Area control in generating smooth and convex grids over general plane regions

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Abstract

A new method (adaptive smoothness functional) to produce convex and smooth grids over general plane regions has been introduced in a recent work by the authors [10]; this method belongs to the variational grid generation approach. Theoretical results showing that a convex grid over a region is obtained when this method is applied, were presented; the basic assumption was that at least one convex grid exists. A procedure to control large cells (bilateral smoothness functional), in addition to smoothness and convexity, was also presented. Experimental results, showing the effectiveness of these methods, were reported; however, no theoretical results were reported assuring that the area control can be always exerted. This article continues the same line of research, introducing a new version of the bilateral smoothness functional that improves the control of large areas. Unlike the former method, theoretical results show the effectiveness of the new bilateral smoothness functional to exert such control. Optimal grids obtained with the new functional are compared with those reached using the older version, demostrating the improvement. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A grid (or mesh) over a plane region can be thought as a subdivision of the region into smaller subregions. We are interested in generating grids suitable for use in finite differences schemes, therefore the subdivision must be in quadrilaterals so the simplest finite differences schemes may be used. For a grid to be useful in this sense, it must have only convex quadrilaterals. This kind of grid is called convex or unfolded.

In the discrete variational approach, a function of the inner points is designed and it is expected that its minimum be attained in a grid with good geometrical properties; that is, the desired properties

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of a grid are to be obtained solving a large scale optimization problem. This kind of functions are called *discrete functionals*. The first studies using this approach were done by Castillo and Steinberg [4, 5]. Using a new form of discretization, Barrera et al. [3, 2, 9] obtained better results than Castillo. In particular, convexity was obtained for a wider class of regions optimizing their functionals; however, the method does not work for complicated regions.

The smoothness deserves special mention, because it produces nice grids; unfortunately if the initial grid is folded, the optimization process will never produce an optimal convex grid. Some modifications of this functional were designed to overcome this drawback: the Ivanenko functional [6] and Barrera's regularized smoothness functional [1, 3]. These functionals were designed attempting to enlarge the class of initial grids suitable for obtaining optimal convex grids. On the one hand, the idea in [1] is easily stated and its functional is easy to use; it produces convex grids for most convex initial grids (generated using other discrete functionals), failing to obtain convexity in complicated regions. On the other hand, Ivanenko's method is complete, in the sense that he is the first to prove positive results concerning the achievement of convexity; however, it requires the user to supply many parameters, both in the functional and in the optimization algorithm; therefore, familiarity with the method is required in order to obtain suitable grids. Moreover, the convergence of this algorithm is very slow.

Tinoco and Barrera [10] introduce a new family of functionals, named *adaptive smoothness functional*, whose optimization produces smooth and convex grids over general plane regions, regardless of whether the initial grid is convex or not. Theoretical results on the effectiveness of this method were presented. The authors introduced a procedure to reduce an effect that is inherent to the smoothness functionals, namely the appearance of some cells with large areas; this reduction is obtained using a new functional termed *bilateral smoothness*; numerical examples show how well this method does its job. However, no theoretical results were presented.

Although the results reached using Tinoco–Barrera's methods are quite good, there remains the fact that no theoretical foundations concerning the reduction of large cells have been given; the material presented in this work is intended to fill this gap. The bilateral smoothness Functional is modified; the resulting functional is more powerful in the reduction of large cells and a sound theoretical basis for it is provided.

The following material is organized as follows. In Section 1, notation and definitions are introduced. In Section 2 the work in [10] is reviewed. The new contribution is introduced in Sections 3 and 4; the new bilateral smoothness functional and its properties are presented in Section 3. Finally, Section 4 is devoted to the presentation and comparison of numerical (and graphical) results.

2. Notation and definitions

Let Ω be a plane region. Ω is a polygonal region if its boundary $(\partial\Omega)$ is a non-self-intersecting closed polygon. In this paper only polygonal regions will be considered, and the term *region* will mean polygonal region.

Let m and n be positive integer numbers, and Ω a region.

Definition 2.1. An $m \times n$ grid G over $\Omega, \{P_{i,j}, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n\}$ is a set of points with the following property:

$$\partial\Omega = \text{Polygonal}\{P_{1,1}, \ldots, P_{1,n}, P_{2,n}, \ldots, P_{m,n}, P_{m,n-1}, \ldots, P_{m,1}, P_{m-1,1}, \ldots, P_{1,1}\}$$

that is, $\partial\Omega$ is the grid boundary $\{P_{i,j}\}$.

Points $\{P_{1,j}, P_{m,j}, \ j = 1, 2, \ldots, n\}$ are on the *horizontal* boundary, while the *vertical* boundary is the set $\{P_{i,1}, P_{i,n}, \ i = 1, 2, \ldots, m\}$. The remaining points; i.e. $\{P_{i,j}, \ i = 2, 3, \ldots, m-1; \ j = 2, 3, \ldots, n-1\}$ are called inner points of the grid. The i, j cell of the grid, denoted by $C_{i,j}$, is the quadrilateral whose vertexes are $P_{i,j}, P_{i+1,j}, P_{i+1,j+1}, P_{i,j+1}$. A grid of $m \times n$ determines $(m-1)(n-1)$ cells. The diagonals of the i, j cell, in turn, determine four triangles $\Delta_{i,j}^{(1)}, \Delta_{i,j}^{(2)}, \Delta_{i,j}^{(3)}, \Delta_{i,j}^{(4)}$, so the total number of triangles determined by the grid cells is $N = 4(m-1)(n-1)$.

The sets $\{P_{1,j}, P_{2,j}, \ldots, P_{m,j}, \ j = 1, 2, \ldots, n\}$ will be called *horizontal coordinate lines*; while the set $\{P_{i,1}, P_{i,2}, \ldots, P_{i,n}, \ i = 1, 2, \ldots, m\}$ will be called *vertical coordinate lines*. When explicit mention of a triangle Δcab is needed, it means that the segments ca and ab are on a coordinate line and that cb is a diagonal of the corresponding cell. With this convention, the following quantities are defined for a triangle in the grid:

$$l(\Delta cab) = \|c - a\|^2 + \|b - a\|^2,$$
$$\alpha(\Delta cab) = (b - a)^t J_2 (c - a),$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

l is the sum of the squares of the lengths of the triangle sides that are on coordinate lines, and α is twice the oriented area of the triangle.

A grid is called *convex* if each of its cells is a convex quadrilateral. It is not difficult to see that the cell $C_{i,j}$ will be convex if and only if the four triangles in it have a positive area. Roughly speaking, a grid is called *smooth* if coordinate lines are smooth; that is, if the slope changes at every vertex are not *too large*.

The main goal in grid generation is to calculate a convex grid over a given plane region, given that points on the boundary are fixed. Other desirable features for a grid are smoothness and small cell area variation. In the discrete variational approach, the required properties for the grid are included in a function of the inner knots that is called *discrete functional* or simply *functional*. Among all the grids of dimension $m \times n$ with the same boundary, the aim is to determine the one whose inner points minimize the given functional. This minimization is performed using an iterative algorithm (e.g., L-BFGS or Truncated Newton), starting with an initial grid, usually constructed using an algebraic method [7]. Results reported in this work were obtained using Truncated Newton with the line search of Moré–Thuente, that has proven to be quite effective for this application [3].

Functionals to be considered here are of a rather special form, namely

$$F(G) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \sum_{p=1}^{4} f(\Delta_{i,j}^{(p)}).$$

If an order is assigned to the triangles in the grid, F can be written as

$$F(G) = \sum_{q=1}^{N} f(\Delta_q).$$

The choice of f depends on the desired properties of the optimal grid.

The following quantities will be used in our formulation. They are useful also in discussing results:

$$\alpha_q = \alpha(\Delta_q),$$

$$l_q = l(\Delta_q),$$

$$\alpha_{-t} = \min\{\alpha_q ; q = 1, 2, \ldots, N\},$$

$$\alpha_{+t} = \max\{\alpha_q ; q = 1, 2, \ldots, N\},$$

$$\alpha_{-C} = \min_{i,j}\{\text{Area}(C_{i,j}); \ C_{i,j}, \ i,j \ \text{cell of} \ G\},$$

$$\alpha_{+C} = \max_{i,j}\{\text{Area}(C_{i,j}); \ C_{i,j}, \ i,j \ \text{cell of} \ G\},$$

$$\bar{\alpha} = \frac{\text{Area}(\Omega)}{(m-1)(n-1)}.$$

Note that $\bar{\alpha}$ is the average of the cell areas of G and also the average of the values α_q. Sometimes, α_- and α_+ will be used instead of α_{-t} and α_{-t}, respectively.

3. The adaptive and bilateral smoothness functionals

In this section, a description of the work in [10] is given. The main limitation of the discrete smoothness functional is that an initial convex grid was needed in order to produce an optimal convex grid; this problem is solved constructing a new family of functionals; the elements of this family are used *to push* a smooth grid towards convexity. We state the main properties of the method.

3.1. The set of convex grids over a region

Let us remember that the optimization of a functional is a process that, starting with any grid, produces other grids in intermediate steps, moving closer to the desirable features. These intermediate grids could be thought as points in a path defined on the set of grids over a region; the final point of this trajectory should belong to the set of convex grids. To be precise, let us consider Ω a fixed plane region, $m, n \in \mathbb{N}$ and the corresponding boundary points on the boundary

$$\{P_{1,j}\}_{j=1,\ldots,m}, \qquad \{P_{m,j}\}_{j=1,\ldots,n}, \qquad \{P_{i,1}\}_{i=1,\ldots,m}, \qquad \{P_{i,n}\}_{i=1,\ldots,m},$$

are fixed. For a grid G over Ω, it has been previously defined the quantity

$$\alpha_-(G) = \min_{\Delta \in G}\{\alpha(\Delta)\},$$

twice the minimum of the areas of triangles in the grid. Let us define now, for a real number ω, the set of all grids such that the minimum area $\alpha_-(G)$ is greater than $-\omega$

$$D_\omega^- = \{G: \ G \in M y \alpha_-(G) > -\omega\},$$

where M is the set of all the $m \times n$ grids defined over Ω. It is obvious that $\omega < \omega'$, implies $D_\omega^- \subset D_{\omega'}^-$ and that for

$$\omega \leqslant -\bar{\alpha} = -\frac{\text{Area}(\Omega)}{(m-1)(n-1)},$$

$D_\omega^- = \emptyset$. It follows that $\{\omega: D_\omega^- \neq \emptyset\}$ is a set of real numbers, nonempty and bounded below. Let us denote by $\omega_- = \inf\{\omega: D_\omega^- \neq \emptyset\}$. If $\omega_- < 0$, the set of convex grids over Ω, D_0^- is not empty. It is not always possible to construct a convex grid with certain dimensions, $\{m, n, \}$, for a given region. A simple example of this fact is presented by Ivanenko [6].

With the notation just introduced, the iterative optimization of a functional starts with a grid G_0 such that $\alpha_-(G_0) > \omega_0$; is $G_0 \in D_{\omega_0}^-$. If grids produced at every step of this optimization are considered as points on a trajectory in the set of grids over Ω, the main goal of grid generation can be restated as the calculation of a trajectory

$$\Gamma: [-\omega_0, 0] \rightarrow M,$$

such that

$$\Gamma(-\omega_0) = G_0$$

and

$$\Gamma(0) = G \in D_0^-,$$

under the assumption that $D_0^- \neq \emptyset$.

The simple observation that the function defined for triangles, $f = l/\alpha$, describing the smoothness functional, can be written as

$$f = \frac{l - 2\alpha}{\alpha} + 2,$$

suggests the examination of the properties of a modification to it, to extend its domain to all D_ω^-, namely

$$f_\omega^- = \frac{l - 2\alpha}{\omega + \alpha},$$

where ω is chosen to make the denominator positive for triangles in the grid. It follows that, for an isolated triangle, the minimum is reached on right, isosceles, positively oriented triangles. This function produces a new discrete functional, having as an ideal optimum a grid with all its cells being squares:

$$F_\omega^-(G) = \sum_{q=1}^{N} \frac{l_q - 2\alpha_q}{\omega + \alpha_q}.$$

3.2. Basic properties of the functional

For a given ω, the just introduced functional has the following properties. The proof can be consulted in [10]:

Lemma 3.1 (Ivanenko [6]). *Let $\{G_q\}$ be a sequence of grids in $D_\omega^-, \omega > 0$, such that $\{G_q\} \to \partial(D_\omega^-)$; i.e. $\{\alpha_-(G_q)\} \to -\omega$. Then $F_\omega^-(G_q) \to \infty$. It follows that, if D_ω is not empty, then F_ω^- has at least a minimum in D_ω^-.*

Lemma 3.2. *Let G_1 in D_ω^-, $\omega > 0$, and $G_\omega \in D_\omega^-$ such that*

$$F_\omega^-(G_\omega) = \min\{F_\omega^-(G): G \in D_\omega^-\}.$$

Then $\alpha_-(G_\omega) \geqslant -\omega'$, with $\omega' = \omega - 2\omega/(\lambda_\omega + 2)$ and $\lambda_\omega = F_\omega^-(G_1)$.

The next lemma suggests a way of calculating ω such that optimizing F_ω^- produces a grid "nearer to convexity". [1]

Lemma 3.3. *Let us suppose that there exist $\bar{G} \in D_0^-$ and G_0 such that $\beta = \alpha_-(G_0) < 0$. Let $\omega = -\beta + \varepsilon$, with $0 < \varepsilon < -\beta/\lambda, \lambda = F_0^-(\bar{G})$. Let G_ω be a grid in D_ω^- where the minimum of F_ω^- is attained.* [2] *Then $\alpha_-(G_\omega) > \beta$.*

This result indicates that the set of values of ω suitable for the optimization to produce a *less nonconvex* grid, depends on the knowledge of the value that the functional takes in a convex grid. In practice, it is not even known if such a grid exists. However, it also follows from the lemma that a value of ω close enough to $-\alpha_-(G_0)$, and smaller than it, will serve for the purpose. As a result of experimentation, it has been found that an adequate way to calculate the value of ω is

$$\omega = \begin{cases} -1.05\alpha_-(G) & \text{if } \alpha_-(G) \leqslant -0.1\bar{\alpha}, \\ -\alpha_-(G) + 0.01\bar{\alpha} & \text{otherwise.} \end{cases}$$

3.3. Adaptive algorithm

Lemma 4.3 says that if the optimization process starts with a nonconvex grid, and a good choice of ω is made, a grid nearer to convexity than the initial one is obtained; i.e. a part of the trajectory of Section 3.1 is constructed. This suggests the following procedure to generate a convex grid over a region, under the assumption that the set of convex grids is not empty:
1. Generate an initial grid G_0

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[1] In fact, it just says that such a value of ω exists. Later on this section it is made more precise the way in which ω is calculated in practice.

[2] It is important to note that the minimum is supposed to be global.

2. Let

$$\omega = - \alpha_-(G_0) + \varepsilon,$$

with ε satisfying the conditions of Lemma 4.3.

3. Calculate G_ω with

$$F_\omega^-(G_\omega) = \min\{F_\omega(G): G \in D_\omega^-\};$$

4. Taking G_ω as new initial grid, repeat steps 2 and 3 until a convex grid is obtained.

The next theorem states that this process really works (this was the main result in [10]). Although in this scheme several functionals are used, one for each value of ω, all of them will be named with a single term: *adaptive smoothness functional.*

Theorem 3.4. *If the set of convex grids is not empty, the process just stated leads to the generation of a convex grid in a finite number of updates of parameter ω.*

In the event that a convex grid does not exist, this algorithm produces a grid as close to convexity as possible. The updating process could be continued until negative values of ω are reached; however there is no guarantee that the minimum areas will increase for these range of values. It is worth noting that, at every step of the algorithm, the optimal grid must be calculated; in a practical implementation, just a few steps of the iterative optimization process are performed; i.e., just an approximation to the optimum is calculated in order to update ω. In order to make the order of magnitude of these functionals independent of the dimension of the grid, it is convenient to use a scaled version, namely

$$F_\omega^-(G) = \frac{1}{2N} \sum_{q=1}^{N} \frac{l_q - 2\alpha_q}{\omega + \alpha_q}.$$

The results of applying the former algorithm are quite satisfactory: convex and smooth grids are obtained, and no pre-treatment on the initial grids is required. The algorithm is capable of obtaining a convex grid in a few minutes, using a personal computer; this is in great contrast to the several hours required in a computer for the same task, as reported by Ivanenko [6]. Another advantage of this scheme is its automaticity; the user is not asked to supply any parameters.

3.4. Area control: the bilateral smoothness functional

Minimizing the adaptive smoothness functional leads to optimal grids having two desirable features: convexity and smoothness. However, for complicated regions, that process produces some cells with very large areas: this effect is undesirable. Unfortunately, in order to diminish it, some concessions must be made: we have the alternative of losing, to some extent, smoothness or convexity; the choice is clear, grids with nonconvex cells are not adequate for the applications. We agree therefore in sacrificing some smoothness. With this in mind, a new functional is proposed to achieve the objective of avoiding large cells. For every triangle in the grid, the following function

is well defined if the denominator is not zero:

$$g = \frac{l - 2\alpha}{\omega - \alpha}.$$

It attains its minimum over right, isosceles, positively oriented triangles, if ω is chosen such that the denominator is positive. The discrete functional associated with this g,

$$\sum_{q=1}^{N} \frac{l_q - 2\alpha_q}{\omega - \alpha_q},$$

can be minimized using an iterative algorithm, and an initial grid with all its triangles of area greater than ω. The result will be another grid with the same property of the triangles. Indeed we would expect the largest area of the optimal grid to be smaller than the corresponding one in the initial one. The empirical results confirm this expectation, but no theoretical result has been proved in this sense. Moreover, in the process of diminishing large cells, convexity can be lost (or never reached, if the initial grid was nonconvex).

If this new functional is combined, in its scaled form, with the adaptive smoothness functional, the discrete functional named *bilateral smoothness* is obtained

$$F_{bs} = \frac{1}{2N} \sum_{q=1}^{N} \left(\frac{l_q - 2\alpha_q}{\omega_1 + \alpha_q} + \frac{l_q - 2\alpha_q}{\omega_2 - \alpha_q} \right),$$

where the values of ω_1 and ω_2 are chosen to make all the denominators positive; i.e., $\omega_1 > -\alpha_-$ and $\omega_2 > \alpha_+$. The minimization of this combination will produce grids such that its triangles have areas between $-\omega_1$ and ω_2. A process of updating the values of ω_1 and ω_2 is then applied. Optimization of the new functionals is performed, until a convex grid without very large cells is obtained. In practice, the value of ω_1 is calculated in the same way that for the adaptive smoothness functional, and the following formula for calculating ω_2 has proven to be adequate:

$$\omega_2 = 1.05\alpha_+. \tag{1}$$

It must be mentioned that if the two parameters are updated simultaneously, convexity is not reached in some difficult cases. Instead, the value of ω_2 is kept fixed until a convex grid is obtained, then the process continues with simultaneous updates of both parameters. This strategy has rendered good results.

4. A new bilateral smoothness functional

Let us consider the following function, defined for triangles

$$f_\omega^+(\Delta) = \frac{l(\Delta)}{\omega - \alpha(\Delta)},$$

with ω such that f_ω^+ is positive for all the triangles in the grid, i.e.,

$$\omega > \alpha_+,$$

which makes $f_\omega^+(\varDelta)$ bounded below by zero. When the discrete functional

$$F_\omega^+(G) = \sum_{q=1}^{N} f_\omega^+(\varDelta_q)$$

is minimized using an iterative procedure with initial grid in

$$D_\omega^+ = \{G \in M : \alpha_+(G) < \omega\},$$

the resulting optimal grid will be in D_ω^+ also, and we would expect that a good choice of ω would produce a grid with its greatest triangle smaller than the corresponding in the initial grid. This is indeed the case, results can be obtained in this direction; however, convexity could not be reached or preserved (if the initial grid was already convex) and this is the reason to use this new functional in conjunction with the smoothness functional, as it was previously made to generate the bilateral smoothness functional. The resulting functional is

$$F_{\omega_1,\omega_2} = \sum_{q=1}^{N} \left(\frac{l_q - 2\alpha_q}{\omega_1 + \alpha_q} + \frac{l_q}{\omega_2 - \alpha_q} \right).$$

It will serve for the same goals as F_{bs}; the procedure to use it (computation and updating of parameters ω_1, ω_2) will also be the same, but there are some major differences:

• Experimental results on this feature are better than those obtained with other functionals.
• We are able to establish theoretical foundations concerning the diminishing of large areas.

In the following paragraphs we state these properties and give the corresponding proofs. Note that the proof technique, with minor changes, is the same used in [9].

4.1. Some properties

Lemma 4.1. *Let $\{G_q\}$ be a sequence of grids belonging to $D_{\omega_2}^+ \cap D_{\omega_1}^-$, $\omega_2 > \bar{\alpha}$, $\omega_1 > 0$. If $\alpha_-(G_q) \to -\omega_1$ and $\omega_+(G_q) \to \omega_2$ then $F_{\omega_1,\omega_2}(G) \to \infty$. Therefore, if $\{G_q\} \to \partial(D_{\omega_2}^+ \cap D_{\omega_1}^-)$, then $F_{\omega_1,\omega_2}(G) \to \infty$ and it follows that if $D_{\omega_2}^+ \cap D_{\omega_1}^- \neq \emptyset$ then F_{ω_1,ω_2} has a minimum there.*

Proof. As a first case, let us suppose that $\alpha_-(G_q) \to -\omega_1$, and let $\varDelta_q$ be a triangle of G_q with minimal area. If $\{F_{\omega_1}^-(G_q)\}$ is bounded, it follows that

$$l(\varDelta_q) - 2\alpha(\varDelta_q) \to 0,$$

that is, $\varDelta_q$ tends to be a right, isosceles triangle with positive area, or to have sides of null length. Neither of these options is possible because of the assumption made on $\alpha_-(G_q)$.

As a second case, if $\alpha_+(G_q) \to \omega_2$ and $F_{\omega_2}^+(G_q)$ is bounded, $\varDelta_q'$ a triangle with maximum area in G_q

$$l(\varDelta_q') = l_q \to 0;$$

this implies that its sides tend to zero; this is false, because $\alpha(\varDelta_q') = \alpha_q > \bar{\alpha}$.

In the general case, if $\{G_q\} \to \partial(D^+_{\omega_2} \cap D^-_{\omega_1})$, there must exist a subsequence approaching $\partial(D^-_{\omega_1})$ or a subsequence tending to $\partial(D^+_{\omega_1})$, in either case, $F_{\omega_1,\omega_2}(G_q) \to \infty$.

Lemma 4.2. *Let* $\omega_1 > 0$, $\omega_2 > \bar{\alpha}$, $G_1 \in D^-_{\omega_1} \cap D^+_{\omega_2}$ *and* $\lambda_{\omega_1,\omega_2} = F_{\omega_1,\omega_2}(G_1)$. *If* $G_{\omega_1,\omega_2} \in D^-_{\omega_1} \cap D^+_{\omega_2}$ *is such that*

$$F_{\omega_1,\omega_2}(G_{\omega_1,\omega_2}) = \min \left\{ F_{\omega_1,\omega_2}(G): \ G \in D^-_{\omega_1} \cap D^+_{\omega_2} \right\},$$

then $\alpha_-(G_{\omega_1,\omega_2}) \geq -\omega'_1$, *and* $\alpha_+(G_{\omega_1,\omega_2}) \leq \omega'_2$, *where*

$$\omega'_1 = \omega_1 - \frac{2\omega_1}{\lambda_{\omega_1,\omega_2} + 2},$$

$$\omega'_2 = \omega_2 - \frac{2\omega_2}{\lambda_{\omega_1,\omega_2} + 2}.$$

Proof. Let $G \in D_{\omega_1,\omega_2}$ such that $\alpha_-(G) < -\omega'_1$ and Δ a triangle in G of minimal area, then

$$F_{\omega_1,\omega_2}(G) \geq \frac{l(\Delta) - 2\alpha(\Delta)}{\omega_1 + \alpha(\Delta)} = \frac{l(\Delta) - 2\alpha_-(G)}{\omega_1 + \alpha_-(G)} = \frac{l(\Delta) + 2\omega_1}{\omega_1 + \alpha_-(G)} - 2$$

$$\geq \frac{2\omega_1}{\omega_1 + \alpha_-(G)} - 2 > \frac{2\omega_1}{\left(\frac{2\omega_1}{\lambda_{\omega_1,\omega_2}+2}\right)} - 2 = \lambda_{\omega_1,\omega_2},$$

and the minimum of F_{ω_1,ω_2} is not attained in G.

Similarly, for $G \in D^-_{\omega_1} \cap D^+_{\omega_2}$ with $\alpha_+(G) > \omega'_2$, let Δ a triangle in G of maximal area. Then

$$F_{\omega_1,\omega_2}(G) \geq \frac{l(\Delta)}{\omega_2 - \alpha(\Delta)} \geq \frac{2\alpha(\Delta)}{\omega_2 - \alpha(\Delta)} > \frac{2\omega'_2}{\omega_2 - \omega'_2}$$

$$= \frac{2\omega'_2}{\left(\frac{2\omega_2}{\lambda_{\omega_1,\omega_2}+2}\right)} = \frac{\omega'_2}{\omega_2}(2 + \lambda_{\omega_1,\omega_2}) = \lambda_{\omega_1,\omega_2};$$

this implies that the minimum of F_{ω_1,ω_2} cannot be attained in G.

Lemma 4.3. *Let us suppose that there exists* $\bar{G} \in D^-_0 \cap D^+_{\overline{\omega_2}}$, *with* $\overline{\omega_2} > \bar{\alpha}$, *and let* $\lambda = F_{0,\overline{\omega_2}}(\bar{G})$. *If* $\beta_1 < 0$, $\beta_2 > \overline{\omega_2}$, *let us choose* $\varepsilon_1, \varepsilon_2$ *to be positive,* $\varepsilon_1 < -\beta_1/\lambda$, $\varepsilon_2 < \beta_2/\lambda$. *Also, make* $\omega_1 = -\beta_1 + \varepsilon_1$, $\omega_2 = \beta_2 + \varepsilon_2$.

Now, let G_{ω_1,ω_2} *be a grid in* $D^-_{\omega_1} \cap D^+_{\omega_2}$ *where the minimum of* F_{ω_1,ω_2} *is attained.*[3] *Then* $\alpha_-(G_{\omega_1,\omega_2}) > \beta_1$ *and* $\alpha_+(G_{\omega_1,\omega_2}) < \beta_2$.

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[3] We are talking about a global minimum.

Proof. Let $\lambda_{\omega_1,\omega_2} = F_{\omega_1,\omega_2}(\overline{G})$. According to Lemma 4.2,

$$\alpha_-(G_{\omega_1,\omega_2}) \geqslant -\omega_1 + \frac{2\omega_1}{\lambda_{\omega_1,\omega_2}+2} \geqslant -\frac{\omega_1 \lambda_{\omega_1,\omega_2}}{\lambda_{\omega_1,\omega_2}+2}$$

$$> \beta_1\left(1+\frac{1}{\lambda}\right)\frac{\lambda_{\omega_1,\omega_2}}{\lambda_{\omega_1,\omega_2}+2} = \left(\frac{\lambda_{\omega_1,\omega_2}}{\lambda}\frac{\lambda+2}{\lambda_{\omega_1,\omega_2}+2}\right)\frac{\lambda+1}{\lambda+2}\beta_1.$$

Now, $\lambda_{\omega_1,\omega_2} \leqslant \lambda$, and it is not difficult to show that the term in parenthesis is less than 1, so

$$\alpha_-(G_{\omega-1,\omega_2}) > \left(\frac{\lambda+1}{\lambda+2}\right)\beta_1 \geqslant \beta_1.$$

Again, using Lemma 4.2, it follows that

$$\alpha_+(G_{\omega_1,\omega_2}) \leqslant \omega_2\left(\frac{\lambda_{\omega_1,\omega_2}}{\lambda_{\omega_1,\omega_2}+2}\right) = (\beta_2+\varepsilon_2)\left(\frac{\lambda_{\omega_1,\omega_2}}{\lambda_{\omega_1,\omega_2}+2}\right)$$

$$< \beta_2\left(1+\frac{1}{\lambda}\right)\left(\frac{\lambda_{\omega_1,\omega_2}}{\lambda_{\omega_1,\omega_2}+2}\right) < \beta_2\left(\frac{\lambda+1}{\lambda+2}\right).$$

This Lemma suggest the following algorithm to obtain a convex grid and reduce the size of the cells in the grid.

1. Generate an initial grid G_0;
2. Let

$$\omega_1 = -\alpha_-(G_0) + \varepsilon_1,$$

$$\omega_2 = \alpha_+(G_0) + \varepsilon_2,$$

with ε_1 and ε_2 satisfying conditions of lemma 4.3;
3. Calculate G_{ω_1,ω_2} with

$$F_{\omega_1,\omega_2}(G_{\omega_1,\omega_2}) = \min\{F_{\omega_1,\omega_2}(G) : G \in D^+_{\omega_2} \cap D^-_{\omega_1}\};$$

4. Taking G_{ω_1,ω_2} as new initial grid, repeat steps 2 and 3 until a convex grid with an acceptable size of cells is obtained.

Theorem 4.4. *If the set $D^-_0 \cap D^{\pm}_{\overline{\omega_2}}$ is not empty, the former process leads to a convex grid with maximum area less than $\overline{\omega_2}$ in a finite number of updates of the parameters ω_1 and ω_2.*

Proof. Suppose that the result does not hold, and let $\{\omega_{1,q}\}, \{\omega_{2,q}\}, \{G_q\}$ be the sequences generated with the process. According to Lemma 4.3

$$0 > \alpha_-(G_q) > \left(\frac{\lambda+1}{\lambda+2}\right)^q \alpha_-(G_0),$$

and, then

$$\lim_{q \to \infty} \alpha_-(G_q) = 0,$$

$$\lim_{q \to \infty} \omega_{1,q} = 0.$$

The last equality holds because $\omega_{1,q} > -\alpha_-(G_q)(1 + 1/\lambda)$. Let us consider

$$D^* = \lim_{q \to \infty} D^-_{\omega_{1,q}} \cap D^+_{\omega_{2,q}} = \bigcap \left(D^-_{\omega_{1,q}} \cap D^+_{\omega_{2,q}} \right).$$

All the grids in this set must be such that none of its cells is strictly non convex, i.e., the areas of all the triangles of a grid in D^* are non negative.

The process produces grids G_q such that

$$F_{\omega_{1,q},\omega_{2,q}}(G_q) \leqslant F_{\omega_{1,q},\omega_{2,q}}(G) \leqslant F_{0,\overline{\omega_2}}(G),$$

where G is a grid in $D_0^- \cap D^+_{\overline{\omega_2}}$.

Let H be a grid in D^* with $\alpha_-(H) = 0$ and $\varDelta$ a triangle of H of minimum area, i.e., $\alpha(\varDelta) = 0$. Then,

$$F_{\omega_{1,q},\omega_{2,q}}(H) \geqslant \frac{l(\varDelta) - 2\alpha(\varDelta)}{\omega_{1,q} + \alpha(\varDelta)} = \frac{l(\varDelta)}{\omega_{1,q}}.$$

If H is obtained as a limit of such a process

$$\lim_{q \to \infty} F_{\omega_{1,q},\omega_{2,q}}(H) \leqslant F_{0,\overline{\omega_2}}(G),$$

then

$$\lim_{\omega_{1,q} \to 0} \frac{l(\varDelta)}{\omega_{1,q}}$$

is finite. But this would imply that $l(\varDelta)$ must be zero. Therefore, the length of the sides of $\varDelta$ must be zero. This, in turn, says that the triangles of H, with a common side with $\varDelta$ must be of null area.

Repeating this reasoning would lead eventually to conclude that the sides of all the cells of H are null, including those on the boundary. This is not possible, therefore the process must produce a convex grid in a finite number of ω_1 updates.

On the other hand,

$$\alpha_+(G_q) < \left(\frac{\lambda + 1}{\lambda + 2} \right)^q \alpha_+(G_0),$$

and thus $\alpha_+(G_q) < \overline{\omega_2}$ for some finite q.

We can normalize this functional in order to make its magnitude independent of the dimension (values of m and n) of grids and the size of the region; the scaling factor used is the same as that for the smoothness functional. Then, the new bilateral smoothness functional is

$$F_{bsl} = \frac{1}{2N} \sum_{q=1}^N \left(\frac{l_q - 2\alpha_q}{\omega_1 + \alpha_q} + \frac{l_q}{\omega_2 - \alpha_q} \right).$$

Table 1
Results for region Havana

Functional	t	it	$\alpha_- t$	$\alpha_+ t$	$\alpha_- c$	$\alpha_+ c$	σ^2	s
F_ω^-	639	85	0.039	15.950	0.080	11.260	1.214	1.733
F_{bs}	1840	126	0.045	6.745	0.094	5.565	0.692	1.753
F_{bs1}	2134	118	0.055	4.363	0.122	3.389	0.413	1.832

Table 2
Result for region England

Functional	t	it	$\alpha_- t$	$\alpha_+ t$	$\alpha_- c$	$\alpha_+ c$	σ^2	s
F_ω^-	1545	210	0.014	16.159	0.051	10.525	1.401	3.241
F_{bs}	4664	293	0.013	5.580	0.056	4.806	0.654	3.333
F_{bs1}	4928	276	0.013	4.850	0.057	3.636	0.519	3.371

5. Results

We have applied our methods to a variety of regions, and we have obtained quite good results. Some of them are presented in Tables 1 and 2. They correspond to optimal grids of dimension 40 times 40 over two regions considered hard to mesh (Havana and England). Initial grids were generated using Transfinite Interpolation (see [7]); these grids have been used as starting point to minimize several functionals, in all the cases we obtained convex grids.

In the tables, t stands for time (in ss) required to obtain the optimal grid;[4] it is the number of iterations performed; quantities $\alpha_- t$, $\alpha_+ t$, $\alpha_- c$, $\alpha_+ c$ defined previously, are expressed in terms of $\bar\alpha$; σ^2 is the variance of the areas of triangles in the grid, also normalized using $\bar\alpha$. The last column gives s, the value of the smoothness functional; to a smaller value of s correspond a smoother grid.

From these tables, we observe that:

- The adaptive smoothness functional requires less time than bilateral smoothness (either version). However, it produces optimal grids with some large cells, and the variance of areas is greater than it is in the other cases.

- Both of the bilateral smoothness functionals produce better control of large cells and a smaller variance of the areas. The second functional leads to better results in this aspect.

- Some smoothness is lost using the bilateral functionals; however, there is no big difference between the two functionals, concerning this feature.

At the end we show some graphs of optimal grids. They correspond to the same regions and functionals that the tables but, for the sake of clarity, the dimensions area is 30 times 30.

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