JOURNAL OF ALGEBRA **191,** 89–108 (1997)
ARTICLE NO. JA966883

Algebras Whose Tits Form Weakly Controls the Module Category

F. Coelho

Departamento de Matemáticas, Universidade de São Paulo, São Paulo, Brazil

J. A. de la Peña

Instituto de Matemáticas, Universidad Nacional Autónoma de México, México 04510, DF, México

and

B. Tomé

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacionol Autónoma de México, México 04518, DF, México

Communicated by Kent R. Fuller

Received February 22, 1996

An important invariant associated to a finite dimensional algebra is its Tits quadratic form. This invariant has been extensively used in the representation theory of algebras to determine the representation type of an algebra (see [4, 11, 16]), to determine the classes of indecomposable modules in the Grothendieck group (see [9, 11, 20]), and as a tool in a wide range of applications (see [16, 21]).

Let $A = kQ/I$ be a basic, finite dimensional algebra over an algebraically closed field k. We will assume that Q is connected and without oriented cycles. We denote by $Q_0 = \{1, \dots, n\}$ (resp. Q_1) the set of vertices (resp. arrows) of Q. Left A-modules are considered as representations of Q satisfying the ideal I (for this point of view, see [12]). By S_i we denote the simple representation associated with the vertex $i \in Q_0$. Observe that $\dim_k \operatorname{Ext}^1_A(S_i, S_j)$ is the number of arrows from i to j in Q. The *Tits form*

0021-8693/97 $25.00

q_A is the integral quadratic form $q_A\colon \mathbb{Z}^n \to \mathbb{Z}$ given by

$$q_A(v) = \sum_{i=1}^{n} v(i)^2 - \sum_{(i\to j)\in Q_1} v(i)v(j) + \sum_{i,j\in Q_0} v(i)v(j)\dim_k \operatorname{Ext}_A^2(S_i, S_j).$$

If all indecomposable A-modules lie in a postprojective component of the Auslander–Reiten quiver Γ_A of A (and hence A is representation-finite), then there is a one to one correspondence $X \mapsto \mathbf{dim}\ X$ between indecomposable modules and positive roots of q_A (that is, vectors $v \in \mathbb{N}^n$ with $q_A(v) = 1$). For a tame algebra A, it is known that the Tits form q_A is weakly non-negative [15]. Moreover, for several important classes of tame algebras (e.g., hereditary algebras [9], domestic tubular and tubular algebras [25], tilted algebras [13]), it has been shown that the Tits form q_A of an algebra A in the family *controls the module category* mod_A, that is:

(i) for every indecomposable module $X \in \operatorname{mod}_A$, $q_A(\mathbf{dim}\ X) \in \{0, 1\}$;

(ii) for any connected vector $v \in \mathbb{N}^n$ with $q_A(v) = 1$, there exists a unique (up to isomorphism) indecomposable A-module X with $\mathbf{dim}\ X = v$;

(iii) for any connected vector $v \in \mathbb{N}^n$ with $q_A(v) = 0$, there exists an infinite family $(X_\lambda)_\lambda$ of pairwise non-isomorphic indecomposable A-modules with $\mathbf{dim}\ X_\lambda = v$ for every λ.

The most precise statements relating indecomposable modules and the Tits form have been obtained for strongly simply connected algebras (see [20, 22, 27]). An algebra A is said to be *strongly simply connected* if every convex (= path closed) subcategory B of A satisfies the separation condition (equivalently, its first Hochschild cohomology group $H^1(B, B)$ vanishes [26]). We will say that q_A *weakly controls* mod_A if for every indecomposable module $X \in \operatorname{mod}_A$, $q_A(\mathbf{dim}\ X) \in \{0, 1\}$. In [23], it was shown that a strongly simply connected algebra A such that q_A weakly controls mod_A is tame of polynomial growth. We shall show in this work the following theorem (for the required definitions, see Section 1).

THEOREM. *Let A be a strongly simply connected algebra. Then the following are equivalent*:

(i) *q_A weakly controls* mod_A.

(ii) *A is of polynomial growth and for every convex subcategory B of A which is a coil algebra, B is a branched-critical algebra with* $\operatorname{gl\,dim} B \leq 2$.

(iii) *q_A is weakly non-negative and for every indecomposable A-module X there is a convex subcategory B of A containing* $\operatorname{supp} X$ *and such that B is either a tilted algebra or a branched-critical algebra with* $\operatorname{gl\,dim} B \leq 2$.

In Section 1 we recall the definitions and results required for the proof of the theorem, which is given in Section 2. In Section 3 we shall consider some algebras A whose Tits form weakly controls the module category but such that A admits non-trivial Galois coverings.

1. BASIC FACTS

Let $A = kQ/I$ be a finite dimensional k-algebra and assume that Q is a connected quiver without oriented cycles. Let $Q_0 = \{1, \dots, n\}$.

1.1. For any representation $X \in \mathrm{mod}_A$, we denote by $\mathbf{dim}\ X \in \mathbb{Z}^n$ (called the dimension vector of X) the class of X in the Grothendieck group $K_0(A) \cong \mathbb{Z}^n$. The Euler characteristic is defined as the (non-symmetric) bilinear form satisfying

$$\langle \mathbf{dim}\ X, \mathbf{dim}\ Y \rangle = \sum_{i=0}^{\infty} (-1)^i\, \mathbf{dim}_k\, \mathrm{Ext}_A^i(X, Y).$$

The associated quadratic form χ_A is called the *Euler form*. Observe that if gl dim $A \leq 2$, then $q_A = \chi_A$ (see [4]). Many of the applications of the Tits form in representation theory make use of this identity, which relates combinatorial data of A and homological information of the A-modules.

1.2. An algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there is a finite family $M_1, \dots, M_s$ of $A - k[t]$-bimodules which are free as right $k[t]$-modules and such that almost every indecomposable A-module X of dimension d is isomorphic to $M_i \otimes_{k[t]} k[t]/(t - \lambda)$ for some $1 \leq i \leq s$ and $\lambda \in k$. The minimal number $\mu(d)$ of bimodules in the definition is called the number of one-parameter families in dimension d. If $\mu(d) \leq c$ for every d, then A is said to be *domestic*; if $\mu(d) \leq d^m$ for some $m \in \mathbb{N}$ and every d, then A is said to be of *polynomial growth*. See [7, 16, 25].

For a tame algebra A, the Tits form is weakly non-negative. In this case, for every $v \in \mathbb{N}^n$ almost every indecomposable A-module X with $\mathbf{dim}$ $X = v$ satisfies

$$q_A(v) \geq \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}_A^1(X, X) \geq 0.$$

See [18].

We briefly recall some "classical" *examples*:

(i) An algebra C is called *critical* if the Auslander–Reiten quiver Γ_C has a postprojective component and every proper quotient of C is representation-finite. In that case, the quiver Γ_C is formed by a postprojective component $\mathscr{P}$, a family $\mathscr{T} = (T_\lambda)_{\lambda \in \mathbb{P}_1 k}$ of tubular components and a preinjective component $\mathscr{I}$. The Tits form q_C is critical, that is, q_C is not weakly positive but every restriction $q_C^{(i)} = q_C(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$: $\mathbb{Z}^{n-1} \to \mathbb{Z}$ is weakly positive. There exists a vector $0 \neq z \in \mathbb{N}^n$ such that $q_C(z) = 0$, and an indecomposable C-module X lies in $\mathscr{P}$ (resp. $\mathscr{T}, \mathscr{I}$) if and only if $\langle z, \mathbf{dim}\ X \rangle < 0$ (resp. $= 0$, > 0). See [9, 25].

(ii) In [25], the *branch extensions* of a critical algebra are defined. An algebra A obtained by a sequence of branch extensions from a critical algebra is tame if and only if A is either a *domestic tubular* or a *tubular algebra*.

Tubular algebras are of polynomial growth; the structure of their Auslander–Reiten quiver is described in [25]. For a domestic tubular or tubular algebra A, the Tits form q_A controls mod_A [25].

(iii) Let A be a tilted algebra. In [13], it was shown that A is tame if and only if q_A is weakly non-negative. In that case, q_A controls mod_A [19].

An important example is the following: let X be an *indecomposable directing* A-module (that is, there is no cycle of non-zero non-isomorphisms $X = X_0 \to X_1 \to \cdots \to X_s \to X$ between indecomposable A-modules), then supp X is convex in Q and the full subcategory B of A induced by supp X is tilted [25].

1.3. An algebra B is said to be *hypercritical* if Γ_B admits a postprojective component and q_B is hypercritical (that is, q_B is not weakly non-negative but every restriction $q_B^{(i)}$ is weakly non-negative). In [17] it is shown that a strongly simply connected algebra A has a weakly non-negative Tits form q_A if and only if A does not admit a convex subcategory B which is a hypercritical algebra.

1.4. A strongly simply connected algebra D is said to be *pg-critical* if it is tame not of polynomial growth but every proper quotient of D is of polynomial growth. These algebras have been classified in [14]. Their importance is due to the following result.

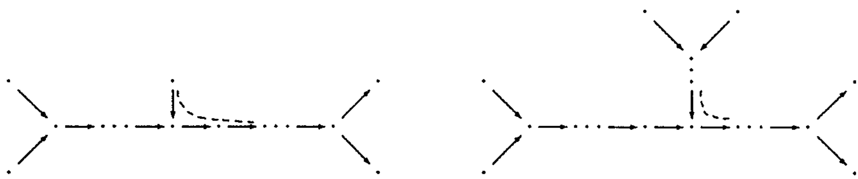
THEOREM [27, 28]. *Let A be a strongly simply connected algebra. Then the following are equivalent*:

(i) *A is of polynomial growth.*

(ii) *q_A is weakly non-negative and A does not admit a convex pg-critical subcategory.*

(iii) *A does not admit a convex subcategory which is hypercritical or pg-critical.*

A *pg*-critical algebra D is tilted equivalent to one of the algebras



where the ideal defining the algebra is generated by the marked paths.

It was shown in [24] that for a *pg*-critical algebra D there is an indecomposable module X with $\dim_k \operatorname{End}_D(X) = 1 = \dim_k \operatorname{Ext}_D^2(X, X)$ and $\operatorname{Ext}_D^1(X, X) = 0$, in particular, $q_D(\mathbf{dim}\ X) = 2$.

1.5. PROPOSITION [23]. *Let A be a strongly simply connected algebra such that q_A weakly controls* mod_A. *Then A is of polynomial growth.*

The proof follows directly using the criterion quoted in (1.4) and the properties of hypercritical and *pg*-critical algebras recalled in (1.3) and (1.4), respectively.

1.6. Let A be a strongly simply connected algebra. By [28], A is of polynomial growth if and only if A is a multicoil algebra, that is, every non-directing indecomposable A-module lies in a coil of a multicoil component of Γ_A; see [1, 2]. The support of a coil is a convex subcategory of A which is a tame coil enlargement of a critical algebra.

In [1], six admissible operations (ad 1), (ad 2), (ad 3) and their duals (ad 1*), (ad 2*), (ad 3*) were introduced. We briefly recall these operations. Let B be an algebra and $\mathscr{C}$ be a standard component of Γ_B. For an indecomposable module X in $\mathscr{C}$, called the *pivot*, the admissible operations are defined depending on the shape of the support of $\operatorname{Hom}_B(X, -)|_{\mathscr{C}}$, in order to obtain a new algebra B'.

(ad 1) If the support of $\operatorname{Hom}_B(X, -)|_{\mathscr{C}}$ is of the form

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

we set $B' = (B \times D)[X \oplus Y_1]$, where D is the full $t \times t$ lower triangular matrix algebra and Y_1 is the indecomposable projective-injective D-module.

(ad 2) If $\operatorname{Hom}_B(X,-)|_{\mathscr{C}}$ is of the form

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

with $t \geq 1$ (so that X is injective), we set $B' = B[X]$.

(ad 3) If the support of $\operatorname{Hom}_B(X,-)|_{\mathscr{C}}$ is of the form

$$\begin{array}{ccccccccc} X = X_0 & \rightarrow & X_1 & \rightarrow & \cdots & X_{t-1} & \rightarrow X_t & \rightarrow & \cdots \\ \downarrow & & \downarrow & & & \downarrow & & & \\ Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & Y_t & & & \end{array}$$

with $t \geq 2$ (so that X_{t-1} is injective), we set $B' = B[X]$.

In all these cases the component $\mathscr{C}'$ of $\Gamma_{B'}$ containing X is a standard component (under certain conditions satisfied in this work).

Following [3], an algebra B is said to be a *coil enlargement* of the critical algebra C if there is a finite sequence of algebras $C = B_0, B_1, \ldots, B_m = B$ such that, for each $0 \leq i < m$, B_{i+1} is obtained from B_i by one of the admissible operations with pivot in a stable tube of the separating tubular family $\mathscr{T}$ or in a component of Γ_{B_i} obtained from a tube in $\mathscr{T}$ by means of the sequence of admissible operations done so far. If B is tame, we say that B is a *coil algebra*.

THEOREM [22]. *Let A be a strongly simply connected algebra of polynomial growth. Let X be an indecomposable A-module. Then there is a convex subcategory B of A containing* supp X *and such that B is either a tilted algebra or a coil algebra, in particular,* gl dim $B \leq 3$.

Moreover, $\operatorname{Ext}_A^r(X,X) = 0$ *for every $r \geq 2$ and*

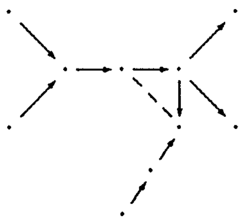
$$q_A(\mathbf{dim}\ X) \geq \chi_A(\mathbf{dim}\ X) = \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}_A^1(X,X) \geq 0.$$

1.7. A coil algebra B which is obtained from a critical algebra C by a sequence of admissible operations of types (ad 1) and (ad 1*) such that each pivot used in the sequence is not both an (ad 1)-pivot and an (ad 1*)-pivot is called a *branched-critical algebra*. In other words, B is a branched-critical algebra if it is obtained from the critical algebra C by a sequence of branch extensions and branch coextensions, and has no full convex subcategory of the form $[M]C'[M]$ with C' a full convex subcategory of B.

2. THE MAIN RESULTS

2.1. First, we give some examples.

Consider the domestic tubular algebra B_0 given by the bound quiver



It is obtained from a critical algebra of type $\tilde{\mathbb{D}}_6$ by a branch coextension (one operation of type (ad 1*)). Consider the unique indecomposable B_0-modules M_1, M_2, M_3, M_4, and M_5 with dimension vectors $v_i = \mathbf{dim}\, M_i$ given as

```
      0           0        0           0
        1   0   0            0   1   0
v1:   0         0   0  v2: 0         0   0
             0                    0
          0                    0

      0           0        0           0
        0   0   1            0   0   1
v3:   0         0   0  v4: 0         1   0
             0                    1
          0                    1

              0           0
                0   0   1
        v5:   0         1   0
                     1
                  0
```

The algebras $B_i = B_0[M_i]$, $1 \leq i \leq 5$, are coil algebras.

We study briefly each case.

B_1 is obtained from B_0 by an operation of type (ad 1). The global dimension of B_1 is 2. It is not difficult to show that q_{B_1} weakly controls mod_{B_1}.

B_2 is obtained from B_0 by an operation of type (ad 1). In this case, gl dim $B_2 = 3$. There is an indecomposable B_2-module X with dimension vector

$$\mathbf{dim}\, X = \begin{array}{cccccc} 1 & & \textcircled{1} & & & 1 \\ & 2 & 2 & & 2 & \\ 1 & & & & 1 & 1 \\ & & & 0 & & \\ & & 0 & & & \end{array}$$

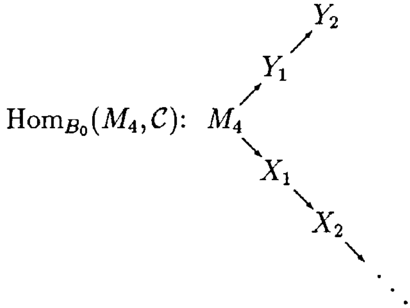
where the encircled number corresponds to the extension vertex. Since $q_{B_2}(\mathbf{dim}\ X) = 2$, then q_{B_2} does not control mod_{B_2}.

B_3 is obtained from B_0 by an operation of type (ad 1). In this case, gl dim $B_3 = 2$. There is an indecomposable B_3-module X with dimension vector

$$\mathbf{dim}\, X = \begin{array}{cccccc} 1 & & \textcircled{1} & & 1 & \\ & 2 & 2 & & 3 & \\ 1 & & & & 1 & 1 \\ & & & 1 & & \\ & & 1 & & & \end{array}$$

such that $q_{B_3}(\mathbf{dim}\ X) = 2$. Hence q_{B_3} does not control mod_{B_3}.

B_4 is obtained from B_0 by an operation of type (ad 2). The module M_4 belongs to a coinserted tube $\mathscr{C}$ of Γ_{B_0} and the vector space category $\mathrm{Hom}_{B_0}(M_4, \mathscr{C})$ has the shape

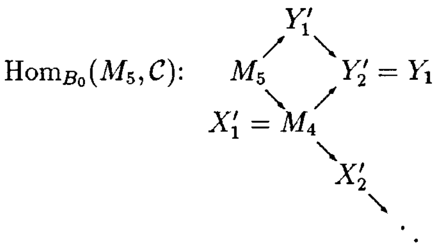


There is an indecomposable B_4-module Y whose restriction to B_0 is $X_7 \oplus Y_2$ and with dimension vector

$$\mathbf{dim}\, Y:\quad \begin{array}{ccccc} 1 & & & & 1 \\ & 2 & 2 & 3 & \\ 1 & & \textcircled{1} & 1 & 1 \\ & & & {}^{1} & \\ & & 2 & & \end{array}$$

Therefore $q_{B_4}(\mathbf{dim}\ Y) = 2$ and q_{B_4} does not control mod_{B_4}.

Finally, B_5 is obtained from B_0 by an operation of type (ad 3). The vector space category $\mathrm{Hom}_{B_0}(M_5, \mathscr{C})$ has the shape



There is an indecomposable B_5-module Z whose restriction to B_0 is $X_8' \oplus Y_2'$ and with dimension vector

$$\mathbf{dim}\, Z:\quad \begin{array}{ccccc} 1 & & & & 1 \\ & 2 & 2 & 3 & \\ 1 & & \textcircled{1} & 1 & 1 \\ & & & {}^{2} & \\ & & 2 & & \end{array}$$

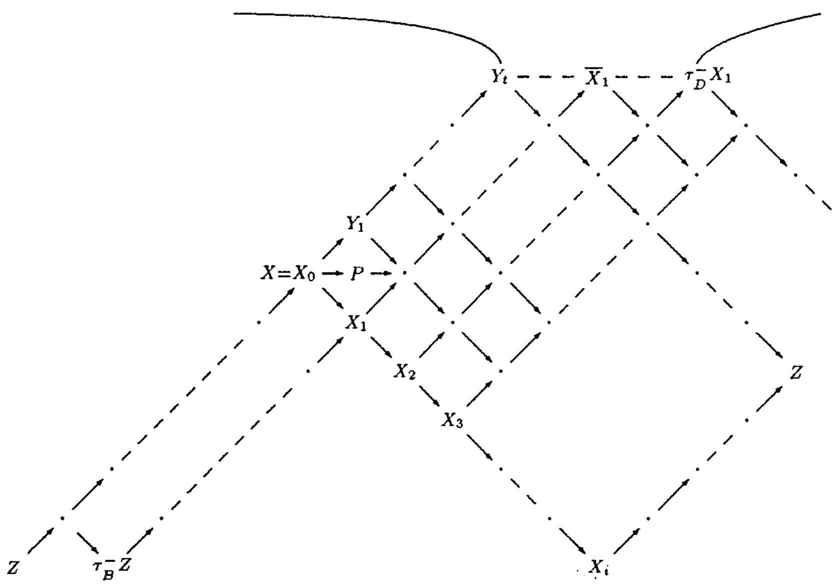
Hence $q_{B_5}(\mathbf{dim}\ Z) = 2$ and q_{B_5} does not control mod_{B_5}.

2.2. PROPOSITION. *Let B be a coil algebra such that q_B weakly controls* mod B. *Then B is a branched-critical algebra.*

Proof. Assume that B is a coil algebra which is not branched-critical. There is a sequence $C = B_0, B_1, \ldots, B_m = B$, where C is a critical algebra and B_{i+1} is obtained from B_i by an admissible operation. Since B is not branched-critical, we may assume that B_n for $n \le m$ is obtained from B_{n-1} by an operation of type (ad 2) or (ad 3) or by an operation of type (ad 1) with pivot a module used previously as an (ad 1*)-pivot, and for every $1 \le i \le n-1$, B_i is obtained from B_{i-1} by an operation of type (ad 1) or (ad 1*). Since B_n is a convex subcategory of B, we may assume that $m = n$. Let $D = B_{m-1}$, X be the pivot of the admissible operation used to obtain B and let $\mathscr{C}$ be the standard coil in Γ_B containing X.

By (1.6), gl dim $B \le 3$ and $\operatorname{Ext}_B^r(Z, Z) = 0$ for every indecomposable B-module Z in $\mathscr{C}$ and $r \ge 2$.

Assume first that X is an (ad 2)-pivot. Then $\mathscr{C}$ has the shape



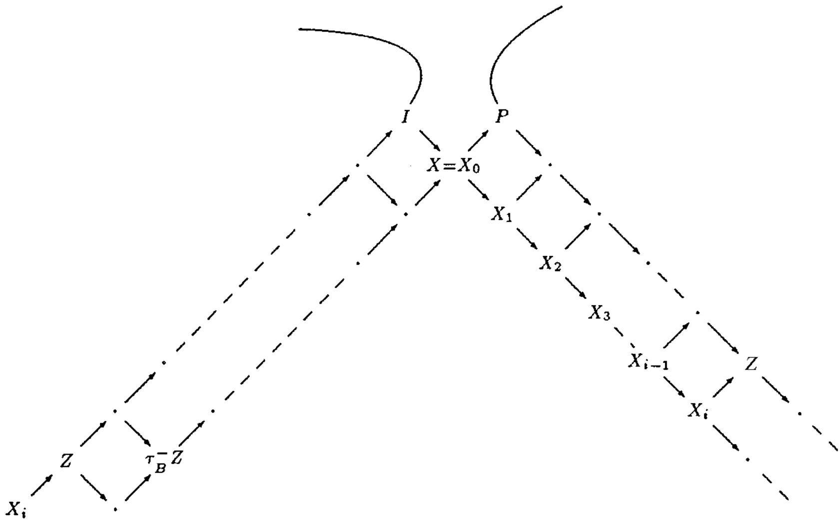
Let Z be the indecomposable B-module whose restriction to D is $X_i \oplus Y_t$, where i is the least positive integer such that $\operatorname{Hom}_D(X_i, Y_t) \neq 0$.

Then $i \dim_B Z \leq 1$ and

$$q_B(\mathbf{dim}\, Z) \geq \chi_B(\dim Z) = \dim_k \operatorname{Hom}_B(Z, Z) - \dim_k \operatorname{Ext}_B^1(Z, Z)$$
$$= \dim_k \operatorname{Hom}_B(Z, Z) - \dim_k \operatorname{Hom}(\tau_B^- Z, Z) = 3 - 1 = 2.$$

The case where X is an (ad 3)-pivot is similar.

Finally, assume that X is both an (ad 1)-pivot and an (ad 1*)-pivot. Then $\mathscr{C}$ has the shape



where P is projective and I is injective.

Let Z be the indecomposable B-module whose restriction to D is X_i, where i is the least positive integer such that $\operatorname{Hom}_D(X_i, I) \neq 0$. Then

$$q_B(\mathbf{dim}\, Z) \geq \chi_B(\mathbf{dim}\, Z) = \dim_k \operatorname{Hom}_B(Z, Z) - \dim_k \operatorname{Ext}_B^1(Z, Z)$$
$$= \dim_k \operatorname{Hom}_B(Z, Z) - \dim_k \overline{\operatorname{Hom}}(Z, \tau_B Z) = 2. \quad ∎$$

2.3. PROPOSITION. *Let B be a branched-critical algebra. Then the following are equivalent*:

(i) q_B *weakly controls* mod_B.

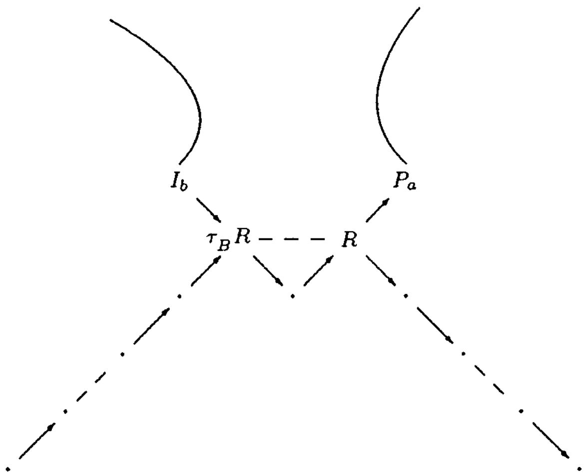
(ii) *For every indecomposable B-module X, we have*

$$q_B(\mathbf{dim}\, X) = \chi_B(\mathbf{dim}\, X) = \dim_k \operatorname{End}_B(X) - \dim_k \operatorname{Ext}_B^1(X, X).$$

(iii) $\operatorname{gl\,dim} B \leq 2$.

Proof. (i) ⇒ (iii) Suppose that gl dim $B > 2$. Then there is an indecomposable summand R of rad P_a for some indecomposable projective B-module P_a, and an indecomposable injective B-module I_b such that $\mathrm{Hom}_B(I_b, \tau_B R) \neq 0$. All modules R, $\tau_B R$, P_a, and I_b lie in a coil $\mathscr{C}$ of Γ_B obtained from a stable tube T in the Auslander–Reiten quiver of a critical algebra C by a sequence of branch insertions and coinsertions.

Since B is branched-critical, the situation in $\mathscr{C}$ may be depicted as



From the description of the modified component after applying an operation of type (ad 1) or (ad 1*), it follows that only the following situations can occur:

(a) $\tau_B R$ and R are both simple regular C-modules.

(b) $E' = \tau_B R|_C$ is a simple regular C-module lying on the coray ending at $\tau_B R$ in $\mathscr{C}$, and $R = \tau_C^- E'$.

(c) $E = R|_C$ is a simple regular C-module lying on the ray starting at R in $\mathscr{C}$, and $\tau_B R = \tau_C E$.

(d) $E' = \tau_B R|_C$ is a simple regular C-module lying on the coray ending at $\tau_B R$ in $\mathscr{C}$, $E = R|_C$ is a simple regular C-module lying on the ray starting at R in $\mathscr{C}$, and $E' = \tau_C E$.

Let Z be the indecomposable regular C-module in T with regular socle E and regular top $\tau_C E$. Then $\mathbf{dim}\ Z = z_0$ is the minimal generator of rad_{q_C}. It is easy to see that in the four cases above, there exists an indecomposable B-module X in $\mathscr{C}$ with $\mathbf{dim}\ X = z_0 + e_a + e_b$. Thus

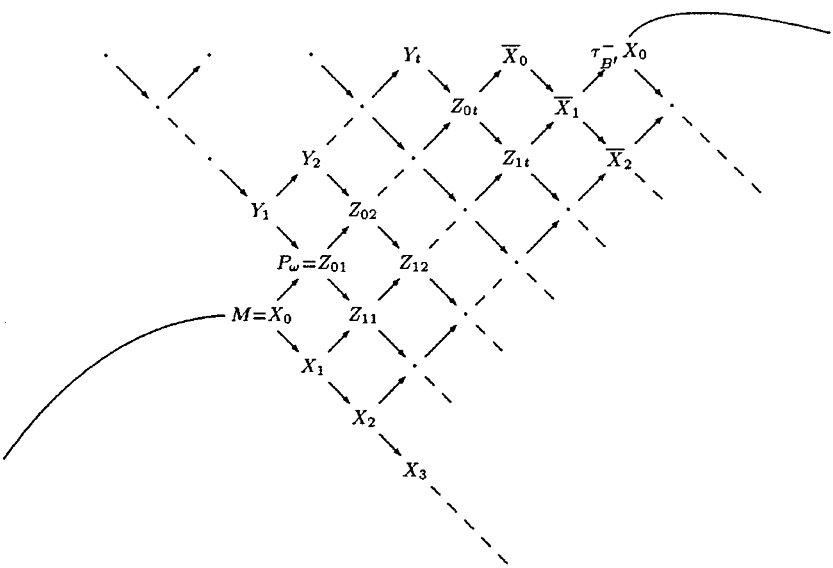
$$q_B(\mathbf{dim}\ X) = q_C(z_0) + 2 + q_B(z_0, e_a) + q_B(z_0, e_b) \geq 2.$$

The inequality due to the fact that $q_B(z_0, e_a) < 0$ (resp. $q_B(z_0, e_b) < 0$) would imply that $q_B(2z_0 + e_a) < 0$ (resp. $q_B(2z_0 + e_b) < 0$), which is impossible since B is tame and therefore q_B is weakly non-negative. This shows that q_B does not control mod_B.

(ii) ⇒ (iii) Assume that gl dim $B > 2$. The indecomposable B-module X constructed above satisfies $\dim_k \mathrm{End}_B(X) = \dim_k \mathrm{End}_C(Z) = 1$, therefore (ii) does not hold.

(iii) ⇒ (ii) Assume that gl dim $B \leq 2$ and let X be an indecomposable B-module. In (1.6) we recalled that $\mathrm{Ext}^2_B(X, X) = 0$, hence (ii) holds for X.

(iii) ⇒ (i) Assume that gl dim $B \leq 2$. By the description of the module category of a coil algebra given in [3], it is enough to show that $q_B(\mathbf{dim}\ X) \in \{0, 1\}$ for every indecomposable B-module X lying in a coil $\mathscr{C}$ of Γ_B containing both projectives and injectives. Hence we may assume inductively that B is obtained from a branched-critical algebra B' by an operation of type (ad 1) with pivot M in a coil $\mathscr{C}'$ of $\Gamma_{B'}$, and that $q_{B'}(\mathbf{dim}\ X) \in \{0, 1\}$ for every indecomposable B'-module X in $\mathscr{C}'$. The situation in $\mathscr{C}$ can be depicted as



where Z_{ij}, $i \geq 0, 1 \leq j \leq t$, is the unique indecomposable B-module whose restriction to B' is $X_i \oplus Y_j$ and whose dimension vector is $\mathbf{dim}\, Z_{ij} = \mathbf{dim}\, X_i + \mathbf{dim}\, Y_j + e_w$, and $\overline{X}_i$, $i \geq 0$, is the unique indecomposable B-module whose restriction to B' is X_i and whose dimension vector is $\mathbf{dim}\, \overline{X}_i = \mathbf{dim}\, X_i + e_w$. Then

$$\begin{aligned} q_B(\mathbf{dim}\, Z_{ij}) = {} & q_B(\mathbf{dim}\, X_i) + q_B(\mathbf{dim}\, Y_j) + 1 + \langle \mathbf{dim}\, X_i, \mathbf{dim}\, Y_j \rangle \\ & + \langle \mathbf{dim}\, Y_j, \mathbf{dim}\, X_i \rangle + \langle \mathbf{dim}\, X_i, e_w \rangle + \langle e_w, \mathbf{dim}\, X_i \rangle \\ & + \langle \mathbf{dim}\, Y_j, e_w \rangle + \langle e_w, \mathbf{dim}\, Y_j \rangle \\ = {} & q_{B'}(\mathbf{dim}\, X_i) + 2 + \langle e_w, \mathbf{dim}\, X_i \rangle + \langle e_w, \mathbf{dim}\, Y_j \rangle \\ = {} & q_{B'}(\mathbf{dim}\, X_i) + 2 - \langle \mathbf{dim}\, X_0, \mathbf{dim}\, X_i \rangle - \langle \mathbf{dim}\, Y_1, \mathbf{dim}\, X_i \rangle \\ & - \langle \mathbf{dim}\, X_0, \mathbf{dim}\, Y_j \rangle - \langle \mathbf{dim}\, Y_1, \mathbf{dim}\, Y_j \rangle \\ = {} & q_{B'}(\mathbf{dim}\, X_i) + 2 - 3 + \dim \operatorname{Ext}^1_{B'}(X_0, X_i). \end{aligned}$$

Since $\operatorname{Ext}^1_{B'}(X_0, X_i) \cong D\, \underline{\operatorname{Hom}}(\tau^-_{B'} X_i, X_0)$, if $\operatorname{Ext}^1_{B'}(X_0, X_i) \neq 0$ then $\dim_k \operatorname{Ext}^1_{B'}(X_0, X_i) = 1$ and $q_B(\mathbf{dim}\, Z_{ij}) = q_{B'}(\mathbf{dim}\, X_i) \in \{0, 1\}$. If $\operatorname{Ext}^1_{B'}(X_0, X_i) = 0$ then $q_{B'}(\mathbf{dim}\, X_i) = \dim_k \operatorname{Hom}_{B'}(X_i, X_i) - \dim_k \operatorname{Ext}^1_{B'}(X_i, X_i) = 1$, and $q_B(\mathbf{dim}\, Z_{ij}) = 0$. Similarly, $q_B(\mathbf{dim}\, \overline{X}_i) \in \{0, 1\}$. ∎

2.4 *Proof of the theorem.* (i) ⇒ (ii) This follows from (1.5), (2.2), and (2.3).

(ii) ⇒ (iii) Since A is tame, then q_A is weakly non-negative. Let X be an indecomposable A-module. If X is directing, then supp X is convex in A and induces a convex subcategory B of A which is a tilted algebra (see [25]). If X is not directing, by (1.6), there is a convex subcategory B of A containing supp X which is a coil algebra. By hypothesis, B is a branched-critical algebra and gl dim $B \leq 2$.

(iii) ⇒ (i) First, observe that the hypothesis implies that A is of polynomial growth. Indeed, q_A is weakly non-negative and A does not admit a convex subcategory B which is *pg*-critical (since such an algebra B is neither tilted nor branched-critical).

Let X be an indecomposable A-module. If X is directing and B is a convex tilted subcategory of A containing supp X, then $q_A(\mathbf{dim}\ X) =$

$q_B(\mathbf{dim}\ X) = 1$. If X is not directing, then by (1.6), there is a convex subcategory B of A containing supp X which is a coil algebra. By hypothesis, B is a branched-critical algebra with gl dim $B \leq 2$. By (2.3), q_B weakly controls mod_B. Hence $q_A(\mathbf{dim}\ X) = q_B(\mathbf{dim}\ X) \in \{0, 1\}$. ▮

2.5. COROLLARY. *Let A be a strongly simply connected algebra such that q_A weakly controls* mod_A. *Then for any indecomposable A-module X, we have*

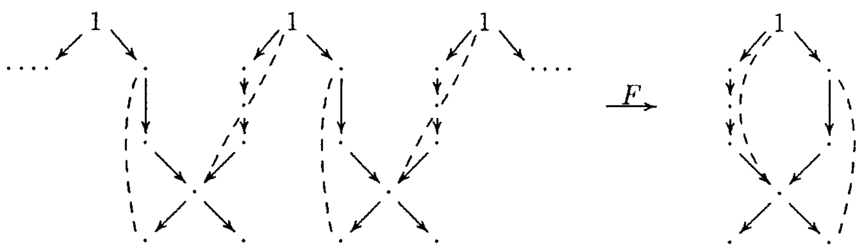
$$q_A(\mathbf{dim}\ X) = \chi_A(\mathbf{dim}\ X) = \dim_k \text{End}_A(X) - \dim_k \text{Ext}_A^1(X, X).$$

Proof. This follows from the proof of (i) ⇒ (iii) in (2.4). ▮

3. ALGEBRAS WITH A SIMPLY CONNECTED GALOIS COVER

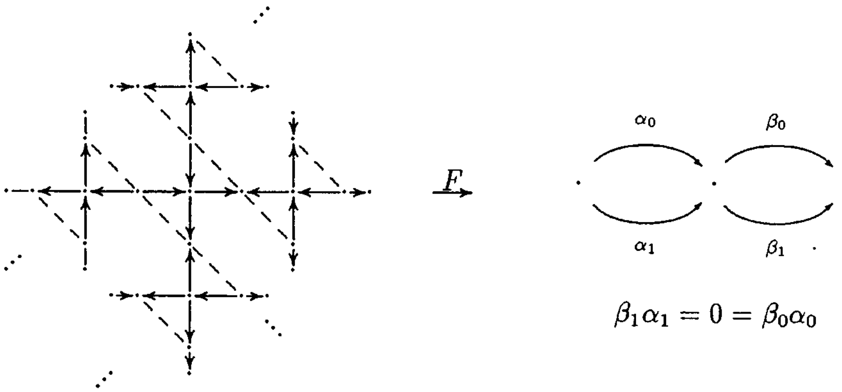
3.1. We are interested in the consideration of algebras A whose Tits form q_A weakly controls mod_A and which admit a Galois covering F: $\tilde{A} \to A$ such that $\tilde{A}$ is strongly simply connected (therefore the situation of the main theorem comes back when $A = \tilde{A}$). For concepts and results on the theory of Galois coverings the reader may see [5, 10].

EXAMPLES. (a) Consider the Galois covering F: $\tilde{A} \to A$ given by the action of the group $\mathbb{Z}$:



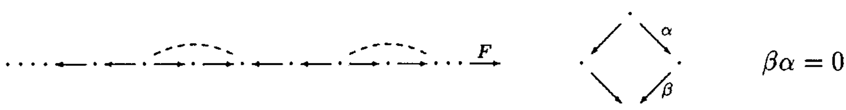
Clearly, q_A weakly controls mod_A. The cover $\tilde{A}$ is of polynomial growth and therefore A is of polynomial growth.

(b) In the next example the Galois covering $F\colon \tilde{A} \to A$ is defined by the action of the free group in two generators.

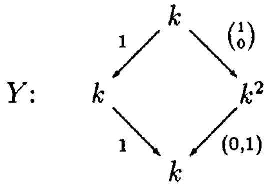


The algebra A is special biserial and q_A weakly controls mod_A [indeed, by [6, 8] indecomposable A-modules are associated to string words—for example, $\alpha_0^{-1}\beta_0^{-1}\beta_1\alpha_0$—and band words—for example, $\alpha_1^{-1}\beta_0^{-1}\beta_1\alpha_0$ or $\alpha_1^{-1}\alpha_0\alpha_1^{-1}\beta_0^{-1}\beta_1\alpha_0$. It is an easy exercise to show that in the first case an indecomposable A-module X has **dim** $X = (a, b, c)$ with $|b - a - c| \leq 1$, and in the second $b = a + c$. Since $q_A(a, b, c) = (a - b + c)^2$, then $q_A(\mathbf{dim}\, X) \in \{0, 1\}$]. Moreover, the algebra A is tame but not of polynomial growth while the cover $\tilde{A}$ is of polynomial growth.

(c) Consider the covering $F\colon \tilde{A} \to A$ with group $\mathbb{Z}$:



Obviously $\tilde{A}$ is locally representation-finite and for every indecomposable $\tilde{A}$-module X we have $q_{\tilde{A}}(\mathbf{dim}\, X) \in \{0, 1\}$. Nevertheless, A is representation-finite and admits an indecomposable module Y as



with $q_A(\mathbf{dim}\ Y) = 2$.

3.2. Let $A = kQ/I$ be an algebra such that Q has no oriented cycle. Let $F: \tilde{A} \to A$ be a Galois covering defined by the action of a group G of automorphisms of $\tilde{A}$. We will assume that G acts freely on the objects of $\tilde{A}$ (that is, the vertices of the quiver associated to $\tilde{A}$). Following [5], we denote by F_λ: $\operatorname{mod}_{\tilde{A}} \to \operatorname{mod}_A$ the push-down functor.

LEMMA. *Let* $X, Y \in \operatorname{mod}_{\tilde{A}}$; *then*

(a) $\operatorname{Ext}^i_A(F_\lambda X, F_\lambda Y) \cong \bigoplus_{g \in G} \operatorname{Ext}^i_{\tilde{A}}(X, Y^g)$,

(b) $\langle \mathbf{dim}\ F_\lambda X, \mathbf{dim}\ F_\lambda Y \rangle_A = \sum_{g \in G} \langle \mathbf{dim}\ X, \mathbf{dim}\ Y^g \rangle_{\tilde{A}}$.

Proof. Obviously (b) is a consequence of (a).

Let $P_1 \xrightarrow{\nu} P_0 \to Y \to 0$ be a projective presentation of Y in $\operatorname{mod}_{\tilde{A}}$; then $F_\lambda P_1 \xrightarrow{F_\lambda \nu} F_\lambda P_0 \to F_\lambda Y \to 0$ is a projective presentation of $F_\lambda Y$ in mod_A. Let Q be a projective $\tilde{A}$-module. Then recalling the definition of a covering functor we get the exact and commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{g\in G} \operatorname{Hom}_{\tilde{A}}(Q, P_1^g) & \to & \bigoplus_{g\in G} \operatorname{Hom}_{\tilde{A}}(Q, P_0^g) & \to & \bigoplus_{g\in G} \operatorname{Hom}_{\tilde{A}}(Q, Y^g) & \to 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
\operatorname{Hom}_A(F_\lambda Q, F_\lambda P_1) & \to & \operatorname{Hom}_A(F_\lambda Q, F_\lambda P_0) & \to & \operatorname{Hom}_A(F_\lambda Q, F_\lambda Y) & \to 0
\end{array}$$

Consider now a projective resolution η: $\cdots \to Q_{i+1} \to Q_i \to \cdots \to Q_0 \to X \to 0$ in $\operatorname{mod}_{\tilde{A}}$. Applying $\operatorname{Hom}_A(-, F_\lambda Y)$ to $F_\lambda \eta$, we get $\operatorname{Hom}_A(F_\lambda \eta, F_\lambda Y) \xleftarrow{\sim} \bigoplus_{g \in G} \operatorname{Hom}_{\tilde{A}}(\eta, Y^g)$. Calculating the ith homology of these sequences we get the result. ∎

3.3. Assume that the group of automorphisms G of $\tilde{A}$ acts freely on objects of $\tilde{A}$ and G is torsion-free. Then G acts also freely on $\operatorname{mod}_{\tilde{A}}$ (see [5]; in fact, if $X^g \cong X$, then supp X is a finite set of vertices fixed by g, then at least one vertex is fixed and $g = 1$). Therefore, by [5], the push-down functor F_λ: $\operatorname{mod}_{\tilde{A}} \to \operatorname{mod}_A$ preserves indecomposability.

The category $\tilde{A}$ may be given in the form $\tilde{A} = k\tilde{Q}/\tilde{I}$. The Tits form $q_{\tilde{A}}$ will be formally considered as taking values on vectors $v \in \mathbb{Z}^{(\tilde{Q}_0)}$, that is, vectors with finite support.

PROPOSITION. *Let $F: \tilde{A} \to A$ be a Galois covering defined by the action of a torsion free group G. Assume that $\tilde{A}$ is strongly simply connected and q_A weakly controls* mod_A. *Then $q_{\tilde{A}}$ is weakly non-negative.*

Proof. Assume $q_{\tilde{A}}$ is not weakly non-negative. Since $\tilde{A}$ is strongly simply connected, then by (1.3), there is a convex hypercritical subcategory B of $\tilde{A}$. Moreover $B = C[M]$ for a convex critical subcategory C of $\tilde{A}$ and an indecomposable postprojective C-module M.

Let Y_1, Y_2, Y_3 be three regular simple C-modules belonging to the mouth of three different (orthogonal) homogeneous tubes in Γ_C. Let $0 \neq f_i \in \operatorname{Hom}_C(M, Y_i)$, $i = 1, 2, 3$. The following $B = C[M]$-modules are indecomposable:

$$Z = \big(k, Y_1 \oplus Y_2, \gamma: k \to \operatorname{Hom}_C(M, Y_1 \oplus Y_2), 1 \mapsto (f_1, f_2)\big)$$

$$Z' = \big(k^2, Y_1 \oplus Y_2 \oplus Y_3, \gamma': k^2 \to \operatorname{Hom}_C(M, Y_1 \oplus Y_2 \oplus Y_3),$$
$$(1,0) \mapsto (f_1, f_2, 0), (0,1) \mapsto (0, f_2, f_3)\big).$$

Therefore $F_\lambda Z$ and $F_\lambda Z'$ are indecomposable A-modules. We shall evaluate $q_A(\mathbf{dim}\, F_\lambda Z)$ and $q_A(\mathbf{dim}\, F_\lambda Z')$.

(1) $q_A(\mathbf{dim}\, F_\lambda Y_1) = 0$: since $F_\lambda Y_1$ is indecomposable, if this were not the case, then $q_A(\mathbf{dim}\, F_\lambda Y_1) = 1$. In the tube of Γ_C where Y_1 sits, there is an indecomposable $\tilde{A}$-module $Y_1[2]$ with $\mathbf{dim}\, Y_1[2] = 2\, \mathbf{dim}\, Y_1$. Therefore $F_\lambda Y_1[2]$ is indecomposable and $q_A(\mathbf{dim}\, F_\lambda Y_1[2]) = q_A(2\, \mathbf{dim}\, F_\lambda Y_1) = 4$, contradicting the fact that q_A weakly controls mod_A.

(2) Let w be the extension vertex in B, that is, $M = \operatorname{rad} P_w$, where P_w is an indecomposable projective B-module. Let $a = F(w)$ be the corresponding vertex in Q. Then $\mathbf{dim}\, Z = 2\, \mathbf{dim}\, Y_1 + e_w$ and $\mathbf{dim}\, F_\lambda Z = 2\, \mathbf{dim}\, F_\lambda Y_1 + e_a$. We have

$$q_A(\mathbf{dim}\, F_\lambda Z) = q_A(2\, \mathbf{dim}\, F_\lambda Y_1 + e_a) = 1 + 2 q_A(\mathbf{dim}\, F_\lambda Y_1, e_a).$$

Since this number is 0 or 1, we get that $q_A(\mathbf{dim}\, F_\lambda Y_1, e_a) = 0$.

(3) Finally, $\mathbf{dim}\, F_\lambda Z' = 3\, \mathbf{dim}\, F_\lambda Y_1 + 2 e_a$ and $q_A(\mathbf{dim}\, F_\lambda Z') = 4 + 6 q_A(\mathbf{dim}\, F_\lambda Y_1, e_a) = 4$, which is a contradiction. This finishes the proof. ∎

3.4. The following is the main result of this section.

PROPOSITION. *Let $F: \tilde{A} \to A$ be a Galois covering defined by the action of a group G. Assume the following conditions*:

(a) *$\tilde{A}$ is strongly simply connected and* $\operatorname{gl\,dim} \tilde{A} \leq 2$.

(b) *For every indecomposable A-module X,*

$$q_A(\mathbf{dim}\ X) = \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}^1_A(X, X) \in \{0, 1\}.$$

Then $\tilde{A}$ is of polynomial growth and A is tame.

Proof. Let X be an indecomposable A-module. Let us observe that $\operatorname{Ext}^2_A(X, X) = 0$. Indeed, since gl $\mathbf{dim}\ \tilde{A} \leq 2$, also gl dim $A \leq 2$. Then

$$\begin{aligned} q_A(\mathbf{dim}\ X) &= \chi_A(\mathbf{dim}\ X) \\ &= \sum_{i=0}^{2} (-1)^i \dim_k \operatorname{Ext}^i_A(X, X) \text{ implies } \operatorname{Ext}^2_A(X, X) = 0. \end{aligned}$$

Since $\tilde{A}$ is strongly simply connected, by [28], G is a torsion free group. In view of (3.3) and (1.4), to show that $\tilde{A}$ is of polynomial growth, we must prove that $\tilde{A}$ does not admit a *pg*-critical convex subcategory. Otherwise, if B is a *pg*-critical convex subcategory of $\tilde{A}$, by (1.4), there is an indecomposable $\tilde{A}$-module Y with $\operatorname{Ext}^2_{\tilde{A}}(Y, Y) \neq 0$. Hence $F_\lambda Y$ is an indecomposable A-module with $\operatorname{Ext}^2_A(F_\lambda Y, F_\lambda Y) \neq 0$ by (3.2), a contradiction. Hence $\tilde{A}$ is of polynomial growth and by [28], A is tame.

ACKNOWLEDGMENTS

Part of this work was done while the second-named author was visiting the first at the University of São Paulo. The authors thank FAPESP, Brazil, and CONACyT and DGAPA, UNAM, México, for their support.

REFERENCES

1. I. Assem and A. Skowroński, Indecomposable modules over multicoil algebras, *Math. Scand.* **71** (1992), 31–61.
2. I. Assem and A. Skowroński, Multicoil algebras *in* "Proceedings, ICRA VI"; *Canad. Math. Conf. Proc.* **14** (1993), 29–67.
3. I. Assem, A. Skowroński, and B. Tomé, Coil enlargements of algebras, *Tsukuba J. Math.* **19**, No. 2 (1995).
4. K. Bongartz, Algebras and quadratic forms, *J. London Math. Soc.* (*2*) **28** (1983), 461–469.
5. K. Bongartz and P. Gabriel, Covering spaces in representation theory, *Invent. Math.* **65,** No. 3, (1981/1982), 331–378.
6. M. C. Butler and C. M. Ringel, Auslander–Reiten sequences with few middle terms and applications to string algebras, *Comm. Algebra* **15** (1987), 145–179.
7. W. W. Crawley-Boevey, On tame algebras and bocses, *Proc. London Math. Soc.* (*3*) **56** (1988), 451–483.
8. W. W. Crawley-Boevey, Functorial filtrations. II. Clans and the Gelfand problem, *J. London Math. Soc.* **40** (1989), 9–30.

9. V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* **173** (1976).
10. P. Dowbor and A. Skowroński, Galois coverings of representation infinite algebras, *Comment. Math. Helv.* **62** (1987), 311–337.
11. P. Gabriel, Indecomposable representation, II, *Sympos. Math. Inst. Naz. Alta Mat.* **11** (1973), 81–104.
12. P. Gabriel and A. V. Roiter, Representations of finite-dimensional algebras, *in* "Algebra VIII," Encyclopaedia Math. Sci., Vol. 73, Springer-Verlag, Berlin/New York, 1992.
13. O. Kerner, Tilting wild algebras, *J. London Math. Soc. Math. Soc.* **39** (1989), 29–47.
14. R. Nörenberg and A. Skowroński, Tame minimal non-polynomial growth strongly simplly connected algebras, *in* "Proceedings, ICRA VII," to appear.
15. J. A. de la Peña, On the representation type of one-point extensions of tame concealed algebras, *Manuscripta Math.* **61** (1988), 183–194.
16. J. A. de la Peña, Quadratic forms and the representation type of an algebra, *in* "Sonderforschungsbereich Diskrete Strukturen in der Mathematik, Bielefeld, 1990" Ergänzungsreihe 343, pp. 90–1003.
17. J. A. de la Peña, Algebras with hypercritical Tits form, *in* "Topics in Algebra," pp. 353–369, Banach Center Publications, Vol. 26, Pt. I, PWN, Warsaw, 1990.
18. J. A. de la Peña, On the dimension of the module-varietes of tame and wild algebras, *Comm. Algebra* **19**, No. 6, (1991), 1795–1807.
19. J. A. de la Peña, Tame algebras with sincere directing modules, *J. Algebra* **161** (1993), 171–185.
20. J. A. de la Peña, On the corank of the Tits form of a tame algebra, *J. Pure Appl. Algebra* **107** (1996), 89–105.
21. J. A. de la Peña, The Tits form of a tame algebra, *in* "Proceedings, Workshop on Representations of Algebras, México, 1994," CMS Proc. Series of AMS, Vol. 19, pp. 159–184, 1996.
22. J. A. de la Peña and A. Skowroński, Geometric and homological characterizations of polynomial growth simply connected algebras, *Invent. Math.* **126** (1996), 287–296.
23. J. A. de la Peña and A. Skowroński, Characterizations of strongly simply connected polynomial growth algebras, *Arch. Math.* **65** (1995), 391–398.
24. J. A. de la Peña and B. Tomé, Tame algebras with a weakly separating family of coils, *in* "Proceedings, ICRA VII," CMS Proc. Ser. Amer. Math Soc., Vol. 18, pp. 555–557, 1996.
25. C. M. Ringel, "Tame Algebras and Integral Quadratic Forms," Lecture Notes in Mathematics, Vol. 1099, Springer-Verlag, Berlin, 1984.
26. A. Skowroński, Simply connected algebras and Hochschild cohomologies, *in* "Proceedings, ICRA IV (Ottawa, 1992)," *Canad. Math. Soc. Proc.* **14** (1993), 431–447.
27. a. Skowroński, Criteria for polynomial growth of algebras, *Bull. Polish Acad. Sci. Math.* **42**, No. 3 (1994), 173–183.
28. A. Skowroński, Tame algebras with simply connected Galois coverings, preprint, Torun, Poland, 1995.