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The number of $\overrightarrow{C_3}$-free vertices on 3-partite tournaments ☆

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ABSTRACT

Let T be a 3-partite tournament. We say that a vertex v is $\overrightarrow{C_3}$-free if v does not lie on any directed triangle of T. Let $F_3(T)$ be the set of the $\overrightarrow{C_3}$-free vertices in a 3-partite tournament and $f_3(T)$ its cardinality. In this paper we prove that if T is a regular 3-partite tournament, then $F_3(T)$ must be contained in one of the partite sets of T. It is also shown that for every regular 3-partite tournament, $f_3(T)$ does not exceed $\frac{n}{9}$, where n is the order of T. On the other hand, we give an infinite family of strongly connected tournaments having $n - 4\overrightarrow{C_3}$-free vertices. Finally we prove that for every $c \geq 3$ there exists an infinite family of strongly connected c-partite tournaments, $D_c(T)$, with $n - c - 1$ $\overrightarrow{C_3}$-free vertices.

1. Introduction

Let c be a non-negative integer, a *c-partite or multipartite tournament* is a digraph obtained from a complete c-partite graph by substituting each edge with exactly one arc. A *tournament* is a c-partite tournament with exactly c vertices. The partite sets of T are the maximal independent sets of T. Let D be an oriented graph. The vertex set and the arc set of an oriented graph D are denoted by $V(D)$ and $A(D)$, respectively. The *out-neighborhood* (*in-neighborhood*, resp.) $N^+(x)$ ($N^-(x)$, resp.) of a vertex x is the set $\{y \in V(D) \mid xy \in A(D)\}$ ($\{y \in V(D) \mid yx \in A(D)\}$, resp.). The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the *out-degree* and the *in-degree* of x, respectively. The *global irregularity* of an oriented graph D is defined as

$$i_g(D) = max_{x,y \in V(D)}\{max\{d^+(x), d^-(x)\} - min\{d^+(y), d^-(y)\}\}.$$

An oriented graph D is *regular* (*almost regular*, resp.) if $i_g(D) = 0$ ($i_g(D) \leq 1$, resp.). Let $X, Y \subseteq V(D)$, X *dominates* Y, denoted by $X \Longrightarrow Y$, if $Y \subseteq N^+(x)$ for every $x \in X$. An oriented graph D is *strongly connected* if for every ordered pair of vertices (x, y) there is a directed path from x to y. An *m-cycle* of an oriented graph is a directed cycle of length m. Let T be a c-partite tournament. We say that a vertex v is $\overrightarrow{C_3}$*-free* if v does not lie on any directed triangle of T. Let $F_3(T)$ be the set of the $\overrightarrow{C_3}$-free vertices in a c-partite tournament and $f_3(T)$ its cardinality. For the standard terminology on digraphs see [1].

The structure of cycles in multipartite tournaments has been extensively studied, see for example [3,4,6,7] and [11]. A very recent survey on this topic [10] appeared with several interesting open problems. For instance, the study of cycles whose length does not exceed the number of partite sets leads to various extensions and generalizations of classic results on tournaments. Bondy [2] proved that each strongly connected c-partite tournament contains an m-cycle for each $m \in$

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$\{3, \ldots, c\}$. In 1994, Guo and Volkmann [5] proved that every partite set of a strongly connected c-partite tournament T has at least one vertex that lies on a cycle of length m for each $m \in \{3, \ldots, c\}$. This result also generalizes a theorem of Gutin [8]. There are examples showing that not every vertex of a strongly connected c-partite tournament is contained in a cycle of length m for each $m \in \{3, \ldots, c\}$ in general [10]. However, Zhou et al. [12] proved that every vertex of a regular c-partite tournament with at least four partite sets ($c \geq 4$) is contained in a cycle of length m for each $m \in \{3, \ldots, c\}$. Volkmann [9] provided the following infinite family of $4p$-regular 3-partite tournaments which shows that the previous theorem is not valid for regular 3-partite tournaments

Example 1 (*Volkmann*)**.** Let $\mathcal{F}$ be the family of tournaments with the partite sets $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ and $W = W_1 \cup W_2$ such that:

1. For every natural number p, the sets have sizes $|W_1| = p$, $|W_2| = 3p$ and $|U_1| = |U_2| = |V_1| = |V_2| = 2p$.
2. The sets $U_1 \cup V_1$ and $U_2 \cup V_2$ generate p-regular bipartite tournaments T_1 and T_2, respectively.
3. $V_1 \Longrightarrow U_2$ and $U_1 \Longrightarrow V_2$.
4. $V(T_1) \Longrightarrow W_1 \Longrightarrow V(T_2)$ and $V(T_2) \Longrightarrow W_2 \Longrightarrow V(T_1)$.

This result leads to the natural question: which is the maximum number of $\overrightarrow{C_3}$-free vertices on regular 3-partite tournaments? In the case of Example 1 we have an infinite family of regular 3-partite tournaments such that $f_3(T) = \frac{|V(T)|}{12}$. In this paper we prove that if T is a regular 3-partite tournament, there is at most one partite set of T containing the vertices in $F_3(T)$. We also show that for every regular 3-partite tournament, $f_3(T)$ does not exceed $\frac{V(T)}{9}$. On the other hand, we give an infinite family of strongly connected tournaments having $n-4$ $\overrightarrow{C_3}$-free vertices. Finally we prove that for every $c \geq 3$ there exists an infinite family of strongly connected c-partite tournaments, $D_c(T)$, with $n-c-1$ $\overrightarrow{C_3}$-free vertices. This examples show that regularity is an important constraint to have lots of vertices that are contained in a directed triangle.

We conclude this section with the following

Remark 1. Let P_1, P_2, P_3 be the partite sets of a regular 3-partite tournament T. Then, $r = |P_1| = |P_2| = |P_3|$ and $d^+(x) = d^-(x) = r$ for all $x \in V(T)$.

2. Tripartite regular tournaments

Let T be a 3-partite regular tournament and P be a partite set of T, $u, v \in V(T)$ and $S \subseteq V(T)$. For the rest of this article we use the following notation:

$P^{+-}(u, v) := N^+(u) \cap N^-(v) \cap P$,
$P^{-+}(u, v) := N^-(u) \cap N^+(v) \cap P$,
$P^+(S) := (\bigcap_{s \in S} N^+(s)) \cap P$,
$P^-(S) := (\bigcap_{s \in S} N^-(s)) \cap P$ and
$P^*(S) := P \setminus (P^+(S) \cup P^-(S))$.

If $S = \{v\}$ for some $v \in V(T)$, then we briefly write $P^+(v)$ and $P^-(v)$ to denote $P^+(\{v\})$ and $P^-(\{v\})$, respectively.

Theorem 2. *For every regular* 3*-partite tournament, there is a partite set containing* $F_3(T)$*.*

Proof. Let U, V, W be the partite sets of the 3-partite regular tournament T. By Remark 1, there exists a positive integer r such that $r = |U| = |V| = |W|$ and $d^+(x) = d^-(x) = r$ for every $x \in V(T)$. Suppose that there exists $S = \{u, v\} \subseteq F_3(T)$ such that $u \in U$, $v \in V$ and $uv \in A(T)$.

If $x \in W^{-+}(u, v)$, u lies on a directed triangle (u, v, x), which is impossible since $S \subseteq F_3(T)$. Thus, $W^{-+}(u, v) = \emptyset$ and

$$W = W^+(S) \cup W^-(S) \cup W^{+-}(u, v). \tag{1}$$

By definition, $W^+(S)$, $W^-(S)$ and $W^{+-}(u, v)$ are pairwise disjoint sets.

If $W^+(S) = \emptyset$, from Eq. (1) we obtain that $r = d^-(v) \geq |W| + |\{u\}| = r + 1 > r$, a contradiction. Thus,

$$W^+(S) \neq \emptyset. \tag{2}$$

Analogously,

$$W^-(S) \neq \emptyset, \tag{3}$$

since $d^-(u) = r < |W| + |\{v\}|$.

The sets $U^-(v)$ and $U^+(v)$ form a partition of U. Notice that $u \in U^-(v)$. By Remark 1 and relation (3), $|W^+(S)| < r$, thus $U^+(v) \neq \emptyset$ because $r = d^+(v) = |W^+(S)| + |U^+(v)|$. By a similar argument, Remark 1 and relation (2), $V^+(u)$ and $V^-(u)$ are non-empty sets.

Let $x \in U^-(v)$ and $w \in W^+(S)$. Since $v \in F_3(T)$, we have that the directed path (x, v, w) is not contained in a directed triangle of T. This implies that $xw \in A(T)$, thus

$$U^-(v) \Longrightarrow W^+(S). \tag{4}$$

Even more, for every $y \in V^-(u)$ and $w \in W^+(S)$, the directed path (y, u, w) is not a directed triangle of T, since $u \in F_3(T)$, thus

$$V^-(u) \Longrightarrow W^+(S). \tag{5}$$

From relations (4) and (5), for every $w \in W^+(S)$ we have that

$$U^-(v) \cup V^-(u) \cup \{v\} \subseteq N^-(w)$$

and

$$|U^-(v)| + |V^-(u)| + 1 \leq r. \tag{6}$$

On the other hand, for every $x \in U^+(v)$ and $w' \in W^-(S)$, we have that the directed path (w', v, x) is not in a directed triangle of T, then

$$W^-(S) \Longrightarrow U^+(v). \tag{7}$$

Moreover, for every $y \in V^+(u)$ and $w' \in W^-(S)$, we have that the directed path (w', u, y) is not in a directed triangle of T, thus

$$W^-(S) \Longrightarrow V^+(u). \tag{8}$$

Thus, from relations (7) and (8), for every $w \in W^-(S)$ we have that

$$U^+(v) \cup V^+(s) \cup \{u\} \subseteq N^-(w')$$

and

$$|U^+(v)| + |V^+(u)| + 1 \leq r. \tag{9}$$

Since T is regular $|U^+(v)| + |U^-(v)| = r$ and $|V^+(u)| + |V^-(u)| = r$. If we sum Eqs. (6) and (9) we get a contradiction. Thus, u and v must be elements of the same partition set of T. □

Remark 3. Let C be a 4-cycle that contains vertices from all the partite sets of a regular 3-partite tournament. Then, the number of $\overrightarrow{C_3}$-free vertices belonging to C is at most one.

Proof. Since C is not an induced cycle, we have that at least three of its vertices lies in a directed triangle, thus at most one of them can be $\overrightarrow{C_3}$-free. □

Theorem 4. *Let T be a 3-partite regular tournament with partite sets V_0, V_1 and V_2. If $u, v \in F_3(T) \cap V_0$, then there exists $i \in \{1, 2\}$ such that $V_i^{+-}(u, v) = V_i^{-+}(u, v) = \emptyset$.*

Proof. Let V_0, V_1 and V_2 be the partite sets of a r-regular 3-partite tournament and $S = \{u, v\} \subseteq F_3(T) \cap V_0$. It is enough to prove that if $V_1^{-+}(u, v) \neq \emptyset$, then $V_2^{+-}(u, v) = V_2^{-+}(u, v) = \emptyset$. Let $x_1 \in V_1^{-+}(u, v)$. If there exists $x_2 \in V_2^{+-}(u, v)$, then (u, x_2, v, x_1) is a directed 4-cycle of T containing two vertices in $F_3(T)$, which by Remark 3 is impossible. Thus,

$$V_2^{+-}(u, v) = \emptyset. \tag{10}$$

If $y_2 \in V_2^{-+}(u, v)$ and $y_1 \in V_1^{+-}(u, v)$, then (v, y_2, u, y_1) is a directed 4-cycle of T containing two vertices of $F_3(T)$. Then,

$$V_1^{+-}(u, v) = \emptyset. \tag{11}$$

By Remark 1,

$$r = |N^+(u)| = |V_1^+(S)| + |V_1^{+-}(u, v)| + |V_2^+(S)| + |V_2^{+-}(u, v)|, \tag{12}$$

$$r = |N^-(v)| = |V_1^-(S)| + |V_1^{+-}(u, v)| + |V_2^-(S)| + |V_2^{+-}(u, v)|, \tag{13}$$

and by relations (10) and (11),

$$2r = \sum_{i=1,2} |V_i^+(S)| + \sum_{i=1,2} |V_i^-(S|)$$

which is a contradiction because

$$r = |V_i^+(S)| + |V_i^{-+}(u, v)| + |V_i^-(S)| \text{ for } i = \{1, 2\}$$

and $V_i^{-+}(u, v) \neq \emptyset$ for $i = \{1, 2\}$. Therefore, $V_2^{-+}(u, v)$ is empty. Thus, we have proved that $V_2^{+-}(u, v) = V_2^{-+}(u, v) = \emptyset$. $\square$

Colorally 1. *Let T be a 3-partite regular tournament such that $F_3(T) \neq \emptyset$. Then, there exists at least one partite set P such that $P = P^+(F_3(T)) \cup P^-(F_3(T))$.*

Proof. Let P_1 and P_2 be partite sets of T not containing $F_3(T)$. Suppose that $P_1 \neq P_1^+(F_3(T)) \cup P_1^-(F_3(T))$ and $P_2 \neq P_2^+(F_3(T)) \cup P_2^-(F_3(T))$.

Then, there exist $u, v \in F_3(T)$ such that $P_1^{+-}(u, v) \neq \emptyset$. By Theorem 4,

$$P_2 = P_2^+(\{u, v\}) \cup P_2^-(\{u, v\}). \tag{14}$$

Since $P_2 \neq P_2^+(F_3(T)) \cup P_2^-(F_3(T))$, there exists $w \in F_3(T)$ such that $P_2 \neq P_2^+(\{u, w\}) \cup P_2^-(\{u, w\})$. Then, by relation (14), $P_2 \neq P_2^+(\{v, w\}) \cup P_2^-(\{v, w\})$. Applying Theorem 4, we have that $P_1 = P_1^+(\{u, w\}) \cup P_1^-(\{u, w\})$ and $P_1 = P_1^+(\{v, w\}) \cup P_1^-(\{v, w\})$. Thus,

$$P_1^{+-}(u, v) \subseteq P_1^+(\{u, w\}) \cap P_1^-(\{v, w\}) \subseteq N^+(w) \cap N^-(w) = \emptyset,$$

which is a contradiction because $P_1^{+-}(u, v) \neq \emptyset$. $\square$

Theorem 5. *Let T be a r-regular 3-partite tournament. Then, $f_3(T) < \frac{r}{3}$.*

Proof. Let T be a 3-partite tournament, by Theorem 2 there exist partite sets P_1 and P_2 of T such that $P_i \cap F_3(T) = \emptyset$ for $i = 1, 2$. Denote by P_i^+ and P_i^- the sets $P_i^+(F_3(T))$ and $P_i^-(F_3(T))$, respectively for $i \in \{1, 2\}$. By Corollary 1, we can assume that $P_1 = P_1^+ \cup P_1^-$. By definition $P_2^* = P_2 \setminus (P_2^+ \cup P_2^-)$, then $P_2 = P_2^+ \cup P_2^- \cup P_2^*$. We may assume $f_3(T) > 0$ for the rest of the proof.

Let $v \in P_1^-$ and $w \in P_2^+ \cup P_2^*$. There exists $x \in F_3(T)$ such that $vx \in A(T)$ and $xw \in A(T)$. Since the directed path (v, x, w) is not in a directed triangle of T, then $vw \in A(T)$. Thus,

$$P_1^- \Longrightarrow P_2^+ \tag{15}$$

and

$$P_1^- \Longrightarrow P_2^*. \tag{16}$$

Let $v \in P_1^+$ and $w \in P_2^- \cup P_2^*$. There exists $x \in F_3(T)$ such that $xv \in A(T)$ and $wx \in A(T)$. Since the directed path (w, x, v) is not in a directed triangle of T, then $wv \in A(T)$. Thus,

$$P_2^- \Longrightarrow P_1^+ \tag{17}$$

and

$$P_2^* \Longrightarrow P_1^+. \tag{18}$$

From relations (15) and (16), we have that P_1^+ and P_2^- are not empty sets, otherwise, $d^-(v) > r$ for every vertex $v \in P_2$ which is impossible. Analogously, from relations (17) and (18), we have that P_1^- and P_2^+ are not empty sets, otherwise, $d^-(v) > r$ for every vertex $v \in P_1$. Let

$$\begin{aligned} s_1 &= |\{uv \in A(T) \mid u \in P_1^+, v \in P_2^+\}|, \\ s_2 &= |\{uv \in A(T) \mid u \in P_2^+, v \in P_1^+\}|, \\ t_1 &= |\{uv \in A(T) \mid u \in P_1^-, v \in P_2^-\}| \text{ and} \\ t_2 &= |\{uv \in A(T) \mid u \in P_2^-, v \in P_1^-\}|. \end{aligned}$$

If $s_1 = 0$, then $d^-(x) > |P_2^+| + |P_2^*| + |P_2^-| + f_3(T) > r$, which is impossible. Thus $s_1 \neq 0$. Analogously, $s_2, t_1, t_2 \neq 0$.

Notice that $s_1 + s_2 = |P_1^+||P_2^+|$ and $t_1 + t_2 = |P_1^-||P_2^-|$. We consider the following cases.

Case 1. $|P_1^+| \leq \frac{r}{2}$

Subcase 1.1. $s_1 \geq \frac{|P_1^+||P_2^+|}{2}$

From the regularity of T, the relations (15), (16) and the definitions of P_2^+ and s_1, we have that

$$r|P_2^+| = |\{uv \in A(T) \mid v \in P_2^+\}| \geq s_1 + f_3(T)|P_2^+| + |P_1^-||P_2^+|.$$

By the hypothesis,

$$r|P_2^+| \geq \frac{|P_1^+||P_2^+|}{2} + f_3(T)|P_2^+| + |P_1^-||P_2^+|.$$

Since $|P_2^+| > 0$, the last equation is equivalent to

$$r \geq \frac{|P_1^+|}{2} + f_3(T) + |P_1^-| = |P_1^+| + f_3(T) + |P_1^-| - \frac{|P_1^+|}{2}.$$

Since T is regular, $|P_1^+| + |P_1^-| = r$. Thus,

$$r \geq f_3(T) + r - \frac{|P_1^+|}{2},$$

which implies that

$$\frac{|P_1^+|}{2} \geq f_3(T).$$

Finally by the hypothesis of Case 1,

$$f_3(T) \leq \frac{r}{4}.$$

Subcase 1.2. $s_2 \geq \frac{|P_1^+||P_2^+|}{2}$

From relations (17) and (18) and the definitions of P_1^+ and s_2, it follows that

$$r|P_1^+| = |\{uv \in A(T) \mid v \in P_1^+\}| \geq s_2 + f_3(T)|P_1^+| + |P_2^-||P_1^+| + |P_2^*||P_1^+|.$$

If we proceed as we did in the previous subcase, by the hypothesis $s_2 \geq \frac{|P_1^+||P_2^+|}{2}$ and the fact that $|P_2^+| + |P_2^*| + |P_2^-| = r$, we obtain that

$$f_3(T) \leq \frac{|P_2^+|}{2}.$$

If $f_3(T) \geq \frac{r}{3}$, from the last relation we obtain that $|P_2^+| \geq \frac{2}{3}r$.

On the other hand, from relations (15) and (16) and the definitions of t_1 and P_1^-, it follows that

$$r|P_1^-| = |\{uv \in A(T) \mid u \in P_1^-\}| \geq t_1 + f_3(T)|P_1^-| + |P_2^+||P_1^-| + |P_2^*||P_1^-|.$$

Since $|P_1^-| > 0$, $|P_2^*| \geq 0$, $t_1 \geq 0$, $|P_2^+| \geq \frac{2}{3}r$ and $f_3(T) \geq \frac{r}{3}$ we have that

$$r \geq |P_2^+| + |P_2^*| + f_3(T) + \frac{t_2}{|P_1^-|} > |P_2^+| + f_3(T) \geq r.$$

which is impossible, thus $f_3(T) < \frac{r}{3}$.

Case 2. $|P_1^-| \leq \frac{r}{2}$

Subcase 2.1. $t_2 \geq \frac{|P_1^-||P_2^-|}{2}$

From the regularity of T, relation (17) and the definitions of P_2^- and t_2 we have that

$$r|P_2^-| = |\{uv \in A(T) \mid u \in P_2^-\}| \geq t_2 + f_3(T)|P_2^-| + |P_1^+||P_2^-|.$$

By the hypothesis, since $|P_2^-| > 0$ and $|P_1^+| + |P_1^-| = r$, the last equation is equivalent to

$$\frac{|P_1^-|}{2} \geq f_3(T).$$

Finally, by the hypothesis of Case 2,

$$f_3(T) \leq \frac{r}{4}.$$

Subcase 2.2. $t_1 \geq \frac{|P_1^-||P_2^-|}{2}$

From relations (15) and (16) and the definitions of P_1^- and t_1, it follows that

$$r|P_1^-| = |\{uv \in A(T) \mid u \in P_1^-\}| \geq t_1 + f_3(T)|P_1^-| + |P_2^+||P_1^-| + |P_2^*||P_1^-|.$$

By the hypothesis, $t_1 \geq \frac{|P_1^-||P_2^-|}{2}$ and the fact that $|P_2^+| + |P_2^*| + |P_2^-| = r$, we have that

$$f_3(T) \leq \frac{|P_2^-|}{2}.$$

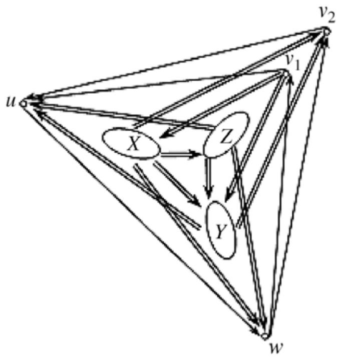

Fig. 1. $T_0(n)$.

If $f_3(T) \geq \frac{r}{3}$, from the last relation we obtain that $|P_2^-| \geq \frac{2}{3}r$.

On the other hand, from relations (17) and (18) and the definitions of s_2 and P_1^+, it follows that

$$r|P_1^+| = |\{uv \in A(T) \mid v \in P_1^+\}| \geq s_2 + f_3(T)|P_1^+| + |P_2^-||P_1^+| + |P_2^*||P_1^+|.$$

Since $|P_1^+| > 0$, $|P_2^*| \geq 0$, $s_2 \geq 0$, $|P_2^-| \geq \frac{2}{3}r$ and $f_3(T) \geq \frac{r}{3}$, then

$$r \geq |P_2^-| + |P_2^*| + f_3(T) + \frac{s_2}{|P_1^+|} > |P_2^-| + f_3(T) \geq r$$

which is impossible. Thus $f_3(T) < \frac{r}{3}$. $\square$

3. Strong c-partite tournaments with lots of $\overrightarrow{C_3}$-free vertices

First we give an infinite family of strongly connected 3-partite tournaments having $n - 4$ $\overrightarrow{C_3}$-free vertices (see Fig. 1).

Example 2. Let $T_0(n)$ be the 3-partite tournament with n vertices and partite sets $X \cup u$, $Y \cup w$ and $Z \cup v_1, v_2$ such that:

$$\begin{aligned} A(T_0(n)) = {} & \{(x, v) \mid x \in X, v \in Y \cup Z \cup \{v_2, w\}\} \cup \{(y, v) \mid y \in Y, v \in \{v_2, u\}\} \\ & \cup \{(z, v) \mid z \in Z, v \in Y \cup \{u, w\}\} \cup \{(v_1, v) \mid v \in X \cup Y \cup \{u\}\} \cup \{(v_2, u), (u, w), (w, v_1), (w, v_2)\}. \end{aligned}$$

Note that $T_0(n)$ is a strongly connected: for every $x \in X$, $y \in Y$ and $z \in Z$ there exists the directed cycle $(v_2, u, w, v_1, x, z, y)$ in $T_0(n)$. Observe that $F_3(T_0(n)) = X \cup Y \cup Z$. Thus, for every n there is a strongly connected 3-partite tournament with $n - 4$ $\overrightarrow{C_3}$-free vertices.

The regularity is also important to prove that every vertex is in a directed triangle in the case of c-partite tournaments, $c \geq 4$. In the following example we give an infinite family of c-partite tournaments on n vertices having $n - c - 1$ $\overrightarrow{C_3}$-free vertices (see Fig. 2).

Example 3. Let $T_0(n - c + 3)$ be the tripartite tournament on $n - c + 3$ vertices of Example 2 and T be a tournament with order $c - 3$. Let $D_c(T)$be the c-partite tournament with vertex set $V(T_0(n - c + 3)) \cup V(T)$ and such that

$$A(D_c(T)) = A(T_0(n - c + 3)) \cup A(T) \cup \{xu \mid x \in V(T)\} \cup \{yx \mid y \in V(T_0(n - c + e)) \setminus \{u\} \text{ and } x \in V(T)\}.$$

It is clear that $D_c(T)$ is strongly connected because $T_0(n - c + 3)$ is strongly connected and for every vertex x of T, xu and wx are arcs of $A(D_c(T))$. If $\overrightarrow{C_3}$ is a directed triangle with at least one vertex in $X \cup Y \cup Z$, then it cannot be contained in $T_0(n - c + 3)$. If $\overrightarrow{C_3}$ has only one vertex $a \in X \cup Y \cup Z$ then a is a source of $\overrightarrow{C_3}$. If $\overrightarrow{C_3}$ has only one vertex of $b \in V(T)$ then b is a sink of $\overrightarrow{C_3}$. Thus, $f_3(D_c(T)) = |X \cup Y \cup Z| = n - c - 1$. So we have proved the following

Theorem 6. *For every $c \geq 3$ there exists an infinite family of strongly connected c-partite tournaments, $D_c(T)$ with $n - c - 1$ $\overrightarrow{C_3}$-free vertices.*

Finally we conjecture that for every tournament T, the maximum number of C_3-free vertices of T is $\frac{|V(T)|}{12}$ as suggested by Example 1.

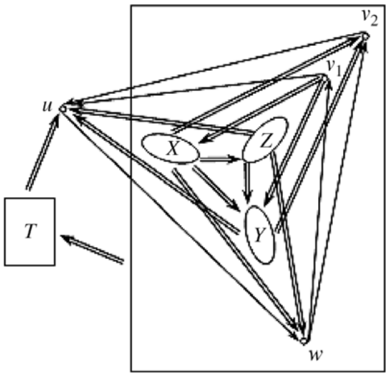

Fig. 2. $D_c(T)$.

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