that the number of arrows of the quiver of a gentle algebra is invariant under derived equivalence. Finally, our invariants separate the derived equivalence classes of gentle algebras with at most one cycle.

MSC: 16G20; 16E30; 18E30

1. Introduction

Let k be a field. For a finite-dimensional connected k-algebra A, the bounded derived category $D^b(A)$ of the module category of finite-dimensional left A-modules, A-mod, contains in some sense the complete homological information about A. So it is natural to classify such algebras up to derived equivalence. Examples of such invariants are the Grothendieck groups, Hochschild (co-)homology and cyclic homology or the neutral component of the group of outer automorphisms; see [12,16]. However it is in general not very easy to calculate these invariants.

In this paper we focus on gentle algebras. This class of algebras is closed under derived equivalence [17]. We propose for gentle algebras new invariants $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ which can be determined easily if A is given as a quiver with relations $kQ/\langle \mathcal{P} \rangle$.

Roughly speaking ϕ_A is obtained as follows: Start with a maximal directed path in Q which contains no relations. Then continue in the opposite direction as long as possible with zero relations. Repeat this until the first path appears again, say after n steps. Then we obtain a pair (n, m) where m is the number of arrows which appeared in a zero relation. Repeat this procedure until all maximal paths without a zero relation have been used; ϕ_A counts then how often each pair $(n, m) \in \mathbb{N}^2$ occurred. See Section 3 for a precise description.

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In order to show that this is in fact a derived invariant, recall first that the repetitive algebra $\hat{A}$ is special biserial and self-injective. We show that ϕ_A describes the action of the suspension functor $\Omega_{\hat{A}}^{-1}$ for the triangulated category $\hat{A}$-mod on the Auslander–Reiten components which contain string modules and Auslander–Reiten triangles, see [10, 1.4], of the form $X \to Y \to \tau_{\hat{A}}^{-1} X \to \Omega_{\hat{A}}^{-1} X$ with Y indecomposable. Note that such components are of the form $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^r\rangle$ for some $r \geq 1$.

Since $D^b(A) \cong D^b(B)$ implies that also $\hat{A}$-mod $\cong \hat{B}$-mod as triangulated categories (even if gl.dim $A = \infty$) we conclude our main result; see Section 6.

Theorem A. *Let A and B be gentle algebras. If A and B are derived equivalent then* $\phi_A = \phi_B$.

We should point out that the argument for showing that ϕ_A is in fact a derived invariant is quite delicate in the case of homogeneous tubes which come from string modules. For this reason we need to analyze the outer automorphism group of a gentle algebra. As a by-product we obtain in Section 4 the following result:

Proposition B. *The number of cycles and the number of arrows are invariant under derived equivalence for gentle algebras.*

Other combinatorial invariants for gentle algebras were recently obtained in [5]. In Section 7 we apply our invariant to the known derived classification of gentle algebras where Q admits at most one cycle (see [3,4,18,6]).

Theorem C. *Let* $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ *and* $B = \mathrm{k}Q'/\langle\mathcal{P}'\rangle$ *be gentle algebras such that* $c(Q), c(Q') \leq 1$. *Then A and B are derived equivalent if and only if* $\phi_A = \phi_B$.

In a forthcoming paper, the first named author will prove that if A is a gentle algebra with two cycles, #Supp $(\phi_A) \in \{1, 3\}$, and will use these invariants to classify those kinds of algebras in the case where # Supp $(\phi_A) = 3$.

2. Preliminaries

2.1. Gentle algebras

A *quiver* is a tuple $Q = (Q_0, Q_1, s, e)$, where Q_0 is the set of vertices, Q_1 the set of arrows and $s, e : Q_1 \to Q_0$ the functions which determine the start and end points respectively of an arrow. For simplicity we assume always that Q is connected. When Q is finite, we say *Q has $c(Q)$ cycles* if $\#Q_1 = \#Q_0 + c(Q) - 1$, in other words if $c(Q)$ is the least number of arrows that we have to remove from Q in order to obtain a tree. A *path* in Q is a sequence of arrows $C = \alpha_n \dots \alpha_2\alpha_1$ such that $s(\alpha_{i+1}) = e(\alpha_i)$ for $1 \leq i \lneq n$; its *length* is n, $l(C) := n$. For $v \in Q_0$ define a *trivial path* 1_v of length zero. We can extend s and e to paths in the obvious way. For $\alpha \in Q_1$ define α^{-1} with $s(\alpha^{-1}) := e(\alpha)$, $e(\alpha^{-1}) := s(\alpha)$ and $(\alpha^{-1})^{-1} := \alpha$; for a path $C = \alpha_n \dots \alpha_2\alpha_1$ define $C^{-1} := \alpha_1^{-1}\alpha_2^{-1} \dots \alpha_n^{-1}$ and $1_v^{-1} := 1_v$ for a trivial path. If C_1 and C_2 are paths in Q define the composition C_2C_1 as the concatenation of the two paths if $s(C_2) = e(C_1)$ or 0 if $s(C_2) \neq e(C_1)$. For a field k let kQ be the corresponding *path algebra*, with paths of Q as a basis and multiplication induced by composition. A non-zero linear combination of paths of length at least 2, with the same start point and end point, is called a *relation in Q*. Let $\mathcal{P}$ be a set of relations in Q and $\langle\mathcal{P}\rangle$ be the ideal of kQ generated by $\mathcal{P}$. In general we consider algebras $\mathrm{k}Q/\langle\mathcal{P}\rangle$ and we identify a path in Q with its corresponding class in $\mathrm{k}Q/\langle\mathcal{P}\rangle$.

The algebras that we are interested in are defined by quivers with relations which fulfill very particular conditions.

Definition 1. We call $\mathrm{k}Q/\langle\mathcal{P}\rangle$ a *special biserial algebra* if the following three conditions hold:

(1) For each $v \in Q_0$, $\#\{\alpha \in Q_1 | s(\alpha) = v\} \leq 2$ and $\#\{\alpha \in Q_1 | e(\alpha) = v\} \leq 2$.
(2) For each $\beta \in Q_1$, $\#\{\alpha \in Q_1 | s(\beta) = e(\alpha)$ and $\beta\alpha \notin \mathcal{P}\} \leq 1$ and $\#\{\gamma \in Q_1 | s(\gamma) = e(\beta)$ and $\gamma\beta \notin \mathcal{P}\} \leq 1$.
(3) For each $\beta \in Q_1$ there is a bound $n(\beta)$ such that any path $\beta_{n(\beta)} \dots \beta_2\beta_1$ with $\beta_{n(\beta)} = \beta$ contains a subpath in $\mathcal{P}$ and any path $\beta_{n(\beta)} \dots \beta_2\beta_1$ with $\beta_1 = \beta$ contains a subpath in $\mathcal{P}$.

A is *gentle* if moreover:

(4) All relations in $\mathcal{P}$ are monomials of length 2.
(5) For each $\beta \in Q_1$, $\#\{\alpha \in Q_1 | s(\beta) = e(\alpha)$ and $\beta\alpha \in \mathcal{P}\} \le 1$ and $\#\{\gamma \in Q_1 | s(\gamma) = e(\beta)$ and $\gamma\beta \in \mathcal{P}\} \le 1$.

Remark 2. A special biserial algebra is finite dimensional if and only if Q is finite; see [15, 1].

Remark 3. A special biserial algebra is called a *string algebra* if $\mathcal{P}$ consists only of paths.

We will make a slight abuse of notation by talking about gentle algebras referring to the quiver with relations which define those algebras.

2.2. *Threads of a gentle algebra*

Let A be a string algebra. A *permitted path* of A is a path $C = \alpha_n \dots \alpha_2\alpha_1$ with no zero relations and it is a *non-trivial permitted thread* of A if for all $\beta \in Q_1$, neither $C\beta$ nor βC is a permitted path. Similarly a *forbidden path* of A is a sequence $\Pi = \alpha_n \dots \alpha_2\alpha_1$ formed by pairwise different arrows in Q with $\alpha_{i+1}\alpha_i \in \mathcal{P}$ for all $i \in \{1, 2, \dots n-1\}$ and it is a *non-trivial forbidden thread* if for all $\beta \in Q_1$, neither $\Pi\beta$ nor $\beta\Pi$ is a forbidden path. The existence of non-trivial permitted threads is due to point (3) in Definition 1 and the existence of non-trivial forbidden threads is due to the restriction of considering pairwise different arrows.

We would like every vertex to be involved in exactly two permitted threads and exactly two forbidden threads so we define also trivial threads for some vertices. Let $v \in Q_0$ be such that $\#\{\alpha \in Q_1 | s(\alpha) = v\} \le 1$, $\#\{\alpha \in Q_1 | e(\alpha) = v\} \le 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = e(\beta)$ then $\gamma\beta \notin \mathcal{P}$; we consider 1_v a *trivial permitted thread* in v and denote it by h_v. Let $\mathcal{H}_A$ be the set of all permitted threads of A, trivial and non-trivial. This set describes completely the algebra A. Similarly, for $v \in Q_0$ with $\#\{\alpha \in Q_1 | s(\alpha) = v\} \le 1$, $\#\{\alpha \in Q_1 | e(\alpha) = v\} \le 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = e(\beta)$ then $\gamma\beta \in \mathcal{P}$; we consider 1_v a *trivial forbidden thread* in v and denote it by p_v. Observe that certain paths can be permitted and forbidden threads at the same time.

For a string algebra it is possible to describe the relations in its quiver by using two functions $\sigma, \varepsilon : Q_1 \to \{1, -1\}$ as in [7, 3], defined by:

(1) If $\beta_1 \ne \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.
(2) If $\gamma_1 \ne \gamma_2$ are arrows with $e(\gamma_1) = e(\gamma_2)$, then $\varepsilon(\beta_1) = -\varepsilon(\beta_2)$.
(3) If β, γ are arrows with $s(\gamma) = e(\beta)$ and $\gamma\beta \notin \mathcal{P}$, then $\sigma(\gamma) = -\varepsilon(\beta)$.

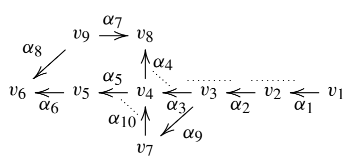
We can extend these functions to threads of A as follows. For $C = \alpha_n \dots \alpha_2\alpha_1$ a non-trivial thread of A define $\sigma(H) := \sigma(\alpha_1)$ and $\varepsilon(H) := \varepsilon(\alpha_n)$. If there is a trivial permitted thread h_v for some $v \in Q_0$, connexity of Q assures the existence of some $\gamma \in Q_1$ with $s(\gamma) = v$ or $\beta \in Q_1$ with $e(\beta) = v$; in the first case we define $\sigma(h_v) = -\varepsilon(h_v) = -\sigma(\gamma)$, and for the second case $\sigma(h_v) = -\varepsilon(h_v) = \varepsilon(\beta)$. If there is a trivial forbidden thread p_v for some $v \in Q_0$, we know that there exists $\gamma \in Q_1$ with $s(\gamma) = v$ or $\beta \in Q_1$ with $e(\beta) = v$; in the first case we define $\sigma(p_v) = \varepsilon(p_v) := -\sigma(\gamma)$, and in the second one $\sigma(p_v) = \varepsilon(p_v) := -\varepsilon(\beta)$.

Remark 4. It is also possible to define threads by using functions σ and ε and the definition of strings in [7], but then four trivial stings will be needed, two corresponding to the permitted case and two corresponding to the forbidden case.

The relations considered are monomials of length 2, so we will indicate them in the quiver by using doted lines, joining each pair of arrows which form a relation:

Example 5.

$A :$



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Define $\sigma(\alpha_1) = \sigma(\alpha_4) = \sigma(\alpha_6) = \sigma(\alpha_8) = \sigma(\alpha_9) = \sigma(\alpha_{10}) = +1$, $\sigma(\alpha_2) = \sigma(\alpha_3) = \sigma(\alpha_5) = \sigma(\alpha_7) = -1$, $\varepsilon(\alpha_3) = \varepsilon(\alpha_7) = \varepsilon(\alpha_8) = +1$ and $\varepsilon(\alpha_1) = \varepsilon(\alpha_2) = \varepsilon(\alpha_4) = \varepsilon(\alpha_5) = \varepsilon(\alpha_6) = \varepsilon(\alpha_9) = \varepsilon(\alpha_{10}) = -1$.

In this case $\mathcal{H}_A$ is formed by α_1, $\alpha_4\alpha_{10}\alpha_9\alpha_2$, $\alpha_6\alpha_5\alpha_3$, α_8, α_7, 1_{v_1}, 1_{v_7} and 1_{v_5}.

2.3. *The repetitive algebra of a gentle algebra*

The information of the following section can be reviewed in [8]; for details see [15]. Let A be a finite-dimensional algebra. The *repetitive algebra of* A is defined as the vector space

$$\hat{A} := (\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} DA)$$

with multiplication given by

$$(\lambda_i, \phi_i)_{i\in\mathbb{Z}} \cdot (\lambda'_i, \phi'_i)_{i\in\mathbb{Z}} := (\lambda_i\lambda'_i, \lambda_{i+1}\phi'_i + \phi_i\lambda'_i).$$

For a gentle algebra, there is a way to describe its repetitive algebra constructing a new quiver from an infinite number of copies of Q. More precisely, let $\mathcal{M}$ the set of non-trivial permitted threads of A, $M := \{1, 2, \ldots, \#\mathcal{M}\}$, and denote its elements by $p_i = \alpha_{i,l(p_i)} \ldots \alpha_{i,2}\alpha_{i,1}$ for each $i \in M$. The *expansion* $\mathbb{Z}(Q, \mathcal{P})$ is defined as the quiver with relations $\mathbb{Z}(Q, \mathcal{P}) := (\hat{Q}, \mathbb{Z}\mathcal{P})$ where $\hat{Q}$ is given by

$$\hat{Q}_0 := \{v[z] | v \in Q_0, z \in \mathbb{Z}\},$$
$$\hat{Q}_1 := \{\alpha[z] | \alpha \in Q_1, z \in \mathbb{Z}\} \cup \{\alpha_{i,0}[z] | i \in M, z \in \mathbb{Z}\}$$

with $\alpha[z] : v[z] \to w[z]$ for each $\alpha : v \to w$, $z \in \mathbb{Z}$ and the connection arrows $\alpha_{i,0}[z] : e(\alpha_{i,l(p_i)})[z+1] \to s(\alpha_{i,1})[z]$ for each $i \in M$, $z \in \mathbb{Z}$. Let $f : \hat{Q}_1 \to M$ be the function $f(\alpha_{i,j}[z]) = i$. Define

$$\mathbb{Z}\mathcal{P} := \{\beta\alpha | \alpha, \beta \in \hat{Q}_1, f(\alpha) \neq f(\beta)\}.$$

There is an automorphism ν in $(\hat{Q}, \mathbb{Z}\mathcal{P})$ given by $\nu(v[z]) = v[z+1]$ and $\nu(\alpha_{i,j}[z]) = \alpha_{i,j}[z+1]$, which corresponds to the Nakayama functor. We say that a path in $\hat{Q}$ not involving any zero relation in $\mathbb{Z}\mathcal{P}$ is *full* if it starts in some $v \in \hat{Q}_0$ and ends in $\nu^{-1}(v)$. Define two sets of relations:

$$\hat{\mathcal{P}} := \mathbb{Z}\mathcal{P} \cup \{p - p' | p, p' \text{ are full with } s(p) = s(p') \text{ and } e(p) = e(p')\}$$
$$\cup \{q | q \text{ is a path in } \hat{Q} \text{ which properly contains a full path}\},$$
$$\bar{\hat{\mathcal{P}}} := \mathbb{Z}\mathcal{P} \cup \{p | p \text{ is a full path}\}.$$

The special biserial algebra $\mathrm{k}\hat{Q}/\langle\hat{\mathcal{P}}\rangle$ is the repetitive algebra of A, $\hat{A}$, while the string algebra $\bar{\hat{A}} := \mathrm{k}\hat{Q}/\langle\bar{\hat{\mathcal{P}}}\rangle$ is obtained from $\hat{A}$ by constructing the quotient over the socles of all projective–injective modules. For each arrow β in $\bar{\hat{A}}$ there are arrows α and γ such that $\beta\alpha$ and $\gamma\beta$ are permitted paths in $\hat{Q}$ not involving any zero relation in $\bar{\hat{\mathcal{P}}}$; we say then that $\bar{\hat{A}}$ is *expanded*. In this case there are only two kinds of vertices:

(1) *Transition vertices*: There is just one arrow α which ends in the vertex, only one arrow β which starts in it and $\beta\alpha \notin \bar{\hat{\mathcal{P}}}$.
(2) *Crossing vertices*: There are exactly two arrows which end in the vertex and exactly two arrows which start in it.

As a consequence of the construction of $\bar{\hat{A}}$, a transition vertex t is the beginning of exactly one non-trivial permitted thread of $\bar{\hat{A}}$, $\mathfrak{p}(t)$, and is the end of exactly one non-trivial permitted thread of $\bar{\hat{A}}$, $\mathfrak{i}(t)$. Meanwhile, a crossing vertex is the beginning and ending of exactly two non-trivial permitted threads of $\bar{\hat{A}}$. Also

$$\mathcal{H}_{\bar{\hat{A}}} = \{p | p \text{ is a non-trivial permitted thread of } \bar{\hat{A}}\} \cup \{t \in \hat{Q}_0 | t \text{ is a transition vertex}\}.$$

For each arrow $\beta \in \hat{Q}_1$ define two elements of $\mathcal{H}_{\bar{\hat{A}}}$ in the following way:

$$\mathfrak{u}(\beta) := \begin{cases} 1_{e(\beta)}, & \text{if } e(\beta) \text{ is a transition vertex} \\ \text{the non-trivial permitted thread which ends in } e(\beta) & \\ \quad \text{and does not contain } \beta, & \text{otherwise.} \end{cases}$$

$$\mathfrak{v}(\beta) := \begin{cases} 1_{s(\beta)}, & \text{if } s(\beta) \text{ is a transition vertex} \\ \text{the non-trivial permitted thread which starts in } s(\beta) & \\ \quad \text{and does not contains } \beta, & \text{otherwise.} \end{cases}$$

Therefore, $\mathcal{H}_{\bar{\hat{A}}}$ is the disjoint union of the sets $\{\mathfrak{v}(\beta) | \beta \in \hat{Q}_1\}$ and $\{\mathfrak{p}(t) | t \text{ is a transition vertex}\}$, or the disjoint union of $\{\mathfrak{u}(\beta) | \beta \in \hat{Q}_1\}$ and $\{\mathfrak{i}(t) | t \text{ is a transition vertex}\}$. We can define a bijection $\tau : \mathcal{H}_{\bar{\hat{A}}} \to \mathcal{H}_{\bar{\hat{A}}}$ by

$$\tau(\mathfrak{r}) := \begin{cases} \mathfrak{u}(\beta) & \text{if } \mathfrak{r} = \mathfrak{v}(\beta) \\ \mathfrak{i}(\nu^{-1}(t)) & \text{if } \mathfrak{r} = \mathfrak{p}(t). \end{cases}$$

The vertices of the stable Auslander–Reiten quiver of $\hat{A}$ which are end terms of Auslander–Reiten sequences with just one indecomposable middle term are naturally parametrized by the set $\mathcal{H}_{\bar{\hat{A}}}$ and τ is the action of the Auslander–Reiten translation; see [7]. If Q is not a tree, the infinite τ-orbits of $\mathcal{H}_{\bar{\hat{A}}}$ parametrize the $\mathbb{Z}A_\infty$ components of the stable Auslander–Reiten quiver of $\hat{A}$, while the finite τ-orbits of $\mathcal{H}_{\bar{\hat{A}}}$ parametrize the $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ components which come from string modules.

3. Combinatorial calculation of invariants

Now we describe a combinatorial algorithm for producing certain pairs of natural numbers, by using only the quiver with relations which defines a gentle algebra A, without needing the Auslander–Reiten quiver of $\hat{A}$. The description of the modules corresponding to Auslander–Reiten sequences with just one indecomposable middle term, see [8, 2.3] and [7, 3], is the key to proving that the algorithm does indeed provide derived equivalent invariants. This is presented in Section 6.

In the algorithm we go through the quiver which defines the algebra, going forward through permitted threads and backwards through forbidden threads in such a way that each arrow is used exactly once and so is its inverse. The procedure is as follows:

(1) (a) Begin with a permitted thread of A, say H_0.
   (b) If H_i is defined, consider Π_i, the forbidden thread which ends in $e(H_i)$ and such that $\varepsilon(H_i) = -\varepsilon(\Pi_i)$.
   (c) Let H_{i+1} be the permitted thread which starts in $s(\Pi_i)$ and such that $\sigma(H_{i+1}) = -\sigma(\Pi_i)$.
   The process stops when $H_n = H_0$ for some natural number n. Let $m = \sum_{1 \le i \le n} l(\Pi_{i-1})$. We obtain the pair (n, m).

(2) Repeat the first step of the algorithm until all permitted threads of A have been considered.

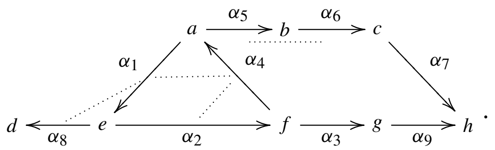
(3) If there are directed cycles in which each pair of consecutive arrows form a relation, we add a pair $(0, m)$ for each of those cycles, where m is the length of the cycle.

(4) Define $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ where $\phi_A(n, m)$ is the number of times that the pair (n, m) arises in the algorithm.

This function is invariant under derived equivalence, as we shall see in Section 6. Note that ϕ_A has always a finite support. Let $\{(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)\}$ be the support of ϕ_A, denote by $[(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)]$ where each (n_j, m_j) is written $\phi_A(n_j, m_j)$ times and the order in which they are written is arbitrary. Define also $\#\phi_A := \sum_{1 \le j \le k} \phi_A(n_j, m_j)$.

Remark 6. The case where there are directed cycles in which each pair of consecutive arrows form a relation corresponds to algebras of infinite global dimension. According to the previous algorithm, for each non-trivial forbidden thread Π determined by such a cycle we add a pair $(0, m)$, where $m = l(\Pi)$. This is indeed a degenerate form of the first step of the process because we go backwards through this forbidden thread of length m but there is no going forward through any permitted thread.

Example 7. Consider

$A:$ 

Let $\sigma(\alpha_1) = \sigma(\alpha_2) = \sigma(\alpha_3) = \sigma(\alpha_7) = \sigma(\alpha_9) = +1$, $\sigma(\alpha_4) = \sigma(\alpha_5) = \sigma(\alpha_6) = \sigma(\alpha_8) = -1$, $\varepsilon(\alpha_4) = \varepsilon(\alpha_7) = +1$ and $\varepsilon(\alpha_1) = \varepsilon(\alpha_2) = \varepsilon(\alpha_3) = \varepsilon(\alpha_5) = \varepsilon(\alpha_6) = \varepsilon(\alpha_8) = \varepsilon(\alpha_9) = -1$.

Let $H_0 = \alpha_5\alpha_4$ and $\Pi_0 = 1_b$ be the trivial forbidden thread in b. Then H_1 is the permitted thread which starts in b and $\sigma(H_1) = -\sigma(\Pi_0)$, that is, $\alpha_7\alpha_6$. Now $\Pi_1 = \alpha_9$ because it is the forbidden thread which ends in h and $\varepsilon(\Pi_1) = -\varepsilon(H_1)$. Then H_2, the permitted thread which starts in $s(\Pi_1)$ and such that $\sigma(H_2) = -\sigma(\Pi_1)$, is the trivial path in g, 1_g. So, Π_2, the forbidden thread which ends in g with $\varepsilon(\Pi_2) = -\varepsilon(H_2)$, is α_3. Then $H_3 = \alpha_5\alpha_4 = H_0$, $n = 3$ and $m = l(\Pi_0)+l(\Pi_1)+l(\Pi_2) = 0+1+1 = 2$. The corresponding pair is (3, 2). We can write this as follows:

$$\begin{array}{lllll} H_0 & = & \alpha_5\alpha_4 & \Pi_0^{-1} = & 1_b \\ H_1 & = & \alpha_7\alpha_6 & \Pi_1^{-1} = & \alpha_9^{-1} \\ H_2 & = & 1_g & \Pi_2^{-1} = & \alpha_3^{-1} \\ H_3 & = & H_0 & & \\ & & & \rightarrow & (3,2). \end{array}$$

If we continue with the algorithm we obtain two other pairs, (2, 4) and (2, 3), in the following way:

$$\begin{array}{lllll} H_0 & = & \alpha_8 & \Pi_0^{-1} = & 1_d \\ H_1 & = & 1_d & \Pi_1^{-1} = & \alpha_2^{-1}\alpha_4^{-1}\alpha_1^{-1}\alpha_8^{-1} \\ H_2 & = & H_0 & & \\ & & & \rightarrow & (2,4) \\ H_0 & = & \alpha_9\alpha_3\alpha_2\alpha_1 & \Pi_0^{-1} = & \alpha_7^{-1} \\ H_1 & = & 1_c & \Pi_1^{-1} = & \alpha_5^{-1}\alpha_6^{-1} \\ H_2 & = & H_0 & & \\ & & & \rightarrow & (2,3). \end{array}$$

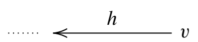
In this case $\phi_A = [(3,2), (2,4), (2,3)]$.

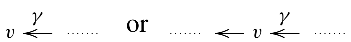
Remark 8. Let $\phi_A = [(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)]$. As these pairs are the ones obtained by using the algorithm we know that $\sum_{1\le j\le k} n_j = \#\mathcal{H}_A$ because each permitted thread appears exactly once. Moreover $\sum_{1\le j\le k} m_j = \#Q_1$, because this corresponds to the sum of lengths of all forbidden threads of A and each arrow arises in exactly one such thread. In other words $\sum_{(n,m)\in\mathbb{N}^2} \phi_A(n,m)p_1(n,m) = \#\mathcal{H}_A$ and $\sum_{(n,m)\in\mathbb{N}^2} \phi_A(n,m)p_2(n,m) = \#Q_1$, where $p_i : \mathbb{N}^2 \to \mathbb{N}$ is the projection in the coordinate i, for $i \in \{1, 2\}$.

Remark 9. In order to prove that ϕ_A is invariant under derived equivalence, let us study all possible cases arising during the algorithm:

(1) For H_i trivial, $H_i = 1_v$ with $v \in Q_0$:
  (a) If $v = s(h)$ for some h non-trivial permitted thread of A, Π_i is trivial and $H_{i+1} = h$.
  (b) If v is not the starting point of a non-trivial permitted thread of A, Π_i is the forbidden thread which ends in γ, the only arrow in Q such that $e(\gamma) = v$.

  Graphically:

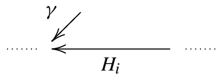
  (a) 

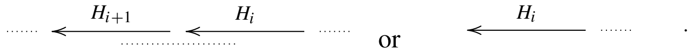
  (b) 

(2) For H_i non-trivial:

(a) If there is $\gamma \in Q_1$ such that $e(H_i) = e(\gamma)$ with $\varepsilon(H_i) = -\varepsilon(\gamma)$, Π_i is the only forbidden thread which final arrow is γ.
(b) If there is no $\gamma \in Q_1$ such that $e(H_i) = e(\gamma)$ with $\varepsilon(H_i) = -\varepsilon(\gamma)$, Π_i is trivial and H_{i+1} is the permitted thread such that $s(H_{i+1}) = e(H_i)$, which is $1_{e(H_i)}$ if $e(H_i)$ is a one-degree vertex.

This corresponds to:

(a) 

(b) 

4. Automorphism groups

We shall prove that for gentle algebras the number of arrows is invariant under derived equivalence. This is necessary for defining in a proper way the characteristic component in Section 5. From now on we consider k an algebraically closed field. By a result presented in [12] and [16], the group $\mathrm{Out}^o(A)$ (which will be defined later) of a finite-dimensional k-algebra is invariant under derived equivalence. We prove that for a gentle algebra it is a group of type $S \ltimes \mathcal{U}$ with $S \cong (\mathrm{k}^*)^{c(Q)}$ and $\mathcal{U}$ a nilpotent subgroup, so the result follows as a consequence.

For a k-algebra A denote by $\mathrm{Aut}(A)$ its automorphism group, by $\mathrm{Inn}(A)$ its inner automorphism group, that is $\mathrm{Inn}(A) := \{\iota_x | x \in A^*\}$ where $\iota_x(a) = x^{-1}ax$ for all $a \in A$, and by $\mathrm{Out}(A)$ its outer automorphism group, $\mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A)$. As $\mathrm{Out}(A)$ is an affine group consider the connected component to which the identity belongs, called its neutral component and denoted by $\mathrm{Out}^o(A)$.

Theorem 10 (*[12,16]*). *Let A be a basic finite-dimensional* k*-algebra. The affine group* $\mathrm{Out}^o(A)$ *is invariant under derived equivalence.*

In order to make some arguments easier we study a subgroup of $\mathrm{Out}(A)$ which has the same neutral component. Following the article [9] define

$$\mathrm{Aut}^l(A) := \{f \in \mathrm{Aut}(A) | f(v) = v\ \forall v \in Q_0\}$$
$$\mathrm{Inn}^l(A) := \mathrm{Inn}(A) \cap \mathrm{Aut}^l(A) = \{\iota_x \in \mathrm{Inn}(A) | x \in \oplus_{v\in Q_0} vAv\}$$
$$\mathrm{Out}^l(A) := \mathrm{Aut}^l(A)/\mathrm{Inn}^l(A).$$

We have

Theorem 11 (*[9, Thm. 15]*). *Let A be a basic finite-dimensional algebra over a field* k*; then* $\mathrm{Out}^l(A)$ *is a subgroup of* $\mathrm{Out}(A)$ *of finite index and has the same neutral component* $\mathrm{Out}^o(A)$.

For a gentle algebra $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ we consider the set of all non-zero paths in A, that is, which are not in $\langle\mathcal{P}\rangle$, as a basis. This collection will be denoted by Γ; let $\Gamma_{\geq 1}$ be the set of elements from Γ of length greater than or equal to 1. Therefore, $A = \oplus_{C\in\Gamma} \mathrm{k}C$ and its Jacobson radical is $\mathcal{J} = \oplus_{C\in\Gamma_{\geq 1}} \mathrm{k}C$. An element f in $\mathrm{Aut}^l(A)$ fixes the vertices of Q, so f is completely determined by the value that it takes in the arrows of the quiver. More precisely, for $f \in \mathrm{Aut}^l(A)$ and $\alpha \in Q_1$,

$$f(\alpha) = \sum_{C\in\Gamma} f_C(\alpha)C = \sum_{C\in e(\alpha)\Gamma s(\alpha)} f_C(\alpha)C$$

where $f_C(\alpha) \in \mathrm{k}$ and the second equality holds because f fixes the vertices of Q. In general, if $D = \alpha_n \dots \alpha_2\alpha_1$,

$$f(D) = \sum_{C_i\in e(\alpha_i)\Gamma s(\alpha_i)} f_{C_n}(\alpha_n) \dots f_{C_2}(\alpha_2) f_{C_1}(\alpha_1) C_n \dots C_2 C_1$$

and we denote by $f_C(D)$ the scalar $f_{C_n}(\alpha_n) \dots f_{C_2}(\alpha_2) f_{C_1}(\alpha_1)$, if $C = C_n \dots C_2 C_1$.

Recall that we identify paths with the corresponding classes in the quotient $kQ/\langle\mathcal{P}\rangle$; anyway, the scalars $f_C(\alpha)$ are well defined because A is gentle and consequently monomial, so there are no two different paths which represent the same element in $kQ/\langle\mathcal{P}\rangle$.

Now, we need to study the neutral component of $\mathrm{Out}^l(A)$ in detail. The next result is the key step for doing so:

Theorem 12. *Let $A = kQ/\langle\mathcal{P}\rangle$ be a gentle algebra, with Q connected, not being the Kronecker quiver. There is a total order $<$ in $\Gamma_{\geq 1}$ such that if $f \in \mathrm{Aut}^l(A)$ with $f_\alpha(\alpha) \neq 0$ for all $\alpha \in Q_1$,*

$$C < D \Rightarrow f_C(D) = 0$$

for $C, D \in \Gamma_{\geq 1}$.

Proof. We define the total order using the path length; if $C, D \in \Gamma_{\geq 1}$ and $l(C) < l(D)$ then we consider $C < D$. It is moreover possible to order paths of the same length linearly according to the above condition. We proceed in several steps:

(1) We prove that for $C, D \in \Gamma$ with $l(C) < l(D)$, $f_C(D) = 0$. First consider the case $D = \alpha \in Q_1$, $C \in \Gamma$ and $\alpha \in Q_1$ with $l(C) < l(\alpha) = 1$, that is $l(C) = 0$. As

$$f(\alpha) = \sum_{E \in e(\alpha)\Gamma s(\alpha)} f_E(\alpha)E,$$

if α is not a loop, this last expression does not involve C because the paths considered on it start and end at a different vertex. If α is a loop, $e(\alpha)\Gamma s(\alpha) = \{1_{s(\alpha)}, \alpha\}$ and $\alpha^2 = 0$, so

$$\begin{aligned} 0 &= f(\alpha^2) = (f(\alpha))^2 = (f_{1_{s(\alpha)}}(\alpha)1_{s(\alpha)} + f_\alpha(\alpha)\alpha)^2 \\ &= (f_{1_{s(\alpha)}}(\alpha))^2 1_{s(\alpha)} + 2f_{1_{s(\alpha)}}(\alpha)f_\alpha(\alpha)\alpha. \end{aligned}$$

Then $(f_{1_{s(\alpha)}}(\alpha))^2 = 0$ and k is a field, so this implies $f_{1_{s(\alpha)}}(\alpha) = 0$. Anyway $l(C) < l(\alpha)$ implies $f_C(\alpha) = 0$. Consider now $D = \alpha_n \ldots \alpha_2\alpha_1$; then $f(D) = f(\alpha_n)\ldots f(\alpha_2)f(\alpha_1)$ and each $f(\alpha_i)$ is an expression

$$\sum_{C_i \in e(\alpha_i)\Gamma s(\alpha_i)} f_{C_i}(\alpha)C_i.$$

Then

$$f(D) = \sum_{C_i \in e(\alpha_i)\Gamma s(\alpha_i)} f_{C_n}(\alpha_n)\ldots f_{C_2}(\alpha_2)f_{C_1}(\alpha_1)C_n\ldots C_2C_1.$$

So $f_C(D) \neq 0$ if and only if $C = C_n \ldots C_2C_1$ for some $C_i \in e(\alpha_i)\Gamma s(\alpha_i)$ with $f_{C_i}(\alpha_i) \neq 0$ for all i; using the previous analysis we know that $l(C_i) \geq l(\alpha) = 1$ and then $l(C) \geq n = l(D)$.

We just proved that if $C, D \in \Gamma_{\geq 1}$ with $l(C) < l(D)$ then $f_C(D) = 0$.

Now consider the case of paths of the same length; by the description of $f(C)$ it is enough to study paths C and D such that $s(C) = s(D)$ and $e(C) = e(D)$. First we study what happens for arrows.

(2) Let $\alpha, \beta \in Q_1$ be such that $s(\alpha) = s(\beta)$ and $e(\alpha) = e(\beta)$. As A is gentle, connected and not the Kronecker one, suppose without loss of generality that there exists $\gamma \in Q_1$ such that $s(\gamma) = e(\alpha)$ with $\gamma\alpha = 0$ and $\gamma\beta \neq 0$ (if this is not the case we use analogous arguments for the dual situation). In fact $f_\beta(\alpha) = 0$ and then we order $\beta < \alpha$ to fulfill the required condition. Indeed

$$0 = f(\gamma\alpha) = f(\gamma)f(\alpha) = \left(\sum_{C \in e(\gamma)\Gamma e(\gamma)} f_C(\gamma)C\right)\left(\sum_{C \in e(\alpha)\Gamma s(\alpha)} f_C(\alpha)C\right)$$

and the coefficient of $\gamma\beta$ is $f_\gamma(\gamma)f_\beta(\alpha)$ (by (1) no trivial paths are involved in the last expression and the algebra is monomial). By hypothesis, $f_\gamma(\gamma) \neq 0$, so $f_\beta(\alpha) = 0$.

(3) Consider now C, D paths of length greater than or equal to 1, with $s(C) = e(D)$ and $e(C) = e(D)$. In fact $f_C(D) = f_D(C) = 0$ and then we can define the order between them as we like. Let $C = \alpha_n \ldots \alpha_2\alpha_1$, and each $f(\alpha_i)$ is an expression $\sum_{E \in e(\alpha_i)\Gamma s(\alpha_i)} f_E(\alpha_i)E$, so $f(C)$ is a linear combination of paths $E_n \ldots E_2E_1$ with coefficient $f_{E_n}(\alpha_n)\ldots f_{E_2}(\alpha_2)f_{E_1}(\alpha_1)$, and each $E_i \in e(\alpha_i)\Gamma s(\alpha_i)$. Let $D = \beta_n \ldots \beta_2\beta_1$. If there is an $i \in \{2, \ldots, n\}$ such that $s(\alpha_i) \neq s(\beta_i)$, D does not appear in the linear combination (again using that

A is monomial). If this is not the case, $s(\alpha_i) = s(\beta_i)$ for all $i \in \{1, 2, \ldots, n\}$, by (2) $f_{\beta_i}(\alpha_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$ and then the coefficient of D in the linear combination, $f_{\beta_n}(\alpha_n) \ldots f_{\beta_2}(\alpha_2) f_{\beta_1}(\alpha_1)$, is zero, that is, $f_D(C) = 0$. Anyway $f_D(C) = 0$, and by symmetry $f_C(D) = 0$. □

Now consider elements f of $\mathrm{Aut}^l(A)$ such that $f_\alpha(\alpha) \neq 0$ for all $\alpha \in Q_1$, that is

$$\mathrm{Aut}^l_*(A) := \{f \in \mathrm{Aut}^l(A) | f_\alpha(\alpha) \neq 0 \; \forall \alpha \in Q_1\}.$$

As a consequence of the previous result it is possible to describe this set:

Corollary 13. *Let $A = \mathrm{k}Q/\langle \mathcal{P} \rangle$ be a gentle algebra, with Q connected, not being the Kronecker quiver.*

(1) *The open set $\mathrm{Aut}^l_*(A)$ is a subgroup of finite index of $\mathrm{Aut}^l(A)$, isomorphic to a lower triangular matrices subgroup.*
(2) $\mathrm{Aut}^l_*(A) = S(Q_1) \ltimes N(A)$ *where*

$$S(Q_1) := \{f \in \mathrm{Aut}^l_*(A) | f(\alpha) \in \mathrm{k}^*\alpha \; \forall \alpha \in Q_1\} \cong (\mathrm{k}^*)^{\# Q_1} \quad \textit{and}$$
$$N(A) := \{f \in \mathrm{Aut}^l_*(A) | f_\alpha(\alpha) = 1 \; \forall \alpha \in Q_1\}.$$

Proof. (1) By Theorem 12 there exists a total order in $\Gamma_{\geq 1}$ such that the matrices describing the elements of $\mathrm{Aut}^l_*(A)$ in terms of the corresponding ordered basis of paths in A are invertible and lower triangular. Then this is a subgroup of $\mathrm{Aut}^l(A)$, because the product of two such matrices is of the same type.

(2) As A is monomial, each element $(c_\alpha)_{\alpha \in Q_1} \in (\mathrm{k}^*)^{\# Q_1}$ defines an automorphism f in $\mathrm{Aut}^l_*(A)$ given by $f(\alpha) = c_\alpha \alpha$ for each $\alpha \in Q_1$ and this correspondence provides an isomorphism between $(\mathrm{k}^*)^{\# Q_1}$ and $S(Q_1)$. Now let us see $\mathrm{Aut}^l_*(A) = S(Q_1) \ltimes N(A)$. Let $f \in \mathrm{Aut}^l_*(A)$. For each $\alpha \in Q_1$, $f(\alpha) = c_\alpha \alpha + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} f_C(\alpha) C$. Consider $g \in S(Q_1)$ defined by $g(\alpha) = c_\alpha \alpha$ for all $\alpha \in Q_1$. But $fg^{-1} \in N(A)$ because

$$\begin{aligned} fg^{-1}(\alpha) &= f(g^{-1}(\alpha)) = f(c_\alpha^{-1}\alpha) \\ &= c_\alpha^{-1} f(\alpha) = 1\alpha + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} c_\alpha^{-1} f_C(\alpha) C \end{aligned}$$

and then $f = fg^{-1}g \in N(A)S(Q_1)$. Also, if $f \in N(A) \cap S(Q_1)$, $f(\alpha) = 1\alpha$ for all $\alpha \in Q_1$, and then $f = id$. Let us prove $N(A) \lhd \mathrm{Aut}^l_*(A)$. Consider $g \in S(Q_1)$ such that $g(\alpha) = c_\alpha \alpha$ with $c_\alpha \in \mathrm{k}^*$ for all $\alpha \in Q_1$, and $h \in N(A)$ such that

$$h(\alpha) = 1\alpha + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} h_C(\alpha) C.$$

Then

$$\begin{aligned} g^{-1}hg(\alpha) &= g^{-1}h(c_\alpha \alpha) = c_\alpha g^{-1} h(\alpha) \\ &= c_\alpha g^{-1} \left(1\alpha + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} h_C(\alpha) C \right) \\ &= c_\alpha \left(1 g^{-1}(\alpha) + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} h_C(\alpha) g^{-1}(C) \right) \\ &= 1\alpha + \sum_{C \in e(\alpha)\Gamma s(\alpha) \setminus \{\alpha\}} h_C(\alpha) c_\alpha g^{-1}(C) \end{aligned}$$

which is an element of $N(A)$ because if $g^{-1}(C) = \alpha$, $C = g(\alpha) = c_\alpha \alpha$ and then $C = \alpha$. So $S(Q_1)$ normalizes $N(A)$. Consider now $f \in \mathrm{Aut}^l_*(A)$ and $h \in N(A)$; f can be written as $f = \tilde{h}g$ for some $\tilde{h} \in N(A)$ and $g \in S(Q_1)$. Then

$$f^{-1}hf = (\tilde{h}g)^{-1}h(\tilde{h}g) = g^{-1}\tilde{h}^{-1}h\tilde{h}g = g^{-1}(\tilde{h}^{-1}h\tilde{h})g$$

which is an element of $N(A)$ because $\tilde{h}^{-1}h\tilde{h} \in N(A)$ and $S(Q_1)$ normalizes $N(A)$. We conclude $N(A) \trianglelefteq \mathrm{Aut}^l_*(A)$. This leads us to $\mathrm{Aut}^l_*(A) = S(Q_1) \ltimes N(A)$. □

Theorem 14. *Let* $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ *be a gentle algebra with* Q *connected and different from the Kronecker quiver.* $\mathrm{Out}^o(A)$ *is isomorphic to a subgroup of lower triangular invertible matrices which has the form* $S \ltimes \mathcal{U}$ *with* $S \cong (\mathrm{k}^*)^{c(Q)}$ *a maximal torus and* $\mathcal{U}$ *a nilpotent subgroup of* $\mathrm{Out}^o(A)$.

Proof. Let $x \in \oplus_{v\in Q_o} vAv$ be invertible. Its component in v is $x_v = c_v 1_v + \sum_{C\in v\Gamma_{\geq 1}v} c_C C$ with $c_v, c_C \in \mathrm{k}$. As the element x is invertible, $c_v \neq 0$ for all $v \in Q_0$, and then $x_v = c_v 1_v(1_v + c_v^{-1}\sum_{C\in v\Gamma_{\geq 1}v} c_C C)$ with $c_v^{-1}\sum_{C\in v\Gamma_{\geq 1}v} c_C C$ a nilpotent element which we denote by $n_v \in vAv$ and $c_v \in \mathrm{k}^*$. Define $s := \sum_{v\in Q_0}(c_v 1_v)$ and $u := \sum_{v\in Q_0}(1_v + n_v)$ and observe $su = x$ because $c_v 1_v 1_w = 0 = c_v 1_v n_w$ for $v, w \in Q_0$ with $v \neq w$ and then $su = \sum_{v\in Q_0} x_v = x$; so $\iota_x = \iota_u\iota_s$ and $\iota_u \in N(A)$, $\iota_s \in S(Q_1)$.

Define

$$\mathrm{Inn}^l_N(A) := \left\{\iota_u | u = \sum_{v\in Q_0}(1_v + n_v), n_v \in vAv \text{ nilpotent}\right\}$$

$$\mathrm{Inn}^l_S(A) := \left\{\iota_s | s = \sum_{v\in Q_0}(c_v 1_v), c_v \in \mathrm{k}^*\right\}.$$

For $f \in \mathrm{Aut}^l_*(A)$, $\iota_x \in \mathrm{Inn}^l(A)$, $y \in A$

$$(f\iota_x f^{-1})(y) = f(\iota_x f^{-1}(y)) = f(x^{-1}f^{-1}(y)x) = f(x^{-1})yf(x) = \iota_{f(x)}(y)$$

and then $f\iota_x f^{-1} = \iota_{f(x)}$. Some remarks can be made at this point:

(1) $\mathrm{Inn}^l_S(A) \trianglelefteq S(Q_1)$ because $\mathrm{Inn}^l_S(A) \subset Z(\mathrm{Aut}^l(A))$. Indeed, if $f\mathrm{Aut}^l(A)$ and $\iota_s \in S(Q_1)$ then $f\iota_s f^{-1} = \iota_{f(s)} = \iota_s$ because s can be expressed as $\sum_{v\in Q_0}(c_v 1_v)$, with $c_v \in \mathrm{k}^*$ and then $f(s) = s$.

(2) $\mathrm{Inn}^l(A) \trianglelefteq \mathrm{Aut}^l_*(A)$ because if $f \in \mathrm{Aut}^l_*(A)$ and $x \in \oplus_{v\in Q_o} vAv$ $f(x) \in \oplus_{v\in Q_o} vAv$, so $f\iota_x f^{-1} = \iota_{f(x)} \in \mathrm{Inn}^l(A)$.

(3) $\mathrm{Inn}^l_N(A) \trianglelefteq N(A)$ because in fact $\mathrm{Inn}^l_N(A)$ is a normal subgroup of $\mathrm{Aut}^l_*(A)$. In order to explain this, let $u = \sum_{v\in Q_0}(1_v + n_v)$ with $n_v = c_v^{-1}\sum_{C\in v\Gamma_{\geq 1}v} c_C C$ a nilpotent element and $f \in \mathrm{Aut}^l_*(A)$; then $f(n_v) = c_v^{-1}\sum_{C\in v\Gamma_{\geq 1}v} c_C f(C)$ which is again nilpotent because no trivial paths appear in $f(C)$ and A is gentle. Therefore, $f(u) = \sum_{v\in Q_0}(1_v + \hat{n}_v)$ with $\hat{n}_v$ nilpotent, and then $f\iota_u f^{-1} = \iota_{f(u)} \in \mathrm{Inn}^l_N(A)$.

It is possible to consider the morphism

$$\varphi : S(Q_1) \to \mathrm{Aut}(N(A))$$

for each $g \in S(Q_1)$; $\varphi(g)$ is defined by $\varphi(g)(h) := g^{-1}hg$ for all $h \in N(A)$. This induces a morphism

$$\bar{\varphi} : S(Q_1)/\mathrm{Inn}^l_S(A) \to \mathrm{Aut}(N(A)/\mathrm{Inn}^l_N(A)).$$

Also define

$$\psi : S(Q_1) \ltimes N(A) \to S(Q_1)/\mathrm{Inn}^l_S(A) \ltimes_{\bar{\varphi}} N(A)/\mathrm{Inn}^l_N(A)$$

by $\psi(gh) = (g\mathrm{Inn}^l_S(A), h\mathrm{Inn}^l_N(A))$. This is a group morphism: if $g, \hat{g} \in S(Q_1)$ and $h, \hat{h} \in N(A)$,

$$\begin{aligned}\psi((gh)(\hat{g}\hat{h})) &= \psi(g\hat{g}(\hat{g}^{-1}h\hat{g})\hat{h}) = (g\hat{g}\mathrm{Inn}^l_S(A), (\hat{g}^{-1}h\hat{g})\hat{h}\mathrm{Inn}^l_N(A))\\ &= (g\hat{g}\mathrm{Inn}^l_S(A), \varphi(g)(h)\hat{h}\mathrm{Inn}^l_N(A))\\ &= (g\mathrm{Inn}^l_S(A), h\mathrm{Inn}^l_N(A))(\hat{g}\mathrm{Inn}^l_S(A), \hat{h}\mathrm{Inn}^l_N(A))\\ &= \psi(gh)\psi(\hat{g}\hat{h})\end{aligned}$$

and $\ker\psi = \mathrm{Inn}^l(A)$. This last statement is because the kernel elements are of the form gh with $g \in \mathrm{Inn}^l_S(A)$ and $h \in \mathrm{Inn}^l_N(A)$ (that is $g = \iota_s$ where $s = \sum_{v\in Q_0}(c_v 1_v)$, $c_v \in \mathrm{k}^*$ and $h = \iota_u$ where $u = \sum_{v\in Q_0}(1_v + n_v)$, $n_v \in vAv$ nilpotent) and, as shown in the first part of the proof, $gh = \iota_s\iota_u = \iota_{su}$ with $su \in \oplus_{v\in Q_o} vAv$ an invertible element; in fact any invertible element in $\oplus_{v\in Q_o} vAv$ can be expressed like this. So

$$\begin{aligned}\mathrm{Aut}^l_*(A)/\mathrm{Inn}^l(A) &= (S(Q_1) \ltimes N(A))/\mathrm{Inn}^l(A)\\ &\cong S(Q_1)/\mathrm{Inn}^l_S(A) \ltimes_{\bar{\varphi}} N(A)/\mathrm{Inn}^l_N(A).\end{aligned}$$

As $N(A)$ is nilpotent, the same is true for $N(A)/\mathrm{Inn}^l_N(A)$. Consider a vertex $v_0 \in Q_0$; if $s = \sum_{v\in Q_0}(c_v 1_v)$, $c_v \in \mathrm{k}^*$ and $s' := 1_{v_0} + \sum_{v\in Q_0\setminus\{v_0\}}(c'_v 1_v)$, $c'_v := c_{v_0}^{-1}c_v$, see that $\iota_s = \iota_{s'}$. Consider now another element of type $s'' = 1_{v_0} + \sum_{v\in Q_0\setminus\{v_0\}}(c''_v 1_v)$ for some $c''_v \in \mathrm{k}^*$; $\iota_{s'} = \iota_{s''}$ implies $c'_v = c''_v$ for all $v \in Q_0 \setminus \{v_0\}$ because Q is connected. This means that each of the automorphisms in $\mathrm{Inn}^l_S(A)$ can be determined by the choice of scalars c_v in the vertices different from v_0 and then $\mathrm{Inn}^l_S(A) \cong (\mathrm{k}^*)^{\#Q_0-1}$. Therefore, $S(Q_1)/\mathrm{Inn}^l_S(A) \cong (\mathrm{k}^*)^{\#Q_1-(\#Q_0-1)} = (\mathrm{k}^*)^{c(Q)}$. By Theorem 11, $\mathrm{Out}^o(A) = (\mathrm{Out}^l)^o(A)$. Also $\mathrm{Aut}^l_*(A)$ is by definition an open set of $\mathrm{Aut}^l(A)$ and using [13, 7.4] we know that it is closed at the same time, so $(\mathrm{Aut}^l(A))^o = \mathrm{Aut}^l_*(A)$. Then $(\mathrm{Out}^l(A))^o = (\mathrm{Aut}^l(A))^o/\mathrm{Inn}^l(A) = \mathrm{Aut}^l_*(A)/\mathrm{Inn}^l(A)$ and the proof is completed. □

Now, we are ready to prove Proposition B stated in the introduction.

By Theorem 10, $\mathrm{Out}^o(A)$ is invariant under derived equivalence. By Theorem 14, $\mathrm{Out}^o(A)$ is solvable and contains a maximal torus of rank $c(Q)$. Finally, for a solvable group all maximal tori are conjugated and have in particular the same rank. We conclude that $c(Q)$ is invariant under derived equivalence. As $\#Q_1 = \#Q_0 + c(Q) - 1$ and the number of vertices is a derived invariant, so is the number of arrows and this completes the proof.

5. Theoretical interpretation of invariants

Let $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ be a gentle algebra, Q not being a tree, and $\hat{A}$ its repetitive algebra. Recall that $\hat{A}$ is self-injective. Consider the stable Auslander–Reiten quiver of $\hat{A}$, which we denote by $\Gamma_{\hat{A},s}$. Let τ be the Auslander–Reiten translation and Ω^{-1} the Heller suspension functor in the stable module category $\hat{A}$-$\underline{\mathrm{mod}}$; as τ and Ω commute, Ω permutes the $\Gamma_{\hat{A},s}$ components. We call *characteristic components of* $\Gamma_{\hat{A},s}$ components of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ which come from string modules; see [8, 2.3]. Note that each component $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, $n \geq 2$, come from string modules; just the homogeneous tubes $\mathbb{Z}A_\infty/\langle\tau\rangle$ consist possibly of band modules.

We define an equivalence relation on the set of characteristic components of $\Gamma_{\hat{A},s}$ as follows: given C_1 and C_2 characteristic components of $\Gamma_{\hat{A},s}$, C_1 is related to C_2 if they are in the same Ω-orbit. We call an equivalence class under the relation a *series of components*.

For each series of components $[C]$ there is an associated pair of natural numbers (n, m), denoted by $i_{[C]}$, obtained as follows. By [8, 2.3] $\Gamma_{\hat{A},s}$ contains a finite number of components $\mathbb{Z}A_\infty$, and therefore each of these components $[C]$ has a finite number of elements; define $i_{[C]} := (n, m)$ where $|n - m| = \#[C]$ and $\Omega^{n-m}(M) = \tau^n(M)$ for all modules M whose isomorphic class belongs to this series of components. For a series of components $[C]$ with C of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, consider $i_{[C]} := (n, n)$; this is in fact a limit case of the previous case. The series consists of a numerable set of components and $\Omega^0(M) = M = \tau^n(M)$ for all modules M whose isomorphic class belongs to that series, because the components are tubes or rank n. These pairs of natural numbers and how many times they occur will be essential in this analysis. Define $N_A : \mathbb{N}^2 \to \mathbb{N}$ with

$$N_A(n, m) = \#\{[C] \text{ series of components of } \Gamma_{\hat{A},s} \,|\, i_{[C]} = (n, m)\}.$$

Just like for the function ϕ_A presented in Section 3, the support of N_A is always finite. Let $\{(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)\}$ be the support of N_A; denote as $[(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)]$ where each (n_j, m_j) is written $N_A(n_j, m_j)$ times and the order in which they are presented is arbitrary. Also define $\#N_A := \sum_{1\leq j\leq k} N_A(n_j, m_j)$ or the number of series of components of A.

6. Justification of the algorithm

According to Section 2.3, for a gentle algebra, a module in the stable Auslander–Reiten quiver of the repetitive algebra which is the start or end of an Auslander–Reiten sequence with just one indecomposable middle term corresponds to a permitted thread of $\hat{A}$; identifying this module with the related permitted threads, now we analyze the action of the Heller suspension functor over the set of permitted threads. By definition, if M is a module over an algebra A, $\Omega^{-1}(M) = \mathrm{coker}(M \to I)$ where I is the injective envelope of M. Let H be a permitted thread of $\hat{A}$. H is trivial or is obtained from a full path by deleting its final arrow, which according to the notation of Section 2.3, can be written as $\alpha_{i,j}[z]$ for some $i \in M$, $j \in \{0, 1, \ldots, l(H)\}$, $z \in \mathbb{Z}$; it also is denoted by $\alpha^H_{i,j}[z]$. In this case, if $j \neq 0$, $\alpha_{i,j} \in Q_1$ and H is identified with $\alpha^H_{i,j}[z]^{-1}$; otherwise, it is a connection arrow (see Section 2.3) and then H corresponds to one of the copies of the permitted thread of A, h, in the quiver associated with $\bar{\hat{A}}$, that is, $H = \alpha_{i,l(H)}[z] \cdots \alpha_{i,2}[z]\alpha_{i,1}[z]$ with $h = \alpha_{i,l(H)} \cdots \alpha_{i,2}\alpha_{i,1}$ a permitted thread of A and $z \in \mathbb{Z}$; define then $H = h[z]$. Because of the structure of $\bar{\hat{A}}$ we get

$$\Omega^{-1}(H) = \begin{cases} \mathfrak{p}(v[z+1]) & \text{if } H = 1_{v[z]} \text{ with } v \in Q_0,\, z \in \mathbb{Z} \\ \mathfrak{v}(\alpha^H_{i,j}[z+1]) & \text{if } H \text{ is non-trivial} \end{cases}$$

or, more precisely,

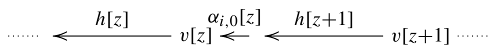
$$\Omega^{-1}(H) = \begin{cases} \mathfrak{p}(v[z+1]) & \text{if } H = 1_{v[z]} \text{ with } v \in Q_0,\, z \in \mathbb{Z} \\ 1_{\nu(e(H))} & \text{if } H \text{ is non-trivial and } \nu(e(H)) \text{ is a transition vertex} \\ \text{the permitted thread starting in} & \\ \quad \nu(e(H)) \text{ which does not involve } \alpha^H_{i,j}[z+1] & \text{if } H \text{ is non-trivial and} \\ \quad \nu(e(H)) \text{ is a crossing vertex} & \end{cases}$$

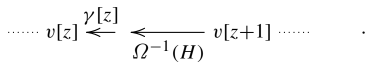
where $\mathfrak{p}$ and $\mathfrak{v}$ are defined as in Section 2.3.

Remark 15. By the previous analysis and the fact that A is gentle, $\Omega^{-1}(H)$ can be described for a permitted thread of $\bar{\hat{A}}$, H, in all cases:

(1) For H trivial, $H = 1_{v[z]}$ with $v \in Q_0$, $z \in \mathbb{Z}$ (in this case 1_v is a trivial thread of A):
   (a) If $v = s(h)$ for some permitted thread of A h, $\Omega^{-1}(H) = h[z+1]$.
   (b) If v is not the start point of a permitted non-trivial thread in A, $\Omega^{-1}(H)$ is identified with $\gamma[z]$ where γ is the only arrow in Q such that $e(\gamma) = v$.

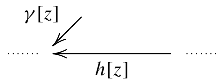
   Graphically:

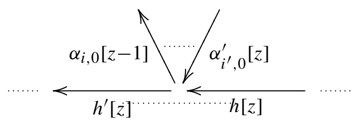
   (a) 

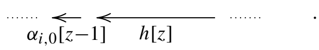
   (b) 

(2) In the case $H = h[z]$ for some permitted non-trivial thread of A, h:
   (a) If there exists $\gamma \in Q_1$ such that $e(h) = e(\gamma)$ with $\varepsilon(h) = -\varepsilon(\gamma)$, $\Omega^{-1}(H)$ is identified with $\gamma[z]^{-1}$, and γ corresponds to the final arrow of a forbidden thread of A.
   (b) If there is no $\gamma \in Q_1$ such that $e(h) = e(\gamma)$ with $\varepsilon(h) = -\varepsilon(\gamma)$, $\Omega^{-1}(H) = h'[z+1]$ where h' is the permitted thread such that $s(h') = e(h)$, which is $1_{\nu(e(H))}$ in the case where $e(h)$ has degree 1.

   Graphically:

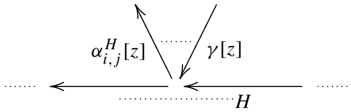
   (a) 

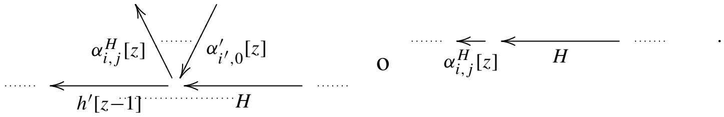
   (b) 

   or 

(3) In the case where H is identified with $\alpha^H_{i,j}[z]^{-1}$:
   (a) If there exists $\gamma \in Q_1$ such that $e(\gamma[z]) = s(\alpha^H_{i,j}[z]) = e(H)$ with $\varepsilon(\gamma) = \sigma(\alpha^H_{i,j})$, $\Omega^{-1}(H)$ is identified with $\gamma[z]^{-1}$ and $\alpha^H_{i,j}\gamma$ is a forbidden thread in A.
   (b) If there is no $\gamma \in Q_1$ such that $e(\gamma[z]) = s(\alpha^H_{i,j}[z]) = e(H)$ with $\varepsilon(\gamma) = \sigma(\alpha^H_{i,j})$, $\alpha^H_{i,j}$ is the starting point of a forbidden thread in A and $\Omega^{-1}(H) = h'[z+1]$ where h' is the permitted thread of A such that $s(h') = s(\alpha^H_{i,j})$ and does not involve $\alpha^H_{i,j}$, that is, such that $\sigma(h') = -\sigma(\alpha^H_{i,j})$, which is $1_{\nu(e(H))}$ if $e(H)$ is a transition vertex.

Graphically:

(a) 

(b) 

After this analysis we get the following result:

Theorem 16. *Let A be a gentle algebra and $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ the function defined in Section 3; then $\phi_A = N_A$.*

Proof. Recall the algorithm of Section 3. Comparing Remarks 9 and 15 we conclude that, if H is a permitted thread of $\bar{\hat{A}}$, the powers $\Omega^i(H)$ are the different permitted threads of A appearing in the algorithm, or identify with the inverses of the arrows in Q which constitute the non-trivial forbidden threads of A, obtained in the same order as in the algorithm. More precisely, let H_0 be a permitted thread of A and $H_0, H_1, \ldots H_n$ the permitted threads obtained in the process starting with H_0. Denote by $\Pi_0, \Pi_1, \ldots \Pi_{n-1}$ the non-trivial forbidden threads of A involved in this part of the algorithm, where

$$\Pi_i = \pi_{i,l(\Pi_i)} \cdots \pi_{i,2}\pi_{i,1}$$

with $\pi_{i,j} \in Q_1$ for $i \in \{0, 1, 2, \ldots, n-1\}$, $j \in \{1, 2, \ldots, l(\Pi_i)\}$, $m = \sum_{0\le i\le n-1} l(\Pi_i)$ and $(n.m)$ the pair of natural numbers obtained from H_0. This part of the algorithm can be described by the array

$$\begin{array}{ll} H_0[0] & \pi^{-1}_{1,1}[0]\cdots\pi^{-1}_{1,l(\Pi_0)}[0] \\ H_1[1] & \pi^{-1}_{2,1}[1]\cdots\pi^{-1}_{2,l(\Pi_1)}[1] \\ H_2[2] & \ldots \\ \vdots & \\ H_n[n] & \end{array}$$

where none of the trivial forbidden threads is written out (and this is indicated by putting an asterisk instead). Identifying permitted threads of $\bar{\hat{A}}$ which contain a connection arrow with its corresponding one $\alpha^H_{i,j}[z]^{-1}$ and using the previous remarks, this is the array

$$\begin{array}{ll} H_0[0] & \Omega^{-l(\Pi_0)}(H_0[0])\cdots\Omega^{-1}(H_0[0]) \\ \Omega^{-l(\Pi_0)-1}(H_0[0]) & \Omega^{-l(\Pi_0)-l(\Pi_1)-1}(H_0[0])\cdots\Omega^{-l(\Pi_0)-2}(H_0[0]) \\ \Omega^{-l(\Pi_0)-l(\Pi_1)-2}(H_0[0]) & \ldots \\ \vdots & \\ \Omega^{-l(\Pi_0)-l(\Pi_1)-\cdots-l(\Pi_{n-1})-n}(H_0[0]) & \end{array}$$

where the last element is $\Omega^{-m-n}(H_0[0])$ because of the way m was defined. As the algorithm stops at the first moment in which $H_n = H_0$, $\Omega^{-n-m}(H_0[0])$ is the first module in the same ν-orbit of $H_0[0]$ obtained by applying a certain number of times Ω^{-1}, in fact $\Omega^{-n-m}(H_0[0]) = H_n[n] = \nu^n(H_0[0])$. As Ω is an equivalence, the information about the modules of the Auslander–Reiten quiver which are extreme points of an Auslander–Reiten sequence with one indecomposable middle term is enough for concluding that $\Omega^{-n-m} = \nu^n$ for every module in the series of components where $H_0[0]$ lies. As $\tau = \Omega^2 \circ \nu$ and Ω commute with ν then

$$\tau^n = (\Omega^2 \circ \nu)^n = \Omega^{2n} \circ \nu^n \Rightarrow \Omega^{-2n} \circ \tau^n = \nu^n = \Omega^{-n-m}$$

which proves that $\tau^n = \Omega^{n-m}$. This is also true if naturals n and m are equal and in this case $\tau^n = \Omega^0 = id$ for modules in the series of components where $H_0[0]$ lies, and n is the least natural for which this happens; then these components are tubes of rank n. If $n \neq m$, evaluating powers of Ω^{-1} at $H_0[0]$, we get modules in the different components forming the series associated with $H_0[0]$ until we get to the component where we started, after exactly $n-m$ steps. Then $|n-m|$ is the number of components of the series. In the case of a gentle algebra of infinite global dimension, there exists a directed cycle in which every pair of consecutive arrows defines a relation. Let α be any of these arrows and H the permitted thread in $\bar{\hat{A}}$ with which we identify $\alpha[0]^{-1}$. By previous analysis, the powers of $\Omega^{-1}(H), \Omega^{-2}(H) \dots \Omega^{-m}(H)$ identify precisely with the arrows conforming the directed cycle just mentioned, with m its length and $\Omega^{-m}(H) = H$. This implies that $\Omega^{-m} = id = \tau^0$ for modules in the series of components where H lies, formed by exactly m components. □

Example 17. For the algebra presented in the example of Section 3, if $H_0 = \alpha_5\alpha_4$ the algorithm is codified by

$$\begin{array}{ll} (\alpha_5\alpha_4)[0] & * \\ (\alpha_7\alpha_6)[1] & \alpha_9[1]^{-1} \\ 1_g[2] & \alpha_3[2]^{-1} \\ (\alpha_5\alpha_4)[3] & * \end{array}$$

and for $H_0 = \alpha_8$ by

$$\begin{array}{ll} \alpha_8[0] & * \\ 1_d[1] & \alpha_2[1]^{-1}\alpha_4[1]^{-1}\alpha_1[1]^{-1}\alpha_8[1]^{-1} \\ \alpha_8[2] & \end{array}$$

and for $H_0 = \alpha_9\alpha_3\alpha_2\alpha_1$ by

$$\begin{array}{ll} (\alpha_9\alpha_3\alpha_2\alpha_1)[0] & \alpha_7[0]^{-1} \\ 1_c[1] & \alpha_5[1]^{-1}\alpha_6[1]^{-1} \\ (\alpha_9\alpha_3\alpha_2\alpha_1)[2] & \end{array}$$

which correspond to the following arrays:

$$\begin{array}{ll} (\alpha_5\alpha_4)[0] & * \\ \Omega^{-1}((\alpha_5\alpha_4)[0]) & \Omega^{-2}((\alpha_5\alpha_4)[0]) \\ \Omega^{-3}((\alpha_5\alpha_4)[0]) & \Omega^{-4}((\alpha_5\alpha_4)[0]) \\ \Omega^{-5}((\alpha_5\alpha_4)[0]) & \end{array}$$

$$\begin{array}{ll} \alpha_8[0] & * \\ \Omega^{-1}(\alpha_8[0]) & \Omega^{-5}(\alpha_8[0])\Omega^{-4}(\alpha_8[0])\Omega^{-3}(\alpha_8[0])\Omega^{-2}(\alpha_8[0]) \\ \Omega^{-6}(\alpha_8[0]) & \end{array}$$

and

$$\begin{array}{ll} (\alpha_9\alpha_3\alpha_2\alpha_1)[0] & \Omega^{-1}((\alpha_9\alpha_3\alpha_2\alpha_1)[0]) \\ \Omega^{-2}((\alpha_9\alpha_3\alpha_2\alpha_1)[0]) & \Omega^{-4}((\alpha_9\alpha_3\alpha_2\alpha_1)[0])\Omega^{-3}((\alpha_9\alpha_3\alpha_2\alpha_1)[0]) \\ \Omega^{-5}((\alpha_9\alpha_3\alpha_2\alpha_1)[0]) & . \end{array}$$

6.1. Main theorem

We prove now Theorem A stated in the introduction.

Consider $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ and $B = \mathrm{k}Q'/\langle\mathcal{P}'\rangle$ two derived equivalent gentle algebras, and ϕ_A and ϕ_B the functions defined by the pairs of natural numbers obtained in the algorithm defined in Section 3.

As A and B are derived equivalent then $\hat{A}$ and $\hat{B}$ are also, see [1, Thm. 1.5] and [2], and because these are self-injective, by [14] we know that $\hat{A}$-$\underline{\mathrm{mod}}$ and $\hat{A}$-$\underline{\mathrm{mod}}$ are equivalent as triangulated categories. Then, a series of components $\mathbb{Z}A_\infty$ which has (n, m) as associated pair corresponds by the equivalence to a series of components of the same type and equal number of elements, $|n-m|$, and in such a way that $\Omega^{n-m}(M) = \tau^n(M)$ for all M modules whose isomorphism class lies in that series. A series of components of tubes of rank n, with $n \gneq 1$, which has (n, n)

as associated pair, corresponds by the equivalence to a series of components of tubes of the same rank, with which we associate also the pair (n, n).

Denote by ϕ'_A the restriction of ϕ_A to $\mathbb{N}^2 \setminus \{(1, 1)\}$; as $\phi'_A = N'_A$ by Theorem 16, ϕ'_A describes the action of Ω^{-1} over the components $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ of $\hat{A}$ with $n \in \mathbb{N} \setminus \{1\}$ and we conclude then that $\phi'_A = \phi'_B$.

Using Remark 8 we know that

$$\sum_{(n,m)\in\mathbb{N}^2} \phi_A(n, m) p_2(n, m) = \#Q_1,$$

so $\phi_A(1, 1) = \#Q_1 - \sum_{(n,m)\in\mathbb{N}^2\setminus\{(1,1)\}} \phi'_A(n, m) p_2(n, m)$ and similarly for B. By the last section the number of arrows is a derived equivalent invariant so $\#Q_1 = \#Q'_1$; also $\phi'_A(n, m) = \phi'_B(n, m)$, and then $\phi_A(1, 1) = \phi_B(1, 1)$. Therefore $\phi_A = \phi_B$ and the proof concludes.

7. Discussion of the results

The first aim of this section is to prove Theorem C, which states that our invariants separate the derived equivalence classes of gentle algebras with at most one cycle.

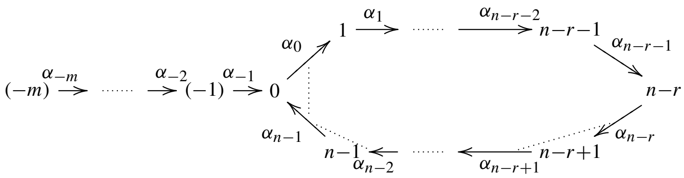
Recall that the derived classification of gentle algebras with at most one cycle is known; see [3,4,18] and [6]. More precisely, we have the following three cases:

(1) Q is a tree if and only if $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ is derived equivalent to $\mathbb{A}_n$ with $n = \#Q_0$, and so $\phi_A = \phi_{\mathbb{A}_n} = [(n + 1, n - 1)]$.

(2) Q has just one cycle and satisfies the clock condition (that is the number of clockwise relations equals the number of anticlockwise relations in the only cycle of A) if and only if $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ is derived equivalent to a hereditary algebra of type $\tilde{\mathbb{A}}_{p,q}$ for some $p, q \in \mathbb{N}$ with $p + q = n$. In this case $\phi_A = \phi_{\tilde{\mathbb{A}}_{p,q}} = [(p, p), (q, q)]$.

(3) Q has just one cycle and $r := |c - a| \neq 0$ where c and a are the number of clockwise and anticlockwise relations respectively in the only cycle of A if and only if $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ is derived equivalent to an algebra of type $\Lambda(r, n, m) = \mathrm{k}Q(r, n, m)/\langle\mathcal{P}(r, n, m)\rangle$ for some $r, n, m \in \mathbb{N}$ with $n \geq r \geq 1$ and $m \geq 0$; graphically:

$\Lambda(r, n, m)$:

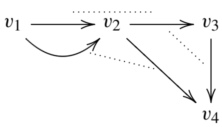


with $\mathcal{P}(r, n, m) = \{\alpha_0\alpha_{n-1}, \alpha_{n-1}\alpha_{n-2}, \ldots, \alpha_{n-r+1}\alpha_{n-r}\}$; see [6]. In this case, $\phi_A = \phi_{\Lambda(r,n,m)} = [(r + m, m), (n - r, n)]$.

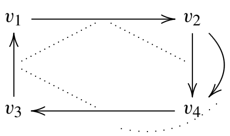
Then if $A = \mathrm{k}Q/\langle\mathcal{P}\rangle$ and $B = \mathrm{k}Q'/\langle\mathcal{P}'\rangle$ are gentle algebras with $c(Q), c(Q') \leq 1$ such that $\phi_A = \phi_B$, they are derived equivalent. This concludes the proof.

Remark 18. By Theorem C our invariants can distinguish two gentle algebras which are not derived equivalent in all known cases. However, consider the following two gentle algebras:

A :



B :



We find $\phi_A = \phi_B = [(3, 5)]$. On the other hand we checked with a computer that it is not possible to transform A into B by a sequence of elementary derived equivalences from [11]. It should be interesting to find another way to decide whether these two algebras are derived equivalent or not.

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