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A simplified approach to the brachistochrone problem

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Abstract
Ever since Johann Bernoulli put forward the challenge “Problema novum ad cujus solutionem Mathematice invitantur” in *Acta Eruditorum Lipsiae* of June, 1696, of finding the minimum time trajectory (the brachistochrone) described by an object moving from one point to another (not directly behind the first one) in a constant uniform gravitational field, many works have been published on this subject, and some books mention it as part of the applications of the Euler–Lagrange formalism. However, we have found only one reference of the problem related to the general inhomogeneous inverse square gravitational field (Supplee and Schmidt 1991 *Am. J. Phys.* **59** 467). Even in this reference, the problem is treated for particular initial conditions. In this work, we develop a simplified method to arrive to the equation of the brachistochrone curve for an arbitrary potential and also to the inverse formulation of the problem: what type of potential energy function is associated with a specified brachistochrone curve?

Introduction

When Johann Bernoulli put forward the challenge, he was thinking of a problem so difficult that he wrote the following:

> *Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise [2].*

It is also known that Johann Bernoulli and Gottfried Leibniz deliberately tempted Isaac Newton with this problem. Among the five correct answers Johann received, there was an anonymous one in which, according to him, ‘one could see the Lion’s paw’, referring to

Newton. The other four answers were from Johann himself, his brother Jacob, Gottfried Leibniz and Guillaume L'Hopital.

Looking at the names of the mathematicians that answered the challenge, it is clear that it was not, and still is not, a trivial problem. With the knowledge accumulated in the many years that have passed since then, at present any undergraduate student can solve this problem. In fact, some books [3–6] state the problem to illustrate the application of the Euler–Lagrange formalism derived from variational calculus, and many papers have been written concerning it [7–13].

In this paper, we develop a simplified way to approach the problem and an extension to the inverse problem is made. For the sake of completeness, in section 1 we present the standard solution of the problem, ending with a not so well-known form of Euler–Lagrange equations. In section 2 we use these equations to restate the brachistochrone problem in a general way, in rectangular and polar coordinates. Section 3 is devoted to two examples: the well-known solution of the uniform gravitational field and the brachistochrone for the inverse square inhomogeneous gravitational field. Finally, in section 4 we present the solution to the inverse problem together with some examples, including 'Galileo's Brachistochrone' [14].

1. The standard solution: variational calculus

The fundamental problem in variational calculus is to find a trajectory $y = y(x)$, between x_1 and x_2, that makes an extremal the curvilinear integral of a function $f(y(x), y'(x), x)$), where $y' = \mathrm{d}y/\mathrm{d}x$. That is,

$$J = \int_{x_1}^{x_2} f(y(x), y'(x), x)\,\mathrm{d}x \tag{1}$$

must be maximum or minimum. The procedure is to take the variation of J equal to zero; that is,

$$\delta J = \delta \int_{x_1}^{x_2} f(y(x), y'(x), x)\,\mathrm{d}x = 0. \tag{2}$$

It can easily be found that the condition for which J is an extremal is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{3}$$

This result can be generalized if f is a function of n independent variables y_i, its derivates y_i' (naturally these variables and their derivates depend on x). In this case the following n conditions[1] must be satisfied:

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} = 0 \qquad \text{for} \quad i = 1, 2, \ldots, n. \tag{4}$$

The standard solution of the brachistochrone problem is provided by variational calculus, and the problem is to find the minimum time required for a particle to travel from $\mathrm{A}(x_1, y_1)$ to $\mathrm{B}(x_2, y_2)$,

$$t = \int_A^B \frac{\mathrm{d}s}{v}, \tag{5}$$

falling under the action of the gravitational uniform field, where $\mathrm{d}s$ is the arc element along the trajectory followed by the body and v is its speed.

[1] In particular, if $y_i = q_i$ is the generalized coordinate $f = T(q_i') - U(q_i) = L(q_i, q_i', x)$ is the Lagrangian and x is time, the equations obtained are the Lagrange equations.

The procedure is to express the arc in Cartesian coordinates,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{6}$$

and the speed v (using the energy conservation) as

$$v = \sqrt{2g(y_0 - y)} \tag{7}$$

where y_0 is the initial position of the particle. Then

$$t_{AB} = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{2g(y_0 - y)}}\, dx \tag{8}$$

and

$$f = \sqrt{\frac{1 + y'^2}{2g(y_0 - y)}}. \tag{9}$$

The following step is to calculate the time variation δt and equate it to zero to obtain the equation that f must satisfy (equation (3)).

Substituting f (equation (9)) in equation (3) one arrives at the following differential equation:

$$2\,(y_0 - y)\, y'' - 1 - y'^2 = 0. \tag{10}$$

The integration of this equation yields the well-known solution of the brachistochrone problem—the parametric equations of a cycloid.

In this paper, we propose another method, which is a short cut to the standard solution. Because $f = f\,[y(x), y'(x), x]$, we can write

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'', \tag{11}$$

from which we obtain, with the help of equation (3),

$$\frac{\partial f}{\partial x} = \frac{df}{dx} - y'\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right). \tag{12}$$

In particular, if the function $f(y, y', x)$ does not depend explicitly on x, the above equation is equal to zero and can be written as

$$f - y'\frac{\partial f}{\partial y'} = C = \text{constant}. \tag{13}$$

This is the key equation for the reformulation of the brachistochrone problem presented in this work.

2. A different way to solve the brachistochrone problem

In Cartesian coordinates, the arc element is expressed as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx. \tag{14}$$

If the potential energy is a function of only one variable $U = U(y)$ and the initial velocity is zero, the speed of the particle in an arbitrary position along its trajectory is

$$v = \sqrt{\frac{2}{m}\left[U\,(y_0) - U\,(y)\right]} = \sqrt{\frac{-2\Delta U}{m}}. \tag{15}$$

Then f is, in this case,

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{-\frac{2\Delta U}{m}}} \tag{16}$$

from which one obtains, after substitution in equation (13),

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{-\left(\frac{A+\Delta U}{\Delta U}\right)} \tag{17}$$

or

$$x - x_0 = \int_{y_0}^{y} \sqrt{-\left(\frac{\Delta U}{A+\Delta U}\right)}\,\mathrm{d}y, \tag{18}$$

where A is constant. The brachistochrone problem is formally resolved in general for Cartesian coordinates.

Now we analyse the case of central potentials (central forces). In this case, the torque is zero so the angular momentum $\mathbf{L}$ remains constant and the movement is in a plane, so the adequate reference system to use is polar coordinates. In this case, the element of arc is

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2} = \sqrt{r^2 + r'^2}\,\mathrm{d}\varphi \qquad \text{where} \quad \mathrm{r}' = \frac{\mathrm{d}r}{\mathrm{d}\varphi}. \tag{19}$$

Repeating the above steps one arrives at

$$\int_{\phi_0}^{\phi} \mathrm{d}\phi = \phi - \phi_0 = \int_{r_0}^{r} \sqrt{\frac{-2A^2\Delta U}{mr^4 + 2A^2r^2\Delta U}}\,\mathrm{d}r. \tag{20}$$

Once again, the problem is formally resolved in general, in this case for any type of central potential.

3. Examples

3.1. The (constant) uniform gravitational field

The solution of the brachistochrone for the uniform gravitational field is straightforward. One only has to substitute $\Delta U = -mg(y - y_0)$ in equation (18),

$$x = \int_{y_0}^{y} \sqrt{\left(\frac{mg(y_0 - y)}{A - mg(y_0 - y)}\right)}\,\mathrm{d}y, \tag{21}$$

from which one readily obtains the parametric equations of the cycloid, and the problem is easily solved in one line.

3.2. The inhomogeneous inverse square gravitational field

As an example we resolve the brachistochrone in the inhomogeneous gravitational field of the earth. The expression for ΔU is

$$\Delta U\,(r) = -GMm\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{-GMm\,(r_0 - r)}{r_0 r}. \tag{22}$$

Substitution in equation (21) yields

$$\phi - \phi_0 = \int_{r_0}^{r} \sqrt{\frac{r_0 - r}{Br^5 - r^2\,(r_0 - r)}}\,\mathrm{d}r. \tag{23}$$

The solution to this equation involves elliptic integrals, so we only show the graphical solution (figure 1) calculated with *Mathematica*. This solution was calculated with r_0 equal to five times the earth's radius ($r_0 = 5R_{\mathrm{Earth}}$).

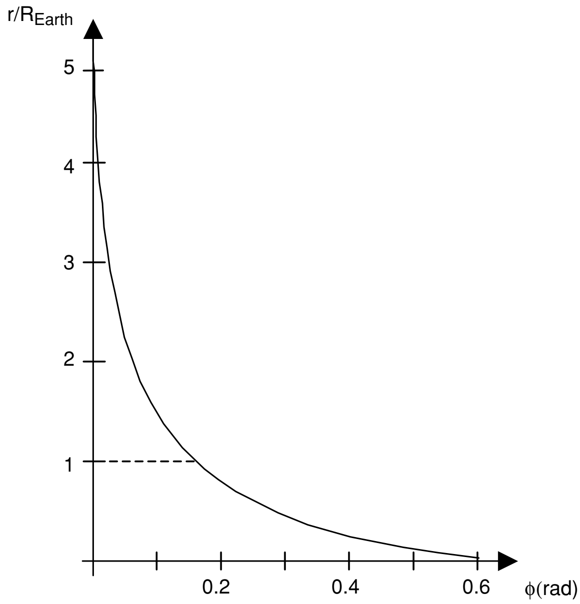

Figure 1. Brachistochrone for the gravitational inverse square law. The dashed line indicates the smallest value of r.

4. The inverse problem

Squaring equation (17) and solving for ΔU one gets

$$\Delta U = U - U_0 = -\frac{A}{1 + y'^2}, \tag{24}$$

which allows one to select an arbitrary curve trajectory and look for the potential energy function that 'makes' this trajectory a brachistochrone. In what follows, we present several examples.

(i) What potential energy function makes a cycloid a brachistochrone?

The parametric equations of a cycloid are

$$x = D(\alpha - \sin\alpha) \qquad y = D(1 - \cos\alpha), \tag{25}$$

from where

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{D\sin\alpha}{D\,(1 - \cos\alpha)} \tag{26}$$

so that

$$\Delta U = U - U_0 = -\frac{Ay}{2D} = -By, \tag{27}$$

which implies a constant force.

(ii) What potential energy function makes a parabola a brachistochrone?

Without loss of generality, we consider a parabola with its axis along the y axis, its vertex at $(0, k)$:

$$y - k = ax^2. \tag{28}$$

Repeating the above calculation one gets

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4a^2x^2 = 4a\,(y-k) \tag{29}$$

and consequently

$$\Delta U = -\frac{A}{1+4a\,(y-k)} = -\frac{B}{C+y}, \tag{30}$$

where B and C are constants. This potential energy is associated with the following force:

$$F = -\frac{B}{(C+y)^2}. \tag{31}$$

(iii) The 'Galilean brachistochrone' (circle).

We consider a circle with its centre at $(0, R)$; that is,

$$x^2 + (y-R)^2 = R^2, \tag{32}$$

from where

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \left(\frac{x}{y-R}\right)^2 = \frac{R^2}{(y-R)^2}; \tag{33}$$

so then

$$\Delta U = -\frac{A\,(y-R)^2}{R^2} = -B\,(y-R)^2 \tag{34}$$

and the force law would be

$$F = 2B\,(y-R). \tag{35}$$

Conclusions

We have developed a simplified method to quickly arrive at the differential equations that define the brachistochrone associated with an arbitrary potential energy function. We also developed the formalism of the inverse problem; that is, to find the potential energy function associated with an arbitrary brachistochrone function. To our knowledge, this is the first time that such an approach has been given.

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