ELSEVIER

Discrete Mathematics 152 (1996) 141–145

DISCRETE MATHEMATICS

Independent sets which meet all longest paths

H. Galeana-Sánchez[a,*], H.A. Rincón-Mejía[b]

[a] *Instituto de Matemáticas, U.N.A.M., †Cindad Universitaria, C.P. 04510, Mexico D.F., Mexico*
[b] *Departamento de Matemáticas, Facultad de Ciencias, U.N.A.M., †Cindad Universitaria, C.P. 04510, Mexico D.F., Mexico*

Received 27 June 1994

Abstract

We prove some sufficient conditions for a directed graph to have the property of a conjecture of J.M. Laborde, Ch. Payan and N.H. Huang (1982): "Every directed graph contains an independent set which meets every longest directed path".

1. Introduction

Let G be a directed graph, and denote by $V(G)$ its vertex-set, by $A(G)$ its arc-set, $X(G)$ denotes its chromatic number, and $\lambda(G)$ the length of the longest directed path. Independently, B. Roy and T. Gallai proved that $X(G) \leqslant \lambda(G)$. Consider an independent set S ('stable' set), and denote by $G - S$ the subgraph of G induced by $V(G) - S$; in 1982, Laborde, Payan and Huang conjectured a plausible looking extension of this result.

Conjecture 1 (Grillet [6]). Every directed graph G contains an independent set S such that $\lambda(G - S) < \lambda(G)$.

A path $\mathscr{M} = (x_1, \ldots, x_k)$ will always be a directed and elementary path; it is a longest path if k is maximum, and a non-augmentable path if for every vertex a, none of the sequences $(a, x_1, x_2, \ldots, x_k)$,$(x_1, x_2, \ldots, x_i, a, x_{i+1}, \ldots, x_k)$ or $(x_1, x_2, \ldots, x_k, a)$ are paths. The anti-path of $\mathscr{M}$ is the sequence $\mathscr{M}^{-1} = (x_k, x_{k-1}, \ldots, x_1)$, which is not necessarily a path.

Undefined terms are in [1].

The problem considered in this paper is: For which graphs do we have $\mathscr{M} \cap S \neq \emptyset$ for some independent set S and for every longest path $\mathscr{M}$?; or for every non-augmentable path $\mathscr{M}$?

---

* Corresponding author.

SSDI 0012-365X(94)00261-4

Remark. It is not true that every maximum independent set meets every longest path. Consider, for example, the graph consisting of two disjoint cycles $[x_0, x_1, x_2, x_3, x_4, x_0]$ and $[y_0, y_1, y_2, y_3, y_4, y_0]$, with the arcs x_1, x_0, x_1x_2, x_2x_3, x_4,x_3, x_4x_0, y_0y_1, y_0y_4, y_2y_1, y_3y_2, y_3y_4; and all the x_iy_j except x_0y_0. Clearly, the independent set $\{x_0y_0\}$ is maximum and does not meet the longest path, which is $(x_1, x_2, x_3, y_3, y_2, y_1)$.

2. Non-augmentable paths and kernels

A graph has a kernel S if S is an independent set and if every vertex which is not in S has at least one successor in S. Many classes of graphs (and in particular those which have no odd circuits) have kernels (see for instance [1, 4]). The following result is a slight generalization of a result proved in [6].

Theorem 1. *Let A be a subset of $V(G)$ which contains every vertex a such that each of the maximal (resp. longest) anti-path starting at a contains all the successors of a. If the subgraph G_A induced by A has a kernel S, then S is an independent set which meets all the non-augmentable (resp. longest) paths and $\lambda(G - S) < \lambda(G)$.*

Let $\mathscr{M}$ be a non-augmentable path which does not meet S, and let z be its terminal vertex. Since $\mathscr{M}$ is non-augmentable, we have $z \in A$, and, consequently, z has a successor in S; this implies $z \in \mathscr{M} \cap S$. A contradiction.

Theorem 2. *Let P denote the graph with vertices a, b, c, d and arcs (a, b), (c, b), (c, d), and let Q denote the graph with vertices a, b, c, d and arcs (a, b), (c, b), (c, d), (b, d). If G is a graph with no pair of parallel arcs, no subgraph isomorphic to P and no subgraph isomorphic to Q, then every maximal independent set meets every non-augmentable path.*

Let $\mathscr{M} = (x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_k)$ be a non-augmentable path which does not meet the maximal independent set S; by the maximality of S, each of these vertices is adjacent to S. By the maximality of $\mathscr{M}$, the number of arcs going from S to x_1 is $m(S, x_1) = 0$, and the number of arcs going from x_k to S is $m(x_k, S) = 0$. Let c be the last vertex x_i of the sequence with $m(S, x_i) = 0$; let b be the next vertex in the sequence. Then $m(S, c) = 0$, $m(S, b) \neq 0$. Let d be a successor of c in S and let a be a predecessor of b in S. Since $\mathscr{M}$ is non-augmentable, $(d, b) \notin A(G)$, $(c, a) \notin A(G)$; and the vertices a, d are distinct and non-adjacent. Thus, the subgraph induced by $\{a, b, c, d\}$ is either isomorphic to P or to Q.

Remark. When the vertices of G are the elements of a poset, and when the arcs of G represent the partial order, we have a stronger result due to Grillet [6], who proved that if every induced subgraph isomorphic to $P = \{(a, b), (c, b), (c, d)\}$ is contained in an induced subgraph isomorphic to $Q = \{(a, b), (c, b), (c, d), (c, e), (e, b)\}$, then every maximal independent set meets every non-augmentable path.

3. The main results

Now, for a graph H, we denote by $I(H)$ the set of initial vertices for the longest paths in H, and by $T(H)$ the set of terminal vertices for the longest paths in H.

We say that a vertex x of H satisfies the property $P(H)$ if for every arc $(y, x) \in H[I(H)]$ (the subgraph of H induced by $I(H)$ which is not a double edge, at least one of the following conditions hold:

(i) every longest path of H with initial vertex y contains x;

(ii) every longest path of H which contains x, and does not start at x, also contains y.

Lemma. *If each subgraph H of G has a vertex in $I(H)$ which satisfies the property $P(H)$, then $I(G)$ contains an independent set S such that $\lambda(G - S) < \lambda(G)$.*

A similar result was proved in [7], and the proof can easily be adapted.

Theorem 3. *If in a graph G every circuit without double edge has a vertex with inner demi-degree $\leqslant 1$ or outer demi-degrees $\leqslant 1$, then $I(G)$ contains an independent set S such that $\lambda(G - S) < \lambda(G)$.*

Proof. By the lemma, it suffices to show that a graph G satisfying the condition has a vertex $x \in I(G)$ with the property $P(G)$.

By contradiction. Suppose that the above statements were false and let x be a vertex in $I(G)$, then there is a vertex $y \in I(G)$ such that (y, x) is not a double edge of G, y is the origin of a longest path of G not containing x; and there exists a longest path of G not starting in x which contains x but does not contain y. Again, there is a vertex $z \in I(G)$ with (z, y) not a double edge of G, z is the origin of a longest path not containing y and there exists a longest path not starting in y which contains y but does not contain z. Continuing this procedure, we obtain a circuit without double edge $\vec{C}_n = (x_0, x_1, \ldots, x_{n-1}, x_0)$ such that for each i $(1 \leqslant i \leqslant n - 1)$, there is:

(1) A longest path with origin x_i not containing x_{i+1} (notation mod. n) and

(2) A longest path not starting in x_i, which contains x_i but does not contain x_{i-1} (notation mod. n). It follows from (1) that the outer demi-degree of each vertex in $\vec{C}_n$ is at least two and (2) implies that the inner demi-degree of each vertex in $\vec{C}_n$ is at least two, contradicting the hypothesis.

In what follows we denote by K_n^* the complete digraph on n vertices and every edge is a double edge. If G and H are isomorphic digraphs we write $D \cong H$. □

Theorem 4. *Let G be a digraph such that every circuit without double edges has a vertex x which satisfies:* $G[\Gamma_G^-(x)] \cong K_{n(x)}^*$ *(where* $n(x) = \delta_G^-(x)$ *the inner demi-degree of x, and* $G[\Gamma_G^-(x)]$ *is the subgraph of G induced by the inner neighbors of x) or* $G[\Gamma_G^+(x)] \cong K_{m(x)}^*$, $m(x) = \delta_G^+(x)$. *Then there exists an independent set* $S \subseteq I(G)$ *with* $\lambda(G - S) < \lambda(G)$.

Proof. We will prove that any digraph satisfying the hypothesis of Theorem 4 has a vertex $x \in I(G)$ which satisfies $P(G)$.

By contradiction. Suppose that the statement were false. Proceeding as in the proof of Theorem 3 we obtain a circuit without double edges $\vec{C}_n = (x_0, x_1, \dots, x_{n-1}, x_0)$ such that for each i, $(0 \leqslant i \leqslant n-1)$ there is:

(1) A longest path starting at x_i not containing x_{i+1} (notation mod. n) and

(2) A longest path not starting at x_i which contains x_i and does not contain x_{i-1} (notation mod. n). Now we analyze the two possible cases:

Case 1. There exists a vertex $x_k \in \vec{C}_n$ with $G[\Gamma_G^-(x_k)] \cong K_{n(x_k)}^*$. $n(x_k) = \delta_G^-(x_k)$. Let $\alpha = (z_0, z_1, \dots, z_p)$ a longest path with $x_k = z_j$ $(0 < j \leqslant p)$ not containing x_{k-1}, then we have $\{(z_{j-1}, x_k), (x_{k-1}, x_k)\} \subseteq A(G)$, hence $\{(z_{j-1}, x_{k-1}), (x_{k-1}, z_{j-1})\} \subseteq A(G)$ and $\alpha' = (z_0, \dots, z_{j-1}, x_{k-1}, x_k, z_{j+1}, \dots, z_p)$ is a directed path with length greater than those of α, contradicting the choice of α.

Case 2. There exists a vertex $x_k \in \vec{C}_n$ with $G[\Gamma_G^+(x_k)] \cong K_{m(x_k)}^*$, $m(x_k) = \delta_G^+(x_k)$. Let $\beta = (y_0 = x_k, y_1, \dots, y_q)$ a longest path starting in x_k and not containing x_{k+1}, then we have $\{(y_1, x_{k+1}), (x_{k+1}, y_1)\} \subseteq A(G)$ and $\beta' = (y_0 = x_k, x_{k+1}, y, \dots, y_q)$ a path longer than β contradicting the choice of β. □

Theorem 5. *Let* $C \subseteq (V(G) - T(G))$. *If* $G - C$ *has a kernel S then* $\lambda(G - S) < \lambda(G)$.

Proof. Suppose that there exists a longest path α with $V(\alpha) \cap S = \emptyset$ and denote by z_0 the endpoint of α. Clearly, $z_0 \in [(V(G) - C) \cap (V(G) - S)]$ and since S is a kernel of $G - C$ there exists $y \in S$ such that $(z_0, y) \in A(G)$ Hence $\alpha' = \alpha \cup (z_0, y)$ is a path longer than α, contradicting the choice of α. □

Theorem 6. *Let* $C \subseteq (V(G) - T(G)) \cup \{x \in V(G) \mid G[\Gamma_G^-(x)] \cong K_{n(x)}^*, n(x) = \delta_G^-(x)\}$. *If* $G - C$ *has a kernel then there exists an independent set* $S \subseteq V(G)$ *such that* $\lambda(G - S) < \lambda(G)$.

Proof. Denote by $C' = C \cap \{x \in V(G) \mid G[\Gamma_G^-(x)] \cong K_{n(x)}^*, n(x) = \delta_G^-(x)\}$. We proceed by induction on the cardinality of C'. If $C' = \emptyset$ then Theorem 6 follows directly from Theorem 5. Suppose that $C' \neq \emptyset$ and let N be a kernel of $G - C$. Since N is an independent set, we can assume that there exists a longest path $\alpha = (z_0, z_1, \dots, z_n)$ such that $N \cap \alpha = \emptyset$.

Case 1. $z_n \in V(G) - C$. We have $z_n \in (V(G) - C) \cap (V(G) - N)$ hence there exists $y \in N$ with $(z_n, y) \in A(G)$, and $\alpha' = \alpha \cup (z_n, y)$ is a path, contradicting the choice of α.

Case 2. $z_n \in C$. Clearly $z_n \in C'$. We prove that $N \cup \{z_n\}$ is an independent set. By contradiction, suppose that there exists $s \in N$ with $\{(s, z_n), (z_n, s)\} \cap A(G) \neq \emptyset$. As in Case 1 we see that $(z_n, s) \notin A(G)$, hence $(s, z_n) \in A(G)$. Now the hypothesis implies $\{(z_{n-1}, s), (s, z_{n-1})\} \subseteq A(G)$ and $\alpha' = (z_0, \ldots, z_{n-1}, s, z_n)$ is a path, contradicting the choice of α. It follows that $N \cup \{z_n\}$ is an independent set. In fact it is a kernel of $G - C_1$, where $C_1 = C - \{z_n\}$ and it follows from the inductive hypothesis that there exists an independent set $S \subseteq V(G)$ with $\lambda(G - S) < \lambda(G)$.

Corollary 1. *Let* G *be a digraph. If there exists a set* $C \subseteq (V(G) - T(G)) \cup \{x \in V(G) \mid G[\Gamma_G^-(x)] \cong K_{n(x)}^*, n(x) = \delta_G^-(x)\}$ *intersecting each odd circuit then there exists an independent set* $S \subseteq V(G)$ *such that* $\lambda(G - S) < \lambda(G)$.

Remark 2. Clearly a digraph G satisfies Conjecture 1 if and only if G^{-1} does it (G^{-1} denotes the reverse digraph of G, obtained from G by reversing the direction of the arcs). Hence by applying the principle of directional duality, we have that for each theorem or corollary, there is a corresponding theorem or corollary obtained by replacing the kernel by cokernel, $I(G)$ by $T(G)$, $\delta_G^+(x)$ by $\delta_G^-(x)$, $\Gamma_G^+(x)$ by $\Gamma_G^-(x)$.

Acknowledgements

The authors wish to thank Professor Claude Berge for many suggestions which improved substantially the final form of this paper.

References

[1] C. Berge, Graphs (North-Holland, Amsterdam 1985).
[2] P. Duchet and H. Meyniel, Une generalization du theoreme de Richardson sur l'existence de noyaux dans les graphes orientes, Discrete Math. 43 (1983) 21–27.
[3] H. Galeana-Sánchez and V. Neumann-Lara, On kernels and semikernels of digraphs, Discrete Math. 48 (1984) 67–76.
[4] H. Galeana-Sánchez and V. Neumann-Lara, On kernel perfect critical digraphs, Discrete Math. 59 (1986) 257–265.
[5] T. Gallai, On directed paths and circuits, in: P. Erdős and G. Katona eds., Theory of Graphs, Proc. Colloq. Tihany, 1966 (Academic Press, New York, 1986) 115–118.
[6] P.A. Grillet, Maximal chains and antichains, Fundam. Math. 65 (1969) 157–167.
[7] J.M. Laborde, C. Payan and N.H. Huang, Independent sets and longest directed paths in digraphs, Graphs and Other Combinatorial Topics, Proc. Third Czechoslovak Symp. on Graph Theory (1982) 173–177.
[8] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, Rev. Francaise Inform. Recherche Oper. 1 (5) (1967) 129–132.