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INVARIANT PSEUDOMETRICS ON PALAIS PROPER G-SPACES

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Abstract. Let G be a locally compact Hausdorff group. It is proved that: (1) on each Palais proper G-space X there exists a compatible family of G-invariant pseudometrics; (2) the existence of a compatible G-invariant metric on a metrizable proper G-space X is equivalent to the paracompactness of the orbit space X/G; (3) if in addition G is either almost connected or separable, and X is locally separable, then there exists a compatible G-invariant metric on X.

1. Introduction and main results

Throughout this paper the letter G will denote a locally compact and Hausdorff topological group, unless stated otherwise. All topological spaces, or merely, spaces are assumed to be Tychonoff (= completely regular and Hausdorff). All equivariant or G-maps are assumed to be continuous. The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in [17], [18] and [21]. Our basic reference on proper group actions is Palais' article [18]. Other good sources are two papers of H. Abels: [1] and [2].

By a G-space we mean a space together with a fixed continuous action of the group G on it.

The notion of a proper G-space under consideration was introduced in 1961 by R. Palais [18] with the purpose to extend a substantial portion of the existing theory of compact group actions to the case of locally compact ones.

A G-space X is called *Palais proper* [18, Definition 1.2.2] if each point of X has a neighborhood V such that for every point of X there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has

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compact closure in G. In such a case the sets V and U are called *thin* relative to each other. Clearly, if G is compact every G-space is proper.

Let us mention yet another related notion. A G-space X is *Bourbaki proper* [7, Ch. III, §4.4] if any two points of X have relative thin neighborhoods. In the language of topological dynamics, the dynamical systems (= $\mathbf{R}$-spaces) satisfying this condition are called *dispersive* [6, Ch. IV]. Note that a G-space is Bourbaki proper iff the map $G \times X \to X \times X$; $(g, x) \mapsto (gx, x)$ is perfect in the sense that it is closed and the inverse image of any compact set is compact [7, §4.4].

Clearly, Palais proper implies Bourbaki proper. For X a locally compact G-space the two notions coincide (see e.g., [18, Theorem 1.2.9]); if in addition G is a discrete group, then we get the classical notion of a *properly discontinuous* action here.

In general a G-space is Palais proper iff it is Bourbaki proper and the orbit space X/G is regular [18, p. 303]. There are Bourbaki proper G-spaces which are not Palais proper [16, p. 303]. The orbit space of any Palais proper G-space is a Tychonoff space [18, Proposition 1.2.8].

Important examples of Palais proper G-spaces are the coset spaces $G/H = \{gH \mid g \in G\}$ with $H \subset G$ a compact subgroup, letting G act on G/H by left translations. The reader can find other interesting examples in [1], [2], [4] and [18].

In the sequel we will use the term "proper G-space" only for Palais proper G-spaces.

In [18] R. Palais proved that if G is a Lie group then on each separable metrizable proper G-space X there is a compatible G-invariant metric. The same holds true for G an arbitrary metrizable group; this was observed by J. de Vries [20]. J. L. Koszul [14, Ch. 1, Theorem 3] proved the existence of a compatible G-invariant metric on a locally compact metrizable proper G-space for an arbitrary G. In the general case of an arbitrary metrizable proper G-space the problem still remains open (even for $G = \mathbf{R}$). Some important questions of equivariant theory of retracts also reduce to this problem (see e.g., [5] and [8]).

One of the purposes of the present paper is to establish the following uniform analogue of the above mentioned result of Palais:

THEOREM A. *On each proper G-space there exists a compatible family of G-invariant pseudometrics.*

Here a pseudometric ρ on a G-space X is called invariant or G-invariant if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$; $x, y \in X$. A family $\{\rho_i\}$ of pseudometrics is called compatible (or consistent) if the topology generated by $\{\rho_i\}$ is the original topology of X, that is to say, the sets of the form $\big\{y \in X \mid \rho_k(y, x) < r\big\}$ constitute a subbase of the topology of X, where $\rho_k \in \{\rho_i\}$, $x \in X$ and $r > 0$.

In other words Theorem A asserts that each proper action is *uniformly equicontinuous* with respect to some compatible uniformity on X (see Proposition 1 below).

It is interesting to compare Theorem A with a result of de Groot [10] (see also [15]) asserting that if G is a second countable group, then on each metrizable (not necessarily proper) G-space there exists a metrizable uniformity μ, compatible with its topology, such that each homeomorphism $g : X \to X$ $(g \in G)$ is μ-uniformly continuous.

It turns out that the existence of a single compatible G-invariant metric on a metrizable proper G-space is conjugated with the paracompactness of the orbit space. Namely, we have

THEOREM B. *Let X be a metrizable proper G-space. Then the following are equivalent:*

(1) *The orbit space X/G is metrizable.*
(2) *The orbit space X/G is paracompact.*
(3) *There is a compatible invariant metric on X.*

The following corollary of Theorem B slightly generalizes Palais' theorem on the existence of invariant metrics [18, Theorem 4.3.4]:

COROLLARY 1. *Let G be either almost connected or separable. Then every metrizable, locally separable, proper G-space X admits a compatible invariant metric.*

Recall that G is called *almost connected* if the group of its connected components is compact. Such a group has a maximal compact subgroup K, i.e., every compact subgroup of G is conjugate to a subgroup of K [1, Theorem A.5]. The corresponding theorem on Lie groups can be found in [12, Ch. XV, Theorem 3.1].

2. Proofs

First of all we recall some necessary definitions.

If X is a G-space, for any $x \in X$ we denote $G_x = \{g \in G \mid gx = x\}$, the stabilizer (or stationary subgroup) of x.

For a subset $S \subset X$ and for a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H,\, s \in S\}$. In particular, $G(x)$ denotes the orbit $\{gx \in X \mid g \in G\}$ of x. The orbit space is denoted by X/G.

DEFINITION [18, p. 305]. Let G be a topological group, H be a closed subgroup of G and X be a G-space. An H-invariant subset $S \subset X$ is called an H-slice in X if $G(S)$ is open in X and there is a G-map $f : G(S) \to G/H$ such that $S = f^{-1}(eH)$, where e denotes the unity of G. The saturation $G(S)$ will be said to be a *tubular set* (more precisely, an *H-tube*). If $G(S)$

$= X$ then we say that S is a *global* H-slice of X. If $H = G_x$ for some $x \in S$ then we say that S is a *slice at the point* x.

It is useful to notice that the map $f : G(S) \to G/H$ is uniquely determined by S as follows: $f(gs) = gH$ for all $gs \in G(S)$ with $g \in G$, $s \in S$.

In [18] Palais established that if G is a Lie group then at each point of a proper G-space X there exists a slice. Using this result, Abels [2] proved that each point $x \in X$ is contained in some H-slice (H depending upon x but not necessarily equal to G_x) even when G is an arbitrary (locally compact) group. This result will play a central rule in our proofs.

LEMMA. *Let H be a compact subgroup of G and X be a proper G-space admitting a global H-slice S. Then there is a compatible family of invariant pseudometrics on X.*

If in addition X is metrizable then there is a compatible invariant metric on X.

PROOF. Let $f : X \to G/H$ be the G-map with $S = f^{-1}(eH)$. Choose a neighborhood W of the point $eH \in G/H$ that has compact closure in G/H. Then the set $U = f^{-1}(W)$ is a *small* subset of X, i.e., each point $x \in X$ has a neighborhood, thin relative to U. It then follows that

$$(1) \qquad \begin{cases} \text{for any compact subset } A \subset X \text{ the set} \\ \langle A, U\rangle = \{g \in G \mid gA \cap U \neq \emptyset\} \quad \text{has compact closure in } G. \end{cases}$$

Take a compatible uniformity $\mathcal{W}$ on X and let $\mathcal{D} = \{d_i\}$ be the family of all bounded pseudometrics on X that are uniformly continuous with respect to the product uniformity on $X \times X$. Then the sets of the form $V(d_i, \varepsilon, x) = \{y \in X \mid d_i(y,x) < \varepsilon\}$ where $d_i \in \mathcal{D}$, $\varepsilon > 0$, $x \in X$, constitute a base for the original topology of X [13, Ch. 6, Theorem 19].

For every $d_i \in \mathcal{D}$, we define

$$r_i(x) = d_i(x, X \setminus U), \quad x \in X.$$

Then for any $x, y \in X$, we have $r_i(x) - r_i(y) \leqq d_i(x,y)$, and hence

$$r_i(x) + r_i(z) \leqq d_i(x,y) + r_i(y) + r_i(z).$$

Therefore, if we write

$$\mu_i(x,y) = \min\left\{ d_i(x,y),\ r_i(x) + r_i(y)\right\}, \quad x, y \in X$$

then it is obvious that μ_i is a pseudometric on X. Define

$$\rho_i(x,y) = \sup\left\{ \mu_i(gx, gy) \mid g \in G\right\}.$$

Clearly, ρ_i is a G-invariant pseudometric on X. Show that the family $\mathcal{P}=\{\rho_i\}$ is compatible with the topology of X. For, let $\{x_\alpha\}$ be a net in X converging to a point $x_0\in X$ relative to the topology generated by $\mathcal{P}$. Take an arbitrary basic neighborhood $V(d_i,\varepsilon,x_0)$ of x_0 in the original topology of X. As $G(U)=X$, there is an element $g_0\in G$ such that $g_0x_0\in U$. As the map $g_0^{-1}: X\to X$ is continuous, there are $d_j\in\mathcal{D}$ and $\delta>0$ such that $V(d_j,\delta,g_0x_0)\subset U$ and $g_0^{-1}\big(V(d_j,\delta,g_0x_0)\big)\subset V(d_i,\varepsilon,x_0)$.

The inclusion $V(d_j,\delta,g_0x_0)\subset U$ implies that $r_j(g_0x_0)\geqq\delta>0$. Since $\{x_\alpha\}$ converges to x_0 in the topology generated by $\mathcal{P}$, there is an index α_0 such that $\rho_j(x_\alpha,x_0)<\delta/2$ for all $\alpha\geqq\alpha_0$. As $\mu_j(g_0x_\alpha,g_0x_0)\leqq\rho_j(x_\alpha,x_0)$, we see that $\mu_j(g_0x_\alpha,g_0x)<\delta/2$. Now, as $r_j(g_0x_\alpha)+r_j(g_0x_0)\geqq r_j(g_0x_0)\geqq\delta$, we infer that $d_j(g_0x_\alpha,g_0x_0)<\delta/2$; so $g_0x_\alpha\in V(d_j,\delta/2,g_0x_0)$. Therefore $x_\alpha\in V(d_i,\varepsilon,x_0)$ for all $\alpha\geqq\alpha_0$, showing that $\{x_\alpha\}$ converges to x_0 relative to the original topology of X.

Conversely, assume that a net $\{x_\alpha\}\subset X$ converges to a point $x_0\in X$ relative to the original topology of X, while $\{x_\alpha\}$ does not converge to x_0 relative to the topology generated by $\mathcal{P}$. Then for some $\varepsilon_0>0$ and for some pseudometric $\rho_i\in\mathcal{P}$ there must be a subnet $\{y_\gamma\}\subset\{x_\alpha\}$ such that $\rho_i(y_\gamma,x_0)\geqq\varepsilon_0$ for all indices γ. Therefore $\mu_i(g_\gamma y_\gamma,g_\gamma x_0)\geqq\varepsilon_0/2$ for a suitable net $\{g_\gamma\}\subset G$. Consequently, $r_i(g_\gamma y_\gamma)+r_i(g_\gamma x_0)\geqq\varepsilon_0/2$, yielding that $\{g_\gamma\}\subset\langle A,U\rangle$, where $A=\{x_0\}\cup\{y_\gamma\}$. As A is compact, it then follows from (1) that $\langle A,U\rangle$ has a compact closure, and hence, $\{g_\gamma\}$ contains a convergent subnet. Without loss of generality we can assume that $\{g_\gamma\}$ itself converges to a limit, say $g\in G$. Then by continuity of the action of G on X, the nets $\{g_\gamma x_0\}$ and $\{g_\gamma y_\gamma\}$ converge to the same limit gx_0, which yields that there is an index γ_0 such that $d_i(g_\gamma y_\gamma,g_\gamma x_0)<\varepsilon_0/2$ whenever $\gamma\geqq\gamma_0$. This contradicts the condition $d_i(g_\gamma y_\gamma,g_\gamma x_0)\geqq\mu_i(g_\gamma y_\gamma,g_\gamma x_0)\geqq\varepsilon_0/2$.

The proof of the second claim is still simpler. □

Proof of Theorem A. For each orbit $G(a)\subset X$ we fix an H_a-tubular neighborhood W_a of $G(a)$ where H_a is a compact subgroup of G; this is possible by [2, Theorem 3.3]. Using the complete regularity of the orbit space X/G [18, pp. 302–303], one can find an invariant neighborhood U_a of $G(a)$ with $\overline{U}_a\subset W_a$, and an invariant function $\varphi_a: X\to[0,1]$ such that $G(a)\subset\varphi_a^{-1}(1)$ and $X\setminus U_a\subset\varphi_a^{-1}(0)$.

By Lemma, each W_a admits a family $\{\rho_i\}$ of G-invariant pseudometrics, generating the topology of W_a. Without loss of generality one can assume that all these pseudometrics are bounded by 1.

Now we extend every pseudometric ρ_i to the whole set X as follows:

$$\widetilde{\rho_i}(x,y)=\begin{cases}\sup\limits_{t\in\overline{U}_a}\left|\varphi_a(x)\rho_i(x,t)-\varphi_a(y)\rho_i(y,t)\right|, & \text{if } x,y\in\overline{U}_a;\\ \sup\limits_{t\in\overline{U}_a}\varphi_a(x)\rho_i(x,t), & \text{if } x\in\overline{U}_a, y\notin\overline{U}_a;\\ 0, & \text{if } x,y\notin\overline{U}_a.\end{cases}$$

It is easily seen that $\widetilde{\rho_i}$ is a pseudometric on X. Its invariance follows directly from the invariance of the pseudometric ρ_i and of the function φ_a. We will denote by $\mathcal{R}$ the totality of all pseudometrics $\widetilde{\rho_i}$, corresponding to all the tubular neighborhoods W_a.

Let τ be the original topology of X and $\widetilde{\tau}$ the topology generated by $\mathcal{R}$. We have to verify that $\tau=\widetilde{\tau}$. For the inclusion $\widetilde{\tau}\subset\tau$ it suffices to prove that for every pseudometric $\widetilde{\rho}\in\mathcal{R}$, every point $b\in X$ and every $\varepsilon>0$ there is a neighborhood $V\in\tau$ of b which is contained in the set $O=\left\{x\in X \mid \widetilde{\rho}(b,x)<\varepsilon\right\}$.

For, let $\widetilde{\rho}$ correspond to a tubular neighborhood W_a, $a\in X$. First, we assume that $b\in\overline{U}_a$. By continuity of the function φ_a and of the pseudometric ρ, one can choose a neighborhood $V\in\tau$ of b such that $\left|\varphi_a(b)-\varphi_a(x)\right|<\varepsilon/4$ and $\rho(b,x)<\varepsilon/4$ for all $x\in V$. Then

$$\left|\varphi_a(x)\rho(x,t)-\varphi_a(b)\rho(b,t)\right|\leqq\left|\varphi_a(b)-\varphi_a(x)\right|\rho(b,t)+\varphi_a(x)\left|\rho(b,t)-\rho(x,t)\right|.$$

Since $\rho(b,t)\leqq 1$, $\varphi(x)\leqq 1$ and $\left|\rho(b,t)-\rho(x,t)\right|\leqq\rho(b,x)$, then for every $x\in V$ we will have

$$\left|\varphi_a(x)\rho(x,t)-\varphi_a(b)\rho(b,t)\right|\leqq\left|\varphi_a(b)-\varphi_a(x)\right|+\rho(b,x)<\varepsilon/4+\varepsilon/4=\varepsilon/2,$$

which implies that $\widetilde{\rho_i}(b,x)<\varepsilon$. Hence, V is as required.

If $b\notin\overline{U}_a$ then $\varphi_a(b)=0$. Again, by continuity of φ_a there is a neighborhood $S\in\tau$ of b such that $\varphi_a(x)<\varepsilon$ for all $x\in V$. Then

$$\widetilde{\rho}(b,x)=\sup_{t\in\overline{U}_a}\varphi_a(x)\rho(x,t)\leqq\varphi_a(x)<\varepsilon \quad\text{whenever}\quad x\in V.$$

So, $V\subset O$.

Let us pass to the inclusion $\tau\subset\widetilde{\tau}$. For, let $Q\in\tau$ and $a\in Q$ be an arbitrary point. Consider the tubular set W_a and its invariant open subset U_a above chosen. As $Q\cap U_a\in\tau$, there are pseudometrics ρ_k, $k=1,\dots,n$, corresponding to W_a such that the set $T=\left\{x\in X \mid \rho_k(a,x)<\varepsilon,\ k=1,\dots,n\right\}$ is contained in $Q\cap U_a$. Now, we observe that

$$\widetilde{\rho}_k(x,a)=\sup_{t\in\overline{U}_a}\left|\varphi_a(x)\rho_k(x,t)-\varphi_a(a)\rho_k(a,t)\right|\geqq\varphi_a(a)\rho_k(x,a)=\rho_k(x,a).$$

Hence the set $\widetilde{T} = \{ x \in X \mid \widetilde{\rho}_k(a,x) < \varepsilon,\ k = 1,\dots,n \}$ is contained in T, and hence, in Q. Consequently, a is an inner point of Q with respect of $\widetilde{\tau}$. Thus $Q \in \widetilde{\tau}$, proving that $\tau \subset \widetilde{\tau}$. □

A useful supplement to Theorem A provides the following standard

PROPOSITION 1. *Let G be a group, X be a G-space and $\mathcal{W}$ be a compatible uniformity on X. Then the following are equivalent:*

(1) *The action of G on X is $\mathcal{W}$-uniformly equicontinuous;*

(2) *$\mathcal{W}$ has a G-invariant base;*

(3) *$\mathcal{W}$ is generated by a family of G-invariant pseudometrics, uniformly continuous on $X \times X$.*

Before passing to the proof, let us recall that if $\mathcal{W}$ is a compatible uniformity on a G-space X given by means of entourages of the diagonal of X (see e.g., [13, Ch. 6]), then an action of G is said to be $\mathcal{W}$-uniformly equicontinuous if for every $U \in \mathcal{W}$ there is a $V \in \mathcal{W}$ such that $(gx, gy) \in U$ for all $(x,y) \in V$ and $g \in G$. A base $\mathcal{B}$ of $\mathcal{W}$ is said to be G-invariant whenever $V = G(V)$ for all $V \in \mathcal{B}$, where $G(V) = \{ (gx, gy) \mid (x,y) \in V,\ g \in G \}$.

PROOF. (1) $\Rightarrow$ (2). For each $U \in \mathcal{W}$, let $\widetilde{U} = \{ (gx, gy) \mid (x,y) \in U, g \in G \}$. Then $\mathcal{B} = \{ \widetilde{U} \mid U \in \mathcal{W} \}$ is a G-invariant base of $\mathcal{W}$.

(3) $\Rightarrow$ (1) is immediate.

(2) $\Rightarrow$ (3). Let $\mathcal{B}$ be a G-invariant base of $\mathcal{W}$ and let $\{d_i\}$ be the family of all bounded pseudometrics, uniformly continuous on $X \times X$. For every $x, y \in X$ we define

$$\widetilde{d}_i(x,y) = \sup \{ d_i(gx, gy) \mid g \in G \}.$$

Clearly, each $\widetilde{d}_i$ is a G-invariant pseudometric and $d_i(x,y) \leqq \widetilde{d}_i(x,y)$ for all $x, y \in X$. So, it remains only to see that each $\widetilde{d}_i$ is uniformly continuous on $X \times X$. For, let $\varepsilon > 0$. As d_i is uniformly continuous, the entourage $U_i = \{ (x,y) \in X \times X \mid d_i(x,y) < \varepsilon/2 \}$ belongs to $\mathcal{W}$ (see [13, Ch. 6, Theorem 11]); hence, there exists a $V \in \mathcal{B}$ with $V \subset U_i$. It then follows from the G-invariance of V that $d_i(gx, gy) < \varepsilon/2$ whenever $(x,y) \in V$ and $g \in G$. Therefore, $\widetilde{d}_i(x,y) \leqq \varepsilon/2 < \varepsilon$ for all $(x,y) \in V$, implying that the entourage $\{ (x,y) \in X \times X \mid \widetilde{d}_i(x,y) < \varepsilon \}$ belongs to $\mathcal{W}$. Hence, $\widetilde{d}_i$ is uniformly continuous (see [13, Ch. 6, Theorem 11]). □

Consequently, it follows from Theorem A and Proposition 1 the following

COROLLARY 2. *On each Palais proper G-space there exists a compatible uniformity with a G-invariant base.*

PROOF OF THEOREM B. The implication (1) $\Rightarrow$ (2) follows from Stone's classical theorem on paracompactness of metrizable spaces.

(3) $\Rightarrow$ (1). If ρ is an invariant metric on the proper G-space X then the function

$$\widetilde{\rho}\big(G(x), G(y)\big) = \inf\{\rho(x', y') \mid x' \in G(x),\ y' \in G(y)\}$$

is a well-defined metric, compatible with the quotient topology of X/G (see [18, Theorem 4.3.4] for details).

(2) $\Rightarrow$ (3). By [2, Theorem 3.3], each orbit in X has a tubular neighborhood, and hence, one can choose a tubular cover $\{W_i\}$ of X. Using the paracompactness of the orbit space X/G and the openness of the orbit map $X \to X/G$, one can choose a partition of unity $\{\varphi_i : X \to [0,1]\}$ such that each φ_i is an invariant function with $\varphi_i^{-1}\big((0,1]\big) \subset W_i$, and that the open covering $\{U_i = \varphi_i^{-1}\big((0,1]\big)\}$ of X is locally finite. By the Lemma, each W_i and hence each U_i, admits a G-invariant metric. According to [3, §3] then there exists a G-embedding $l_i : U_i \to X_i$ into a normed linear G-space X_i, equipped with a linear and isometric action of G.

Consider the direct sum $E_i = \mathbf{R} \oplus X_i$ endowed with the norm $\|(t,z)\| = |t| + \|z\|$, where $(t,z) \in \mathbf{R} \oplus X_i$. We make E_i an isometric linear G-space by letting G act on it according to the rule: $g(t,z) = (t, gz)$; $g \in G$. Now, the map $h_i : U_i \to E_i$ defined by $h_i(x) = \big(1, l_i(x)\big) / \big\|\big(1, l_i(x)\big)\big\|$, $x \in U_i$ is an equivariant embedding of U_i in the unit sphere of E_i. For each index i, we set

$$f_i(x) = \begin{cases} \varphi_i(x) h_i(x), & \text{if } x \in U_i, \\ 0, & \text{if } x \in X \setminus U_i. \end{cases}$$

One readily sees that f_i is a G-map with $X \setminus U_i = f_i^{-1}(0)$. Moreover, the restriction $f_i|_{U_i}$ is an embedding. Indeed, it is clear that $f_i|_{U_i}$ is continuous and injective. Let us check the continuity of the inverse map. For, let $x_0 \in U_i$ be a point, $\{x_k\}$ be a sequence in U_i such that $f_i(x_k) \rightsquigarrow f_i(x_0)$, i.e., $\varphi_i(x_k)h_i(x_k) \rightsquigarrow \varphi_i(x_0)h_i(x_0)$. Then $\|\varphi_i(x_k)h_i(x_k)\| \rightsquigarrow \|\varphi_i(x_0)h_i(x_0)\|$. But $\varphi_i(x_k) = \|\varphi_i(x_k)h_i(x_k)\|$ and $\varphi_i(x_0) = \|\varphi_i(x_0)h_i(x_0)\|$, because $\|h_i(x_k)\| = \|h_i(x_0)\| = 1$; so we get that $\varphi_i(x_k) \rightsquigarrow \varphi_i(x_0)$. This, in turn, together with $\varphi_i(x_k)h_i(x_k) \rightsquigarrow \varphi_i(x_0)h_i(x_0)$ implies that $h_i(x_k) \rightsquigarrow h_i(x_0)$. Since $h_i|_{U_i}$ is an embedding, we conclude that $x_k \rightsquigarrow x_0$. This proves that the map $\big(f_i|_{U_i}\big)^{-1}$ is continuous as well, and hence, $f_i|_{U_i}$ is an embedding.

Denote by E the subset of the product $\prod E_i$ consisting of all points $v = (v_i)$ such that $v_i \neq 0$ only for finitely many indices i. Define a norm on E by the following rule:

$$\|v\| = \sum \|v_i\|, \quad \text{where} \quad v = (v_i) \in E.$$

It is easily seen that E becomes a normed linear G-space if we define the diagonal action of G on it. Furthermore, this action is also isometric.

Next, we consider the map $f : X \to E$ defined by $f(x) = \big(f_i(x)\big)$. Since $\{U_i\}$ is a locally finite open covering of X, we infer that f is a well-defined continuous map. Its equivariance follows from the equivariance of the maps f_i. We claim that f is an embedding. Indeed, let $x, y \in X$ and $x \neq y$. Choose a tube $U_j \in \{U_i\}$ with $x \in U_j$. If $y \in U_j$ then $f_j(x) \neq f_j(y)$ as $f_j|_{U_j}$ is injective. If $y \notin U_j$ then $f_j(y) = 0$, while $f_j(x) \neq 0$. Thus $f(x) \neq f(y)$ which means that f is injective. Check that the map $f : X \to f(X)$ is open. For, let Q be an open subset of X and $x \in Q$ be an arbitrary point. As above, let $U_j \in \{U_i\}$ be a tube containing the point x. Since the restriction $f_j|_{U_j}$ is an embedding, we see that $f_j(Q \cap U_j)$ is open in $f_j(U_j)$. Choose $0 < \varepsilon < \big\| f_j(x) \big\|$ such that $f_j(U_j) \cap O\big(f_j(x), \varepsilon\big) \subset f_j(Q \cap U_j)$, where $O\big(f_j(x), \varepsilon\big)$ denotes the open ε-ball in E_j, centered at the point $f_j(x)$.

We claim that $f(X) \cap O\big(f(x), \varepsilon\big) \subset f(Q)$, where $O\big(f(x), \varepsilon\big)$ is the open ε-ball in E, centered at $f(x)$. Indeed, let $y \in X$ be such that $\big\| f(y) - f(x) \big\| < \varepsilon$. Then $\big\| f_j(y) - f_j(x) \big\| \leqq \big\| f(y) - f(x) \big\| < \varepsilon$, and therefore $f_j(y) \neq 0$, which is equivalent to $y \in U_j$. Hence $f_j(y) \in f_j(U_j) \cap O\big(f_j(x), \varepsilon)\big)$, yielding that $f_j(y) \in f_j(Q \cap U_j)$. Since f_j is injective on U_j, we conclude that then $y \in U_j \cap Q$. Consequently $f(y) \in f(Q)$, and hence, $f(X) \cap O\big(f(x), \varepsilon\big) \subset f(Q)$. This gives that $f(x)$ is an interior point of $f(Q)$ in $f(X)$, and hence, $f(Q)$ is open in $f(X)$. Thus, f is a homeomorphism of X onto $f(X)$. It remains only to observe that the metric on X induced from E is the required one. $\square$

PROPOSITION 2. *Let G be an almost connected group and X be a locally Lindelöff, paracompact proper G-space. Then the orbit space X/G is paracompact.*

PROOF. 1. First, we consider the case of a connected group G. As it is observed in [11, Proposition 13], X is a discrete sum of its open-closed Lindelöff subsets X_α. By connectedness of G all the orbits in X are connected; hence, each X_α is invariant in X. Therefore, $X/G = \bigoplus(X_\alpha/G)$, the discrete sum. Since X_α is also a proper G-space, by [18, Proposition 1.2.8] the orbit space X_α/G is completely regular. Therefore X_α/G, being a continuous image of a Lindelöff space, is itself Lindelöff [9, Theorem 3.8.6]. Since every regular Lindelöff space is paracompact [9, Theorem 5.1.2], the result now follows from [9, Theorem 5.1.30].

2. Let G be almost connected, i.e., the factor group G/G_0 is compact, where G_0 is the connected component of the identity of G. Since X can be regarded as a proper G_0-space (with the induced action), the orbit space X/G_0 is paracompact by Case 1. As G/G_0 is compact, the G/G_0-orbit map $X/G_0 \to X/G = \frac{X/G_0}{G/G_0}$ is perfect. Consequently, by Michael's Theorem the

paracompactness of X/G_0 yields the paracompactness of $\frac{X/G_0}{G/G_0}$. It remains only to observe that the two orbit spaces X/G and $\frac{X/G_0}{G/G_0}$ are naturally homeomorphic. $\square$

PROOF OF COROLLARY 1. If G is almost connected then by Proposition 2, X/G is paracompact. If G is separable then each orbit $G(x)$ in X, being a continuous image of G, is itself separable. Since the orbit map $X \to X/G$ is open, it then follows from a result of A. H. Stone [19] that X/G is metrizable. Thus, in both cases the orbit space X/G is paracompact. The result now follows from Theorem B.

In connection with Theorem B and Proposition 2, it is worthy to mention here an unsolved conjecture due to O. Hájek [11] (the case $G = \mathbf{R}$) and H. Abels [1] (the general case) to the effect that the orbit space X/G of any paracompact proper G-space X is paracompact provided G is connected. In fact, all the difficulty here lies in the normality of the orbit space. Namely, we have

PROPOSITION 3. *The following are equivalent:*

(1) *For each paracompact proper G-space X the orbit space X/G is paracompact.*

(2) *For each paracompact proper G-space X the orbit space X/G is normal.*

PROOF. One need to prove the implication (2) $\Rightarrow$ (1) only. Let X be a paracompact proper G-space. By Tamano's theorem [9, Theorem 5.1.39], it suffices to show that for every compact space Y, the product $(X/G) \times Y$ is normal. To this end, we consider the G-space $X \times Y$, letting G act on it by $g(x,y) = (gx,y)$. Then $X \times Y$ is a proper G-space [18, Proposition 1.3.3]; besides, it is paracompact [9, Theorem 5.1.36]. Consequently, by the hypothesis, the orbit space $(X \times Y)/G$ is normal. It remains only to observe that $(X/G) \times Y = (X \times Y)/G$. $\square$

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