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Travelling Wave Phenomena in Some Degenerate Reaction–Diffusion Equations

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In this paper we study the existence of travelling wave solutions (t.w.s.), $u(x, t) = \phi(x - ct)$ for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u), \tag{*}$$

where the *reactive part* $g(u)$ is as in the Fisher–KPP equation and different assumptions are made on the *non-linear diffusion term* $D(u)$. Both functions D and g are defined on the interval $[0, 1]$. The existence problem is analysed in the following two cases.

Case 1. $D(0) = 0$, $D(u) > 0$ $\forall u \in (0, 1]$, D and $g \in C^2_{[0,1]}$, $D'(0) \neq 0$ and $D''(0) \neq 0$. We prove that if there exists a value of c, c^*, for which the equation (*) possesses a travelling wave solution of *sharp type*, it must be unique. By using some continuity arguments we show that: for $0 < c < c^*$, there are no t.w.s., while for $c > c^*$, the equation (*) has a continuum of t.w.s. of front type. The proof of uniqueness uses a monotonicity property of the solutions of a system of ordinary differential equations, which is also proved.

Case 2. $D(0) = D'(0) = 0$, D and $g \in C^2_{[0,1]}$, $D''(0) \neq 0$. If, in addition, we impose $D''(0) > 0$ with $D(u) > 0$ $\forall u \in (0, 1]$, we give sufficient conditions on c for the existence of t.w.s. of front type. Meanwhile if $D''(0) < 0$ with $D(u) < 0$ $\forall u \in (0, 1]$ we analyse just one example ($D(u) = -u^2$, and $g(u) = u(1-u)$) which has oscillatory t.w.s. for $0 < c \leq 2$ and t.w.s. of front type for $c > 2$.

In both the above cases we use higher order terms in the Taylor series and the *Centre Manifold Theorem* in order to get the local behaviour around a *non-hyperbolic point* of *codimension one* in the phase plane. © 1995 Academic Press, Inc.

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1. Introduction

The first mention of travelling waves as solutions for a certain reaction–diffusion equation was in a report due to Luther in 1906. He drew an analogy between the conduction of a nerve pulse and a crystallisation process. A modern version of his paper can be seen in Showalter and Tyson (1987). In 1937 an important contribution was made in two separate works due to Fisher and Kolmogorov *et al.*, respectively; both of which are related to the description of the space–time distribution of an advantageous gene in a population which lives in a one-dimensional domain. In Kolmogorov *et al.*, the authors introduced a formal way in which one can analyse the existence and the stability of the travelling wave for the case when it is a solution of a type of parabolic equation. They stated their results on existence for the equation $u_t = Du_{xx} + f(u)$ with $D > 0$ and $f \in C^1_{[0,1]}$ satisfying: $f(0) = f(1) = 0$, $f(u) > 0$ $\forall u \in (0, 1)$ $f'(0) > 0$, and $f'(1) < 0$. With a Heaviside function as initial condition they proved a theorem on convergence to the travelling wave.

Since these seminal papers, much research has been carried out in an attempt to extend the original results to more complicated equations which arise in several fields. For example in ecology the first systematic treatment of dispersion models of biological populations (due to Skellam in 1951) assumed random movement. Here the probability that an individual which at time $t = 0$ is at the point x_1 moves to the point x_2 in the interval of time Δt is the same as that of moving from x_2 to x_1 during the same time interval. So the probability, p, is a symmetric function; i.e.,

$$p(x_1, x_2) = p(x_2, x_1). \tag{1}$$

On this basis were constructed the classical models of population dispersion in which the diffusion coefficient appears as constant.

There is, however, considerable evidence that some species engage in non-random movement. In a very general way, this phenomenon can be divided into two types:

1. *Spatial Characteristics.* Some insects move in response to olfactory or visual stimuli. Obviously here the probability is not symmetric. To model this type of movement, McMurtrie (1978) considered the case where attractive and repulsive forces are the cause of movement of one species. He assumed that both forces could be measured by a function which depends on position. Letting $\alpha(x)$ and $\beta(x)$ be the concentration at the point x of the attractive and repulsive substances respectively, the probability takes the form

$$p(x_1, x_2) = p(\alpha(x_1), \beta(x_2)). \tag{2}$$

In this case the population density, $u(x, t)$, satisfies the following diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\psi(x)\frac{\partial u}{\partial x} + \left\{\frac{\partial\psi}{\partial\beta}\frac{d\beta}{dx} - \frac{\partial\psi}{\partial\alpha}\frac{d\alpha}{dx}\right\}u\right] \tag{3}$$

where ψ is the variance of the motion. (For full details see McMurtrie, 1978). If we include both space-dependent diffusion and non-linear rate of growth the model takes the general form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right] + g(u). \tag{4}$$

Shiguesada *et al.* (1986) derived and studied a "logistic" model for a dispersing population in a heterogeneous environment in which they also included space variation in the intrinsic rate of growth. Their equation is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right] + u[\varepsilon(x) - \mu u], \tag{5}$$

where $\mu > 0$, and D and ε are periodic functions in space with period l.

2. *Density-Dependent Characteristics.* Some species migrate from densely populated areas into sparsely populated areas to avoid crowding. Thus overcrowding increases population dispersion. Other species have social behaviour such that the population only moves from one place to another until its density attains a certain value. Myers and Krebs (1974) studied density-dependent dispersion as a regulatory mechanism of the cyclic changes in the density of some small rodents. In these cases, the probability that an animal moves from the point x_1 to x_2 depends on the density at x_1. Here the population density satisfies the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right]. \tag{6}$$

The details of the derivation of (6) can be found in McMurtrie (1978).

When both density-dependent diffusion and non-linear rate of growth are present, we have the general one-dimensional model for a species[1]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u). \tag{7}$$

[1] In a more realistic situation we have that the diffusion coefficient depends on both the distance and the density.

The first model of this type was derived by Gurney and Nisbet (1975) in an ecological context. They used a probabilistic approach to construct the following three-dimensional model for a Malthusian rate of growth

$$\frac{\partial u}{\partial t} = D\,\nabla \cdot (u\,\nabla u) + ru, \tag{8}$$

where D and r are positive. Meanwhile Gurtin and MacCamy (1977) adopted a continuum approach and deduced the model

$$\frac{\partial u}{\partial t} = \nabla^2[\varphi(u)] + g(u) \tag{9}$$

where $\varphi'(0) = 0$, $\varphi'(u) > 0$ for $u > 0$, to describe the growth of one species. However, their analysis was developed for the particular equation

$$\frac{\partial u}{\partial t} = \nabla^2(u^\alpha) + \mu u \tag{10}$$

with $\alpha \geqslant 0$. They transformed (10) into

$$\frac{\partial w}{\partial \tau} = \nabla^2(w^\alpha), \tag{11}$$

where

$$u = we^{\mu t} \qquad \text{and} \qquad \tau = \frac{e^{\mu(\alpha-1)t} - 1}{\mu(\alpha - 1)}$$

to prove the existence of weak solutions.

In another context, if the state variable $w(\mathbf{r}, \tau)$ in (11) is interpretated as the concentration of a substance, then such an equation is called the *porous media equation* (see Aronson, 1980, 1986).

The problem of existence of t.w.s. for equation (7) when the non-linear diffusion coefficient $D(u)$ is strictly positive in the interval $[0, 1]$ and the reactive part g is as in the Fisher–KPP equation has been studied completely. Hadeler (1981) gives sufficient conditions on the speed c for the existence of solutions of front type satisfying the boundary conditions $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. In a later paper Hadeler (1983) shows a very important relationship between the ordinary differential equation (ODE) system for the travelling wave solution of a Fisher–KPP equation and the corresponding ODE system for the travelling wave solution of the more general Eq. (7). For the same equation Engler (1985) gives necessary and sufficient conditions for the existence of front type solutions. These conditions are given in terms of a relationship between the t.w.s. for the equation $u_t = u_{xx} + D(u)\,g(u)$ and those for (7).

When the density-dependent diffusion is zero at one point ($u=0$, for instance) and strictly positive in $(0, 1]$ for the same kinetic term as above, the problem of existence of t.w.s. for Eq. (7) has been analysed only in a few particular cases. Aronson (1980) considered the equation $u_t = [mu^{m-1}u_x]_x + u(1-u)$ and gave a plausible reasoning for the existence of a critical value of c, $c^* = c^*(m)$, for which this equation: (i) has no t.w.s. for $0 < c < c^*(m)$, (ii) possesses a travelling wave solution of *sharp* type for $c = c^*(m)$, and (iii) has a continuum of t.w.s. of front type for $c > c^*(m)$. He illustrated this with numerical solutions for the case $m = 2$.

For the same equation with $m = 2$, Murray (1989) found that $c^*(2) = 1/\sqrt{2}$ and gave explicitly the travelling wave of sharp type.

De Pablo and Vázquez (1991) considered the equation $u_t = [mu^{m-1}u_x]_x + \lambda u^n(1-u)$ with $m > 1$ and $\lambda > 0$. They proved a result on the existence of a critical value of c, c^*, as above for $m > 1$, $\lambda = 1/m$, and $m + n \geqslant 0$.

For the same diffusion term but for reactive part g satisfying: $g(0) = g(\alpha) = g(1) = 0$, $g(u) < 0$ $\forall u \in (0, \alpha)$, and $g(u) > 0$ $\forall u \in (\alpha, 1)$, Hosono (1985) gave necessary and sufficient conditions for the existence of t.w.s. He also proved a result on stability of such t.w.s.

In this paper we study the existence of t.w.s. for Eq. (7) when the nonlinear diffusion coefficient $D(u)$ vanishes at $u = 0$ and the reactive term g is as in the Fisher–KPP equation. In Section 2 we add the conditions: D and $g \in C^2_{[0,1]}$, $D(u) > 0$ $\forall u \in (0, 1]$, $D'(0), D''(0) \neq 0$, and prove that if Eq. (7) has a travelling wave solution of sharp type for a certain value of c, c^*, it must be unique. The proof uses a monotonicity property of the solutions which is also proved in Section 2. Using arguments of continuity of the solutions with respect to the parameter c we show that: (i) for $0 < c < c^*$ Eq. (7) has no t.w.s., and (ii) for $c > c^*$ Eq. (7) possesses a continuum of t.w.s. of front type. In Section 3 we consider the same smoothness conditions on the functions D and g but set $D'(0) = 0$. When $D''(0) > 0$ we give sufficient conditions on the speed c for the existence of t.w.s. of front type. The case $D''(0) < 0$ implies that at least locally D must be negative. We consider $D(u) < 0$ $\forall u \in (0, 1]$ to show the appearance of t.w.s. of oscillatory type.

2. A Uniqueness Result of Sharp Type Solutions for Some Degenerate Equations

In this section we will consider the existence of travelling wave solutions $u(x, t) = \phi(x - ct) = \phi(\xi)$ for the one-dimensional reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u); \qquad (x, t) \in R \times R^+, \tag{12}$$

where the following assumptions are made:

1. $g(0) = g(1) = 0$, $g(u) > 0$ for all $u \in (0, 1)$
2. $g \in C^2_{[0,1]}$ with $g'(0) > 0$ and $g'(1) < 0$
3. $D(0) = 0$ with $D(u) > 0$ for all $u \in (0, 1]$
4. $D \in C^2_{[0,1]}$ with $D'(0), D''(0) \neq 0$.

We require that the function ϕ satisfies the conditions

$$\phi(-\infty) = 1 \qquad \text{and} \qquad \phi(+\infty) = 0 \tag{13}$$

with $0 \leqslant \phi(\xi) \leqslant 1$ $\forall \xi \in (-\infty, +\infty)$.

We start our analysis by introducing the following definition.

DEFINITION 2.1. If there exists a value of u, u^*, in the domain of D such that $D(u^*) = 0$, then Eq. (12) is called degenerate.

Note that, for all $u \neq u^*$ for which D is defined, Eq. (12) is parabolic of second order but, for $u = u^*$, (12) *degenerates* to first order.

Now we will calculate the sign of the speed c of the possible t.w.s. for Eq. (12). Equation (12) can be written as

$$\frac{\partial u}{\partial t} = D(u)\frac{\partial^2 u}{\partial x^2} + D'(u)\left[\frac{\partial u}{\partial t}\right]^2 + g(u). \tag{14}$$

If a travelling wave solution of (14) exists, it must satisfy the following second order ODE[2]

$$D(\phi(\xi))\,\phi'' + c\phi'(\xi) + D'(\phi(\xi))[\phi'(\xi)]^2 + g(\phi(\xi)) = 0. \tag{15}$$

Multiplying both sides of (15) by $D(\phi)\,\phi'$ (not by ϕ' as in the Fisher–KPP equation) we get

$$D^2(\phi)\,\phi''\phi' + cD(\phi)[\phi']^2 + D(\phi)\,D'(\phi)[\phi']^3 + g(\phi)\,D(\phi)\,\phi' = 0. \tag{16}$$

Note that

$$\frac{d}{d\xi}\left\{\frac{1}{2}[D(\phi(\xi))\,\phi'(\xi)]^2\right\} = D^2(\phi)\,\phi''\phi' + [\phi']^3\,D'(\phi)\,D(\phi)$$

so (16) becomes

$$\frac{d}{d\xi}\left\{\frac{1}{2}[D(\phi(\xi))\,\phi'(\xi)]^2\right\} + cD(\phi(\xi))[\phi'(\xi)]^2 + g(\phi(\xi))\,D(\phi(\xi))\,\phi'(\xi) = 0$$

[2] Here we are abusing the notation since $'$ on the function D means derivative with respect to ϕ, while on ϕ it means derivative with respect to ξ.

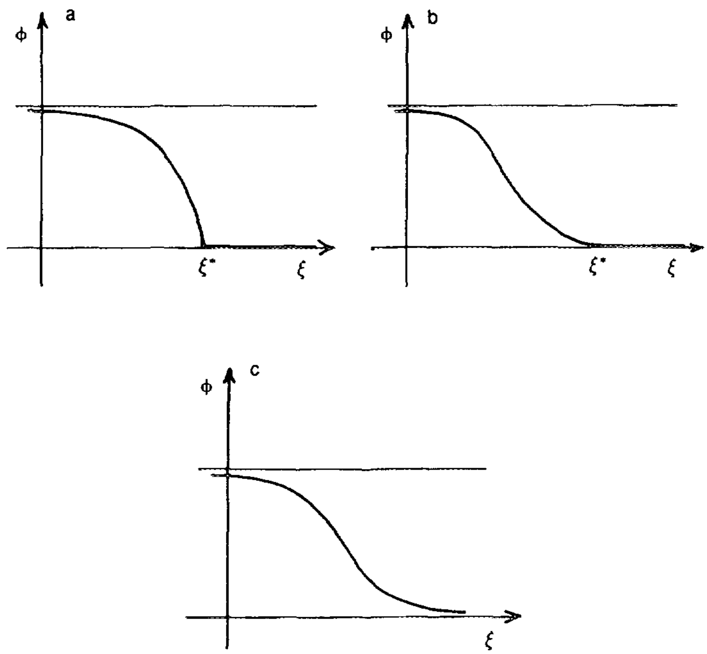


FIG. 2.1. Different behaviours of travelling wave solutions to (12). (a) Discontinuous derivative in ξ^*. The left derivative tends to $\phi'(\xi^{*-}) \neq 0$, the right derivative tends to $\phi'(\xi^{*+}) = 0$. (b) Continuous derivative in ξ^*, i.e., $\phi'(\xi^{*-}) = \phi'(\xi^{*+}) = 0$. (c) Here $\lim_{\xi \to +\infty} \phi(\xi) = \lim_{\xi \to +\infty} \phi'(\xi) = 0$.

We integrate the above equation with respect to ξ from $-\infty$ to ξ^*, where $\xi^* \in (-\infty, +\infty]$. Thus for t.w.s. such that

$$\phi(-\infty) = 1 \qquad \text{and} \qquad \phi(\xi) = 0 \qquad \forall \xi \in [\xi^*, +\infty]; \tag{17}$$

the above equation, after integration, becomes

$$\frac{1}{2}\left[D(\phi(\xi))\,\phi'(\xi)\right]^2\Big|_{-\infty}^{\xi^*} + c\int_{-\infty}^{\xi^*} D(\phi(s))[\phi'(s)]^2\,ds - \int_0^1 g(w)\,D(w)\,dw = 0. \tag{18}$$

Now we analyse the above equality for the three hypothetical behaviours[3] of ϕ sketched in Fig. 2.1. Each of them satisfy, for different values of ξ^*, the conditions (17). We write explicitly the first term of the last equality as

$$\tfrac{1}{2}\left[D(\phi(\xi))\,\phi'(\xi)\right]^2\Big|_{-\infty}^{\xi^*} = \tfrac{1}{2}\left\{\left[D(\phi(\xi^*))\,\phi'(\xi^*)\right]^2 - \left[D(\phi(-\infty))\,\phi'(-\infty)\right]^2\right\}.$$

[3] The reason why we take just these three cases will be made clear later in this section.

Since $\phi'(-\infty)=0$ and $D(\phi(-\infty))=D(1)<\infty$, the second term on the right in the above equality is zero. Meanwhile for the first term on the right, we have the following cases corresponding to Fig. 2.1:

1. $D(\phi(\xi^*))=D(0)=0$ and $\phi'(\xi^*)\neq 0$ (finite)
2. $D(\phi(\xi^*))=D(0)=0$ and $\phi'(\xi^*)=0$
3. $D(\phi(\xi^*))=D(0)=0$ and $\phi'(+\infty)=0$.

Therefore regardless of the exact behaviour of ϕ, (18) simplifies to

$$c\int_{-\infty}^{\xi^*} D(\phi(s))[\phi'(s)]^2\,ds-\int_0^1 g(w)\,D(w)=0,$$

hence

$$c=\frac{\int_0^1 g(w)\,D(w)\,dw}{\int_{-\infty}^{\xi^*} D(\phi(s))[\phi'(s)]^2\,ds}\geqslant 0. \tag{19}$$

We introduce the following definition

DEFINITION 2.2. If there exists a value of the speed c, c^*, and a value of ξ, $\xi^*\in(-\infty,+\infty]$, such that $\phi(x-c^*t)=\phi(\xi)$ satisfies

1. $D(\phi)\,\phi''+c^*\phi'+D'(\phi)[\phi']^2+g(\phi)=0$, $\forall\xi\in(-\infty,\xi^*)$,
2. $\phi(-\infty)=1$, $\phi(\xi^*)=0$ and $\phi'<0$, $\forall\xi\in(-\infty,\xi^*)$,
3. $\phi'(\xi^*)=-c^*/D'(0)$ and $\phi(\xi)=0$ $\forall\xi\in(\xi^*,+\infty]$,

then the function $u(x,t)=\phi(x-c^*t)$ is called a travelling wave solution of sharp type for Eq. (12).

To study the existence of t.w.s. for Eq. (12) we will analyse the corresponding phase portrait of the ODE system associated with (15). Setting $v=\phi'$ we have the following system of ODEs

$$\begin{aligned}\phi'&=v\\ D(\phi)\,v'&=-cv-D'(\phi)\,v^2-g(\phi).\end{aligned} \tag{20}$$

Since $D(0)=0$ this system possesses a singularity at $\phi=0$. We can remove it by introducing the parameter τ (Aronson, (1980)) such that

$$\frac{d\tau}{d\xi}=\frac{1}{D(\phi(\xi))}\Rightarrow\tau(\xi)=\int_0^{\xi}\frac{ds}{D(\phi(s))}. \tag{21}$$

Except at $\phi=0$ where $d\tau/d\xi$ is not defined, $d\tau/d\xi>0$. Thus τ has an inverse τ^{-1} which in principle can be obtained from (21). Then we have

$$\phi(\xi)=\phi(\tau(\xi))\qquad\text{and}\qquad v(\xi)=v(\tau(\xi)) \tag{22}$$

and we obtain

$$\phi'(\xi) = \frac{d\phi}{d\tau}(\tau)\,\frac{1}{D(\phi(\xi))} \qquad \text{and} \qquad v'(\xi) = \frac{dv}{d\tau}(\tau)\,\frac{1}{D(\phi(\xi))}. \tag{23}$$

Substituting $\phi'(\xi)$ and $v'(\xi)$ into (20) we have the new system without the singularity[4]

$$\begin{aligned} \dot{\phi} &= D(\phi)\, v \equiv F(\phi, v) \\ \dot{v} &= -cv - D'(\phi)\, v^2 - g(\phi) \equiv G(\phi, v), \end{aligned} \tag{24}$$

where the dot denotes differentation with respect to τ.

Note that systems (20) and (24) are topologically equivalent in the positive half plane $\{(\phi, v) \mid \phi > 0,\ -\infty < v < +\infty\}$. This occurs since (22) defines a re-parametrization of the trajectories which, according with (23), preserves the orientation.

Since $D'(0) \neq 0$ the system (24) has the following three equilibrium points in the region $\{(\phi, v) \mid 0 \leqslant \phi \leqslant 1,\ -\infty < v < +\infty\}$: $P_0 = (0, 0)$, $P_1 = (1, 0)$ and $P_c = (0, -c/D'(0))$. The local behaviour of the trajectories of the system (24) can be obtained as usual by using the linear approximation of (24) around each stationary point. The Jacobian matrix for all points (ϕ, v) is

$$J[F, G]_{(\phi, v)} = \begin{bmatrix} D'(\phi)\, v & D(\phi) \\ -D''(\phi)\, v^2 - g'(\phi) & -c - 2D'(\phi)\, v \end{bmatrix}. \tag{25}$$

If we evaluate (25) at P_0 we get

$$J[F, G]_{(0, 0)} = \begin{bmatrix} 0 & 0 \\ -g'(0) & -c \end{bmatrix}, \tag{26}$$

from which we have $\operatorname{tr} J[F, G]_{(0, 0)} = -c < 0$ and $\det J[F, G]_{(0, 0)} = 0$. Hence the linear system is inadequate to give us the local behaviour around P_0. Since the eigenvalues of (26) are $\lambda_1 = 0$ and $\lambda_2 = -c$, P_0 is a *non-hyperbolic point of codimension one* (Arrowsmith and Place, 1990). The corresponding eigenvectors are $\mathbf{v}_1 = (c, -g'(0))^T$ and $\mathbf{v}_2 = (0, 1)^T$, respectively.

In order to determine the phase portrait of the system (24) around P_0 we are required to use higher order terms in the Taylor series as well as the *Centre Manifold Theorem.*

For the system (24) the second order terms are sufficient. Thus we have the quadratic approximation to (24) around P_0 as

$$\begin{aligned} \dot{\phi} &= D'(0)\, \phi v \equiv F_2(\phi, v) \\ \dot{v} &= -g'(0)\, \phi - cv + G_1(\phi, v) \equiv G_2(\phi, v), \end{aligned} \tag{27}$$

[4] In fact, we resolve the singularity at $\phi = 0$ into two equilibrium points.

where $G_1(\phi, v) = -\frac{1}{2}g''(0)\,\phi^2 - D'(0)\,v^2$. Now, we follow the technique developed by Andronov *et al.* (1973) (see Appendix A). For systems such as (24) for which

$$\frac{\partial F}{\partial \phi}(0, 0) = \frac{\partial F}{\partial v}(0, 0) = 0$$

it is necessary to define $\bar{\tau}$, $\bar{\phi}$ and $\bar{v}$ as

$$\bar{\tau} = \kappa\tau, \qquad \bar{\phi} = \phi, \qquad \text{and} \qquad \bar{v} = \frac{1}{c}\,g'(0)\,\phi + v \tag{28}$$

where κ is a non-zero constant. Using (28) and choosing $\kappa = -c$, the system (27) takes the form

$$\begin{aligned} \frac{d\bar{\phi}}{d\bar{\tau}} &= -\frac{D'(0)}{c}\,\bar{\phi}\left[\bar{v} - \frac{g'(0)}{c}\,\bar{\phi}\right] \equiv \bar{F}_2(\bar{\phi}, \bar{v}) \\ \frac{d\bar{v}}{d\bar{\tau}} &= \bar{v} + \left[\frac{g''(0)}{2c} + \frac{2g'^2(0)\,D'(0)}{c^3}\right]\bar{\phi}^2 - \frac{3D'(0)\,g'(0)}{c^2}\,\bar{\phi}\bar{v} \\ &\quad + \frac{D'(0)}{c}\,\bar{v}^2 \equiv \bar{v} + \bar{G}_2(\bar{\phi}, \bar{v}). \end{aligned} \tag{29}$$

Denote by $\sigma(\bar{\phi}, \bar{v})$ the divergence of the vector field defined in (29) at the point $(\bar{\phi}, \bar{v})$. Then we have $\sigma(0, 0) = 1$ so that in a small neighbourhood $V_\delta(0, 0)$ of the origin we can suppose that $\sigma(\bar{\phi}, \bar{v}) \neq 0$. By continuity arguments we have that $\sigma(\bar{\phi}, \bar{v}) > 0$ for all $(\bar{\phi}, \bar{v}) \in V_\delta(0, 0)$. Thus, by Bendixon's Test, $V_\delta(0, 0)$ contains neither closed paths nor loops. Therefore the point P_0 is not a centre and it has no elliptic sectors. Hence there must exist semi-trajectories of the system (29) that end at the equilibrium point P_0. Andronov *et al.* give the qualitative behaviour around P_0 in terms of the first non-zero coefficient in the power series of the function $\psi(\bar{\phi})$, which is defined as

$$\psi(\bar{\phi}) = \bar{F}_2(\bar{\phi}, \varphi(\bar{\phi}))$$

where $\varphi(\bar{\phi})$ is the solution of the equation $\bar{v} + \bar{G}_2(\bar{\phi}, \bar{v}) = 0$ i.e., $\varphi(\bar{\phi}) + \bar{G}_2(\bar{\phi}, \varphi(\bar{\phi})) = 0$. Thus if

$$\psi(\bar{\phi}) = \Delta_m\bar{\phi}^m + \Delta_{m+1}\bar{\phi}^{m+1} + \cdots,$$

where $m \geq 2$ and $\Delta_m \neq 0$, then depending on whether the subscript m is even or odd we have different phase portraits around the point P_0. Specifically the above authors conclude[5]:

[5] Their theorem is stated in Appendix A.

1. If m is odd and $\Delta_m > 0$, P_0 is a topological node.

2. If m is odd and $\Delta_m < 0$, P_0 is a topological saddle point, two of whose separatrices tend to P_0 in the directions 0 and π, the other two in the directions $\pi/2$ and $3\pi/2$.

3. If m is even P_0 is a *saddle-node*, i.e., an equilibrium state whose canonical neighbourhood is the union of one parabolic and two hyperbolic sectors. If $\Delta_m < 0$, the hyperbolic sectors contain a segment of the positive horizontal axis bordering the point P_0 and if $\Delta_m > 0$ they contain a segment of the negative horizontal axis.

Before we apply the method developed in Andronov *et al.* to our particular case, for notational convenience we define A, B, and E as follows:

$$A = \frac{g''(0)}{2c} + \frac{2g'(0)\,D'(0)}{c^3}, \qquad B = -\frac{3D'(0)\,g'(0)}{c^2},$$

and

$$E = \frac{D'(0)}{c}.$$

Now let $\mathscr{F}: R^2 \to R$ be a function defined by

$$\mathscr{F}(\bar\phi, \bar v) = \bar v + A\bar\phi^2 + B\bar\phi\bar v + E\bar v^2$$

From the definition it is clear that $\mathscr{F}$ satisfies:

1. $\mathscr{F}(0, 0) = 0$.
2. $\partial\mathscr{F}/\partial\bar\phi$ and $\partial\mathscr{F}/\partial\bar v$ are continuous for all $(\bar\phi, \bar v)$.
3. $\partial\mathscr{F}/\partial\bar v(0, 0) = 1$.

Then, by the Implicit Function Theorem, there exists a neighbourhood $V_\varepsilon(0, 0)$ of the origin in which the equality $\mathscr{F}(\bar\phi, \bar v) = 0$ defines one unique function $\varphi: V_\varepsilon(0, 0) \to R$ such that $\bar v = \varphi(\bar\phi)$ satisfies:

1. $\mathscr{F}(\bar\phi, \varphi(\bar\phi)) = 0$.
2. $\varphi'(\bar\phi) = -(\partial\mathscr{F}/\partial\bar\phi)(\bar\phi, \bar v)/(\partial\mathscr{F}/\partial\bar v)(\bar\phi, \bar v)$.
3. $\varphi(0) = \varphi'(0) = 0$.

We ensure that

$$\varphi(\bar\phi) = \frac{1}{2}\left\{ -\left[\frac{B\bar\phi + 1}{E}\right] + \sqrt{\left[\frac{B\bar\phi + 1}{E}\right]^2 - \frac{4A}{E}\bar\phi^2} \right\}$$

is such a function. Obviously φ satisfies 1. Since φ was obtained by solving this equality. Also $\varphi(0)=0$ and

$$\varphi'(\bar{\phi}) = -\frac{2A\bar{\phi}+B\varphi(\bar{\phi})}{1+B\bar{\phi}+2E\varphi(\bar{\phi})} \Rightarrow \varphi'(0)=0.$$

If we define $\psi(\bar{\phi})$ as

$$\psi(\bar{\phi}) = \bar{F}_2(\bar{\phi}, \varphi(\bar{\phi})) = -E\bar{\phi}\left[\varphi(\bar{\phi}) - \frac{g'(0)}{c}\bar{\phi}\right];$$

then, since $\psi(0)=\psi'(0)=0$ but $\psi''(0)=2Eg'(0)/c\neq 0$, we have that in the power series of ψ around $\bar{\phi}=0$ the first term different to zero is $\Delta_2 = Eg'(0)/c = D'(0)\,g'(0)/c$. By hypothesis, $D(u)>0$ $\forall u\in(0,1]$ with $D(0)=0$. This means that $D'(0)>0$ and by assumption 1 we conclude that $\Delta_2>0$. Hence by the result of Andronov *et al.* (see Appendix A), the point P_0 is a *saddle-node* for the system (29). Since the last two equations in (28) define a linear transformation from the ϕv-plane to the $\bar{\phi}\bar{v}$-plane we have that P_0 is also a saddle-node point for (27) and therefore for (24).

To complete the analysis around P_0 we have, by a straightforward application of the *Centre Manifold Theorem*[6] (Carr, 1981, and Arrowsmith and Place, 1990), that the system (27) has a unique one-dimensional invariant stable manifold locally tangent to the eigenvector $\mathbf{v}_2=(0,1)^T$ and a one-dimensional invariant centre manifold locally tangent to the eigenvector $\mathbf{v}_1=(c,-g'(0))^T$. Both of these manifolds contain P_0. Moreover, Carr's theorems (see Appendix B) also guarantee that any trajectory of the system (27) in the vicinity of P_0 except those on the stable manifold tend rapidly to the centre manifold. In other words; the dynamics around P_0 is given by the dynamics on the centre manifold.

To find an approximation to the centre manifold we use Carr's theorems. The equation for the centre manifold of the system (27) takes the form

$$[M\tilde{h}](\phi) = \tilde{h}'(\phi)[D'(0)\,\tilde{h}(\phi)\,\phi] + g'(0)\,\phi + c\tilde{h}(\phi) + \frac{g''(0)}{2}\phi^2 + D'(0)\,\tilde{h}^2(\phi). \tag{30}$$

If we write $\tilde{h}(\phi)=o(\phi^k)$ for $k>1$, Eq. (30) becomes

$$[M\tilde{h}](\phi) = o(\phi^{2k}) + g'(0)\,\phi + c\tilde{h}(\phi) + \frac{g''(0)}{2}\phi^2.$$

[6] We state this theorem in Appendix B.

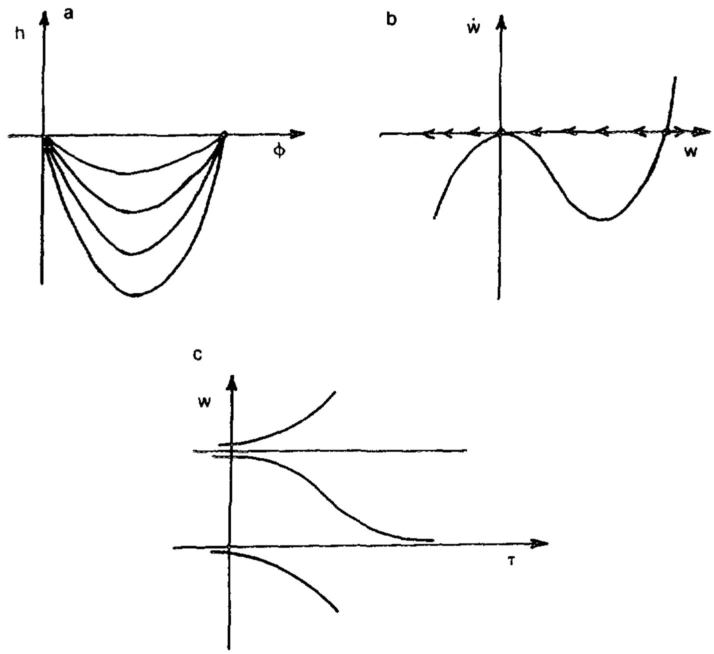


FIG. 2.2. (a) Behaviour of the centre manifold for the system (27) when the speed c changes. (b) Phase portrait of (*). (c) The behaviour of the solutions (*) gives the dynamics on the centre manifold.

If we choose $\tilde{h}(\phi)$ as

$$\tilde{h}(\phi)=\frac{1}{c}\left[-\frac{g''(0)}{2}\phi^2-g'(0)\,\phi\right],$$

then $[M\tilde{h}](\phi)=o(\phi^{2k})$. Therefore, in a neighbourhood of the origin the centre manifold for (27) (and therefore for (24)) can be approximated by (Theorem B.3, Appendix B)

$$h(\phi)=\frac{1}{c}\left[-\frac{g''(0)}{2}\phi^2-g'(0)\,\phi\right]+o(\phi^{2k}).$$

Note the behaviour of the centre manifold $h(\phi)$ with respect to the parameter c. As c decreases, $h(\phi)$ becomes more negative in $0<\phi<1$; while for c large $h(\phi)$ tends to the horizontal axis (see Fig. 2.2(a)).

The flow on the centre manifold is given by the equation

$$\dot{w}=\frac{D'(0)}{c}\left\{\phi\left[-\frac{g''(0)}{2}\phi^2-g'(0)\,\phi\right]+o(\phi^{2k+1})\right\}$$

whose first term, for ϕ sufficiently small, gives us the dynamics. In Figs. 2.2(b) and (c) we sketch the dynamics from the equation

$$\dot{w} = \frac{D'(0)}{c}\left[-\frac{g''(0)}{2}\phi^3 - g'(0)\,\phi^2 \right]. \qquad (*)$$

The above analysis shows that all trajectories of (27) with initial conditions (ϕ_0, v_0) such that $\sqrt{\phi_0^2 + v_0^2}$ is very small:

1. tend to P_0 along the centre manifold for $\phi_0 > 0$
2. move away from P_0 tending to the centre manifold for $\phi_0 < 0$.

If we collect all the above analyses we conclude that the nodal sector of P_0 is on the right side of the v-axis and the saddle region of P_0 is on the left side of the v-axis. The phase portrait of (24) around P_0 is illustrated in Fig. 2.3. Now we will complete the local analysis of the trajectories of the system (24). Evaluating (25) at P_1 we obtain

$$J[F, G]_{(1,0)} = \begin{bmatrix} 0 & D(1) \\ -g'(1) & -c \end{bmatrix}, \qquad (31)$$

from which it follows that $\operatorname{tr} J[F, G]_{(1,0)} = -c < 0$ and $\det J[F, G]_{(1,0)} = g'(1)\,D(1)$ which, by assumptions 2 and 3 at the beginning of this section, is negative. Therefore P_1 is a saddle point. The roots of the characteristic polynomial of (31) are

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[-c \pm \sqrt{c^2 - 4g'(1)\,D(1)}]$$

with $\lambda_1 > 0$ and $\lambda_2 < 0$. The corresponding eigenvectors are $\mathbf{v}_1 = (1, \lambda_1)^T$ and $\mathbf{v}_2 = (1, \lambda_2)^T$, respectively.

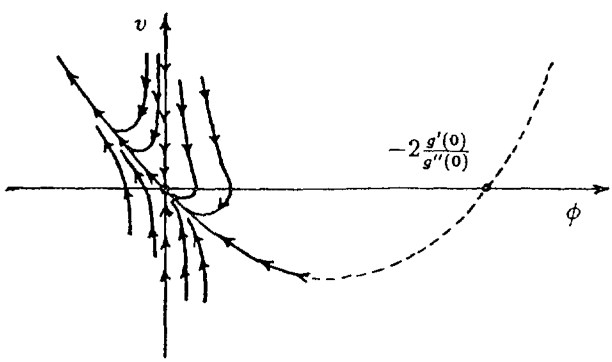


FIG. 2.3. Local behaviour around the non-hyperbolic point P_0. This shows the centre manifold of the system (27) and the nodal and saddle regions of P_0.

The Jacobian matrix at P_c is

$$J[F, G]_{(0, -c/D'(0))} = \begin{bmatrix} -c & 0 \\ -\frac{D''(0)\,c^2}{(D'(0))^2} - g'(0) & c \end{bmatrix}. \tag{32}$$

Setting $v_c = -c/D'(0)$, we have $\operatorname{tr} J[F, G]_{(0, v_c)} = 0$ and $\det J[F, G]_{(0, v_c)} = -c^2 < 0$. Hence P_c is also a saddle point. The eigenvalues of (32) are $\lambda_1 = -c$ and $\lambda_2 = c$. The associated eigenvectors are $\mathbf{v}_1 = (1, -r/2c)^T$ and $\mathbf{v}_2 = (0, 1)^T$, where $r = -D''(0)\,c^2/(D'(0))^2 - g'(0)$.

The local behaviour around P_1 and P_c is sketched in Fig. 2.4. The particular cases discussed in Aronson (1980), Murray (1989) and de Pablo and Vázquez (1991) as well in our Example 2.1 (which is below) provide insight to the global behaviour of the trajectories of (24). All of them suggest the existence of a unique *bifurcating value* of the speed c, c^*, such that:

1. For $0 < c < c^*$ the system (24) has no heteroclinic connections and therefore there are no travelling wave solutions for Eq. (12).

2. For $c = c^*$ the system (12) has only one heteroclinic connection between the points P_1 and P_c (saddle–saddle connection) and so equation (12) possesses a unique travelling wave solution of sharp type.

3. For each $c > c^*$ the system (24) possesses only a heteroclinic connection between the points P_1 and P_0 (a saddle–saddle-node connection) and therefore the reaction–diffusion equation (12) has for each c a travelling wave solution of front type satisfying $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$.

Unfortunately we do not yet have a result which gives us the conditions under which there exists a saddle–saddle connection. There are particular

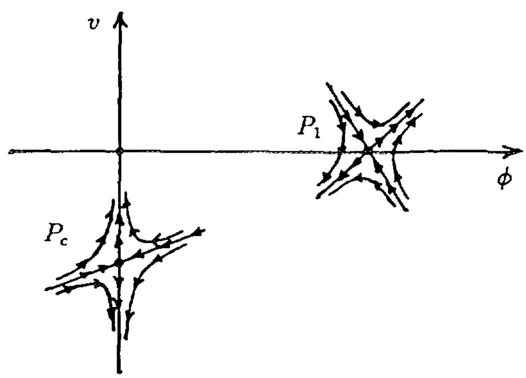


FIG. 2.4. Local behaviour of the trajectories of the system (27) around P_c and P_1.

cases, which will be discussed later (in fact, the equations discussed by the above authors belong to the class of our equation), which show us the possibility of the above behaviour. At the moment, we will present a general uniqueness result which is only a partial solution to the whole problem.

We start by writing the first order ODE for the trajectories of (24):

$$\frac{dv}{d\phi} = \frac{-cv - D'(\phi)\, v^2 - g(\phi)}{D(\phi)\, v} \tag{33}$$

and note that $v = v(\phi)$ also satisfies the condition $v(1) = 0$.

Let $v_1(\phi)$ and $v_2(\phi)$ be two solutions of (33) corresponding to two values of c, c_1 and c_2, respectively. Suppose that v_1 and v_2 satisfy $v_1(1) = v_2(1) = 0$. Thus we have

$$\frac{dv_1}{d\phi} = \frac{-c_1 v_1 - D'(\phi)\, v_1^2 - g(\phi)}{D(\phi)\, v_1} \tag{34}$$

$$\frac{dv_2}{d\phi} = \frac{-c_2 v_2 - D'(\phi)\, v_2^2 - g(\phi)}{D(\phi)\, v_2}. \tag{35}$$

Subtracting (34) from (35) we obtain

$$v_2' - v_1' = \frac{(c_1 - c_2)\, v_1 v_2 + (v_1^2 v_2 - v_2^2 v_1)\, D'(\phi) + (v_2 - v_1)\, g(\phi)}{D(\phi)\, v_1 v_2}. \tag{36}$$

Now we define the function $\mathscr{G}(\phi)$ as

$$\mathscr{G}(\phi) = (v_2 - v_1) \exp\left\{ -\int_{\tilde{\phi}}^{\phi} \frac{(v_1^2 v_2 - v_2^2 v_1)\, D'(s) + g(s)(v_2 - v_1)}{v_1 v_2 (v_2 - v_1)\, D(s)}\, ds \right\}, \tag{37}$$

where $\phi \in (0, 1)$ and $\tilde{\phi}$ is an arbitrary point.

The following proposition holds:

PROPOSITION 2.1. *Let v_1 and v_2 be two solutions of (33), corresponding to the speeds c_1 and c_2, satisfying:* (i) $v_1(1) = v_2(1) = 0$ *and* (ii) $v_1(\phi)\, v_2(\phi) > 0$ *for all $\phi \in (0, 1)$. Then $\mathscr{G}(\phi) \to 0$ when $\phi \to 1$.*

Proof. The integral in (37) can be decomposed into the sum of two integrals:

$$\int_{\tilde{\phi}}^{\phi} \frac{(v_1^2 v_2 - v_2^2 v_1)\, D'(s)}{v_1 v_2 (v_2 - v_1)\, D(s)}\, ds \tag{38}$$

and

$$\int_{\tilde{\phi}}^{\phi} \frac{g(s)(v_2 - v_1)}{v_1 v_2(v_2 - v_1) D(s)} ds.$$

We observe that

$$\frac{(v_1^2 v_2 - v_2^2 v_1)}{v_1 v_2(v_2 - v_1)} = -1$$

and, using this equality in (38), the sum of the two above integrals becomes

$$-\int_{\tilde{\phi}}^{\phi} \frac{D'(s)}{D(s)} ds + \int_{\tilde{\phi}}^{\phi} \frac{g(s)}{v_1 v_2 D(s)} ds$$

$$= -\ln D(\phi) + \ln D(\tilde{\phi}) + \int_{\tilde{\phi}}^{\phi} \frac{g(s)}{v_1 v_2 D(s)} ds.$$

Using the above, (37) can be written as

$$\mathscr{G}(\phi) = \left\{ [v_2(\phi) - v_1(\phi)] \frac{D(\phi)}{D(\tilde{\phi})} \right\} \exp \left\{ -\int_{\tilde{\phi}}^{\phi} \frac{g(s)}{v_1 v_2 D(s)} ds \right\}.$$

Since $v_2(\phi)\, v_1(\phi) > 0$, $g(\phi) > 0$ in the interval $(0, 1)$ and $D(\phi) > 0$ for all $\phi \in (0, 1]$, the primitive of $g(\phi)/v_1(\phi)\, v_2(\phi)\, D(\phi)$ is greater than zero and so the last exponential is bounded. Therefore the *limit* of the exponential as $\phi \to 1$ exists, and since the *limit* of the other term enclosed in braces also exists, we have

$$\lim_{\phi \to 1} \mathscr{G}(\phi) = \lim_{\phi \to 1} \left\{ \frac{[v_2(\phi) - v_1(\phi)]\, D(\phi)}{D(\tilde{\phi})} \right\} \lim_{\phi \to 1} \exp \left\{ -\int_{\tilde{\phi}}^{\phi} \frac{g(s)}{v_1 v_2 D(s)} ds \right\} = 0 \quad \blacksquare$$

The following Lemma gives us a monotonicity property of the solutions of equation (33) with respect to the speed c for ϕ close to 1.

LEMMA 2.1. *Let $v_1(\phi)$ and $v_2(\phi)$ be two solutions of (33) corresponding to c_1 and c_2 respectively. Suppose that v_1 and v_2 satisfy;* (a) $v_1(1) = v_2(1) = 0$ *and* (b) $v_1(\phi)\, v_2(\phi) > 0$ *for all ϕ. Then for all ϕ in the interval* $(0, 1)$:

1. $v_1(\phi) = v_2(\phi)$ *if* $c_1 = c_2$
2. $v_1(\phi) > v_2(\phi)$ *if* $c_1 > c_2$
3. $v_1(\phi) < v_2(\phi)$ *if* $c_1 < c_2$.

Proof. From (37) and using (36) we have

$$\mathscr{G}'(\phi) = \frac{(c_1 - c_2)}{D(\phi)} \exp \left\{ -\int_{\tilde{\phi}}^{\phi} \frac{(v_1^2 v_2 - v_2^2 v_1)\, D'(s) + g(s)(v_2 - v_1)}{v_1 v_2(v_2 - v_1)\, D(s)} ds \right\}.$$

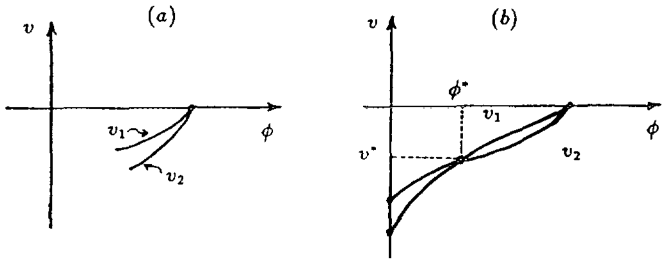


FIG. 2.5. (a) Geometrical interpretation of the result given in Lemma 2.1. (b) The supposition in the demonstration of Lemma 2.2 leads to a contradiction (see text for details).

Consider (1) to (3) separately. If $c_1 = c_2$, $\mathscr{G}'(\phi) = 0$ which implies that $\mathscr{G}(\phi) = constant$, for all ϕ. Using Proposition 2.1 we have $\mathscr{G}(\phi) = 0$ for all ϕ, and by (37) we conclude (1). If $c_1 > c_2$ the derivative $\mathscr{G}'(\phi)$ is positive, but since $\mathscr{G}(\phi) \to 0$ when $\phi \to 1$ this means that $\mathscr{G}(\phi) < 0$ for all $\phi \in (0, 1)$. Hence $v_1(\phi) > v_2(\phi)$. If $c_1 < c_2$, $\mathscr{G}'(\phi) < 0$ thus $\mathscr{G}(\phi)$ is decreasing for all $\phi \in (0, 1)$ and using the behaviour of $\mathscr{G}$ when $\phi \to 1$ we have that $\mathscr{G}(\phi) > 0$, therefore $v_1(\phi) < v_2(\phi)$ for all ϕ such that $0 < \phi < 1$. ∎

Remark 2.1. In geometrical terms, Lemma 2.1 gives us the relative position of the graphs of $v_1(\phi)$ and $v_2(\phi)$ for $\phi < 1$ with $(\phi - 1) \approx 0$. For example if $c_1 > c_2$ (both positive) the corresponding graphs are as illustrated in Fig. 2.5(a).

Remark 2.2. Lemma 2.1 can be verified, for instance along the left unstable manifold of P_1, on which v and ϕ are related by

$$v(\phi) = \left[\frac{-c + \sqrt{c^2 - 4g'(1)\,D(1)}}{2}\right](\phi - 1).$$

Thus for fixed ϕ in a neighbourhood of P_1 we have

$$\frac{dv}{dc} = \frac{(\phi - 1)}{2}\left[\frac{c - \sqrt{c^2 - 4g'(1)\,D(1)}}{\sqrt{c^2 - 4g'(1)\,D(1)}}\right],$$

from which, for $\phi < 1$, $dv/dc > 0$.

With Lemma 2.1 and Remark 2.1, we can demonstrate the following result on uniqueness of the heteroclinic trajectory for the system (24) connecting P_1 with P_c.

LEMMA 2.2. *There exists at most one heteroclinic trajectory for the system* (24) *connecting* P_1 *with* P_c.

Proof. Suppose that $v_1(\phi)$ and $v_2(\phi)$ are two solutions of equation (33) corresponding to the speeds c_1 and c_2 respectively, and satisfying the boundary conditions:

$$v_1(1) = v_2(1) = 0 \qquad \text{and} \qquad v_1(0) = -\frac{c_1}{D'(0)}, \qquad v_2(0) = -\frac{c_2}{D'(0)}.$$

Without loss of generality consider $c_1 > c_2$. By Lemma 2.1 the graph of v_2 is below that of v_1 as in Fig. 2.5(b). Now, the local analysis around P_c tells us that the relationship between v and ϕ along the right stable manifold of P_c is

$$v(\phi) = \left[\frac{c^2 D''(0) + g'(0)(D'(0))^2}{2c(D'(0))^2}\right]\phi - \frac{c}{D'(0)}.$$

For fixed ϕ, we have

$$\frac{dv}{dc} = \frac{c\, D''(0)}{2(D'(0))^2}\phi - \frac{g'(0)}{2c^2}\phi - \frac{1}{D'(0)},$$

from which $dv/dc \to -(1/D'(0)) < 0$ as $\phi \to 0$. Hence $v(\phi)$ along this manifold is decreasing and therefore the graph of v_1 is below that of v_2 (as can be seen in Fig. 2.5(b)). This means the existence of at least one point, (ϕ^*, v^*), in the semi-strip $\{(\phi, v) \mid 0 < \phi < 1, -\infty < v < 0\}$ in which both graphs intersect tranversally. This contradicts Lemma 2.1, hence our hypothesis at the beginning of the proof is false. If $c_1 < c_2$, the role of the graphs is reversed. Therefore, if there exists a heteroclinic trajectory of (24) between P_1 and P_c, it must be unique. ∎

Remark 2.3. If there exists a value of c, c^*, for which the system (24) possesses a heteroclinic connection between P_1 and P_c then, since equation (33) is invariant under the transformation $v \to -v$, $c \to -c$, it follows that for $c = -c^*$ the system (24) has a trajectory which goes from P_c to P_1, i.e., a solution of (33) satisfying the boundary conditions $v(0) = c^*/D'(0)$ and $v(1) = 0$.

For c^* as above, we have the following theorem whose proof is a consequence of Lemma 2.2:

THEOREM 2.1. *If the functions D and g in (12) satisfy the conditions given at the beginning of this section, then the reaction-diffusion Eq. (12) possesses at most a travelling wave solution $u(x, t) = \phi(x - c^*t)$ of sharp type such that*:

1. *For* $c^* > 0$: $\phi(-\infty) = 1$, $\phi(\xi) = 0$ *for* $\xi \geqslant \xi^*$; $\phi'(-\infty) = 0$, $\phi'(\xi^{*-}) = -c^*/D'(0)$, $\phi'(\xi^{*+}) = 0$ *and* $\phi'(\xi) = 0$ *for* $\xi > \xi^*$.

2. *For* $-c^*$: $\phi(\xi) = 0$ *for* $-\infty < \xi \leqslant \xi^*$, $\phi(+\infty) = 1$; $\phi'(+\infty) = 0$, $\phi'(\xi) = 0$ *for* $-\infty < \xi \leqslant \xi^*$, $\phi'(\xi^{*+}) = c^*/D'(0)$ *and* $\phi'(\xi^{*-}) = 0$.

EXAMPLE 2.1. Consider the degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(\beta u + u^2)\frac{\partial u}{\partial x}\right] + u(1-u), \tag{39}$$

where $\beta > 0$. Clearly the functions $D(u) = (\beta u + u^2)$ and $g(u) = u(1-u)$ satisfy the conditions imposed in this section and therefore all the above results hold.

Suppose that (39) has a travelling wave solution $u(x, t) = \phi(x - ct) = \phi(\xi)$. Setting $v = \phi'(\xi)$ we have that the second order ODE for ϕ can be written as the following singular system

$$\begin{aligned} \phi' &= v \\ v' &= \frac{1}{(\beta\phi + \phi^2)}\left[-cv - (\beta + 2\phi)\, v^2 - \phi(1-\phi)\right]. \end{aligned} \tag{40}$$

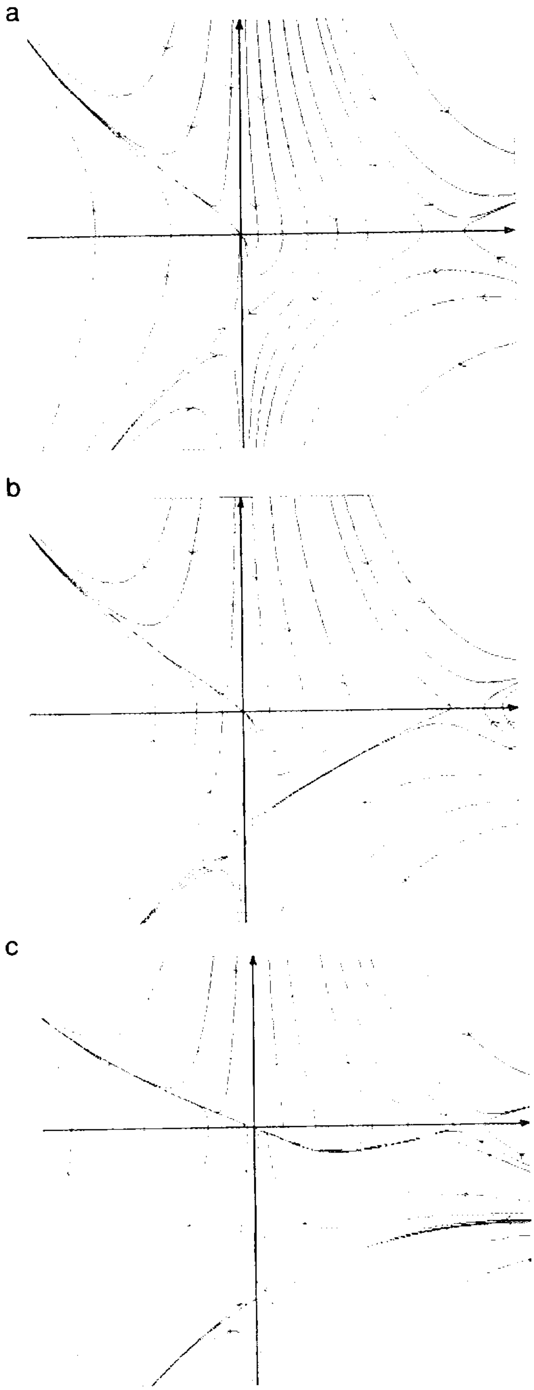
Introducing the new parameter τ as in (21) with $D(\phi) = \beta\phi + \phi^2$, we get a new system without singularity

$$\begin{aligned} \dot{\phi} &= (\beta\phi + \phi^2)\, v \\ \dot{v} &= -cv - (\beta + 2\phi)\, v^2 - \phi(1-\phi) \end{aligned} \tag{41}$$

whose equilibrium points in the region $\{(\phi, v) \mid 0 \leqslant \phi \leqslant 1, -\infty < v < +\infty\}$ are $P_0 = (0, 0)$, $P_1 = (1, 0)$ and $P_c = (0, v^*)$ where $v^* = -c/\beta$.

In Fig. 2.6 we show the phase portraits of the system (41) for $\beta = 2$ and for different values of c, which, of course, agree with our results. In particular, for values of c close to 1.1 one can suspect that Eq. (39) possesses a travelling wave solution of sharp type which, according to Theorem 3.1, must be unique (see Fig. 2.6(b)). Figures 2.6(a), (c) show the other two behaviours which have been predicted by the analysis of this section.

FIG. 2.6. Phase portraits of the system (41). (a) Behaviour for $0 < c < 1.1$. Here we take $c = 0.81$. (b) Behaviour for $c = 1.1$. (c) Behaviour for $c > 1.1$. Here we take $c = 2$.


a
b
c

3. A Result on Existence of Solutions of Front-Type for Some Degenerate Equations

The constraints $D'(0) \neq 0$ and $D''(0) \neq 0$ in the previous section led to a result on uniqueness of the travelling wave solution of sharp type for equation (12).

In this section we will consider the degenerate parabolic equation (12) in which the reactive part g satisfies the conditions (1) and (2) and the density-dependent diffusion coefficient, as well as satisfying condition (3) and the first part of condition (4) at the beginning of Section 2, must also satisfy:

$$D'(0) = 0 \qquad \text{and} \qquad D''(0) \neq 0.$$

We will investigate the existence of t.w.s. for such a reaction–diffusion equation using the same methodology employed in Section 2.

The system (24) now has only two equilibrium points: $P_0 = (0, 0)$ and $P_1 = (1, 0)$. The analysis at the beginning of the last section holds here. In particular the eigenvalues and the eigenvectors at P_0 are unchanged which, of course, is also a non-hyperbolic point. The first difference appears in the h.o.t. in the Taylor series. Since the first non-zero term is that of third order in the first equation of the system (24), we have the following non-linear system

$$\begin{aligned} \dot{\phi} &= \frac{D''(0)}{2}\phi^2 v \equiv F_1(\phi, v) \\ \dot{v} &= -g'(0)\,\phi - cv - \frac{g''(0)}{2}\phi^2 \equiv G_1(\phi, v) \end{aligned} \tag{42}$$

which approximates (24) in a neighborhood of P_0.

Following the methodology developed by Andronov *et al.*, we can still use the transformation (28) for this case. Thus, using (28) with $\kappa = -c$, the system (42) becomes

$$\begin{aligned} \frac{d\bar{\phi}}{d\bar{\tau}} &= A_1 \bar{v}\bar{\phi}^2 + A_2\bar{\phi}^3 \equiv \bar{F}_1(\bar{\phi}, \bar{v}) \\ \frac{d\bar{v}}{d\bar{\tau}} &= \bar{v} + B_1\bar{\phi}^2 + B_2\bar{v}\bar{\phi}^2 + B_3\bar{\phi}^3 \equiv \bar{v} + \bar{G}_1(\bar{\phi}, \bar{v}) \end{aligned} \tag{43}$$

where

$$A_1 = -\frac{D''(0)}{2c}, \qquad A_2 = \frac{D''(0)\,g'(0)}{2c^2}, \qquad B_1 = \frac{g''(0)}{2c}$$

$$B_2 = -\frac{g'(0)\,D''(0)}{2c^2}, \qquad \text{and} \qquad B_3 = -\frac{(g'(0))^2\,D''(0)}{2c^3}.$$

By the same arguments given in Section 2, in a neighbourhood of the point P_0 there are no closed paths of the system (43) (likewise for (42)), so the trajectories cannot be spirals. Therefore there must exist semipaths of (43) (or of (42) and so of (24)) which end in the equilibrium P_0.

Now we will determine the local behaviour of the trajectories of (43) and so of (42). To do this, consider the function $\mathscr{F}: R^2 \to R$ such that for $(\bar{\phi}, \bar{v})$

$$\mathscr{F}(\bar{\phi}, \bar{v}) = \bar{v} + B_1\bar{\phi}^2 + B_2\bar{v}\bar{\phi}^2 + B_3\bar{\phi}^3.$$

From its definition, $\mathscr{F}$ satisfies the hypothesis of the Implicit Function Theorem in a neighbourhood of P_0 and hence the equality $\mathscr{F}(\bar{\phi}, \bar{v}) = 0$ defines a unique solution $\bar{v} = \varphi(\bar{\phi})$ which satisfies several conditions (see Section 2). In this case

$$\varphi(\bar{\phi}) = \frac{-B_1\bar{\phi}^2 - B_3\bar{\phi}^3}{(1 + B_2\bar{\phi}^2)}.$$

Now we define ψ as

$$\psi(\bar{\phi}) = \bar{F}_1(\bar{\phi}, \varphi(\bar{\phi})) = A_1\bar{\phi}^2\left[\frac{-B_1\bar{\phi}^2 - B_3\bar{\phi}^3}{(1 + B_2\bar{\phi}^2)}\right] + A_2\bar{\phi}^3.$$

It can be shown that $\psi(0) = \psi'(0) = \psi''(0) = 0$, but

$$\psi'''(0) = \frac{3D''(0)\, g'(0)}{c^2} \neq 0.$$

Therefore the first term different to zero in the power series of ψ around $\bar{\phi} = 0$ is of third order. By the hypothesis $g'(0) > 0$ the sign of

$$\Delta_3 = \frac{\psi'''(0)}{6} = \frac{D''(0)\, g'(0)}{2c^2}$$

is the same as that of $D''(0)$. As the first term in the series of ψ is of odd order the behaviour depends on the sign of Δ_3 (Andronov *et al.*, 1973). Thus it is necessary to distinguish the following two cases:

1. If $D''(0) > 0$ then the point P_0 is node-like (in the topological sense).

2. If $D''(0) < 0$, the point P_0 is saddle-like, two of whose separatrices tend to P_0 in the directions $\pi/2$ and $3\pi/2$, respectively.

In order to complete the analysis around the point P_0 we observe that the Centre Manifold Theorem (see Appendix B) implies that the system (42), under the new assumptions at the beginning of this section, possesses a one-dimensional invariant centre manifold locally tangent to the eigenvector $\mathbf{v}_1 = (c, -g'(0))^T$ and the whole dynamic of the system (24) around P_0 is given in terms of the dynamic on its centre manifold. From Carr's Theorems (see Appendix B) the equation for the centre manifold for (42) is

$$[M\tilde{h}](\phi) = \tilde{h}'(\phi)\left[\frac{D''(0)}{2}\phi^2\tilde{h}(\phi)\right] + g'(0)\,\phi + c\tilde{h}(\phi) + \frac{g''(0)}{2}\phi^2.$$

Setting $\tilde{h}(\phi) = o(\phi^k)$ with $k > 1$, this equality takes the form

$$[M\tilde{h}](\phi) = g'(0)\,\phi + c\tilde{h}(\phi) + \frac{g''(0)}{2}\phi^2 + o(\phi^{2k+1})$$

and if we take

$$\tilde{h}(\phi) = -\frac{1}{2}\left[g'(0)\,\phi + \frac{g''(0)}{2}\phi^2\right]$$

then, by Carr's Theorems, we have the following approximation to the centre manifold:

$$h(\phi) = \frac{1}{c}\left[-\frac{g''(0)}{2}\phi^2 - g'(0)\,\phi\right] + o(\phi^{2k+1}). \tag{44}$$

The flow on the centre manifold is given by the equation

$$\dot{w} = F_1(\phi, h(\phi)) = \frac{D''(0)}{2c}\left[-\frac{g''(0)}{2}\phi^4 - g'(0)\,\phi^3\right] + o(\phi^{2k+1}).$$

For ϕ very close to zero, the dynamic is given by the ODE

$$\dot{w} = \frac{D''(0)}{2c}\left[-\frac{g''(0)}{2}\phi^4 - g'(0)\,\phi^3\right]. \tag{45}$$

The phase portrait for (45) is sketched in Fig. 3.1(a), and Fig. 3.1(b) illustrates the qualitative behaviour of the solutions of (45) for $D''(0) > 0$. The corresponding behaviour for $D''(0) < 0$ can be seen in Figs. 3.1(c) and (d).

We summarize all the above analyses as follows:

For $D''(0) > 0$ all the trajectories of (24) (including the hypothesis made at the beginning of this section) tend to P_0 along the centre manifold and P_0 looks like a node. For $D''(0) < 0$ the centre manifold acts as the unstable manifold (left and right) of P_0 which looks like a saddle point. In a neighbourhood of P_0, except on the v-axis (the stable manifold of P_0), all

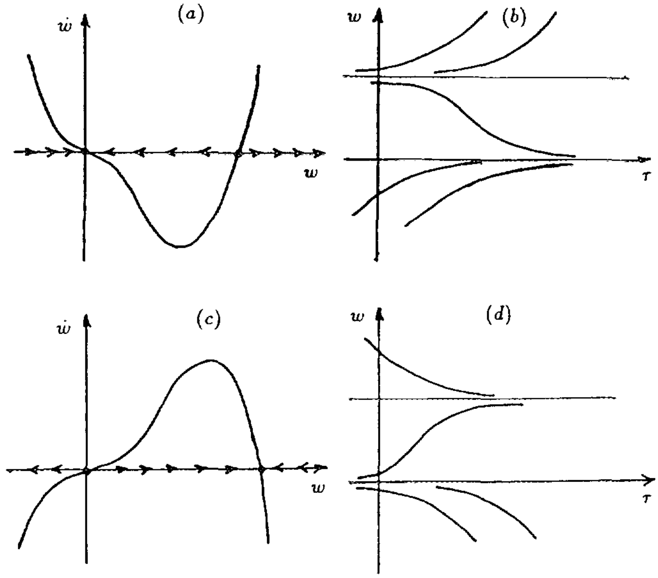


FIG. 3.1. (a) Graph of (45) for $D''(0) > 0$. (b) Solutions of (45) for $D''(0) > 0$. (c) Phase portrait of (45) for $D''(0) < 0$. (d) Behaviour of the solutions of (3.43) with $D''(0) < 0$.

the trajectories of (24) leave P_0 through the centre manifold. We sketch the qualitative behaviour around P_0 in Fig. 3.2 for both cases.

To continue our local analysis, we evaluate (25) at P_1:

$$J[F, G]_{(1,0)} = \begin{bmatrix} 0 & D(1) \\ -g'(1) & -c \end{bmatrix} \tag{46}$$

from which $\operatorname{tr} J[F, G]_{(1,0)} = -c$ and $\det J[F, G]_{(1,0)} = g'(1)\, D(1)$. The eigenvalues of (46) are

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[-c \pm \sqrt{c^2 - 4g'(1)\, D(1)}] \tag{47}$$

and the corresponding eigenvectors are $\mathbf{v}_1 = (1, \lambda_1)^T$ and $\mathbf{v}_2 = (1, \lambda_2)^T$.

Here it is necessary to consider the following two sub-cases:

Sub-Case 1. $D(0) = D'(0) = 0$ and $D''(0) > 0$. With these properties D has a local minimum; but since $D(u) > 0$ for all $u \in (0, 1]$, it is, in fact, the absolute minimum. As $g'(1) < 0$ and $D(1) > 0$, the eigenvalues (47) are real with $\lambda_1 > 0$ and $\lambda_2 < 0$ so P_1 is a saddle point. With this we have finished the local analysis for this sub-case, and we will start the global analysis.

From the above analysis we have concluded that P_0 is qualitatively a node and P_1 is a saddle point. With this observation we can draw an analogy with the analysis made by Hadeler (1981) and attempt to

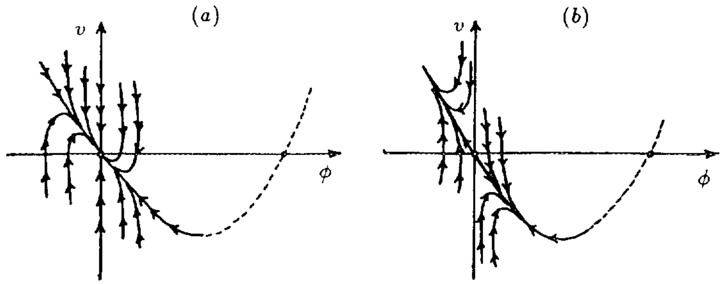


FIG. 3.2. (a) Dynamics around P_0 for $D''(0) > 0$. (b) Dynamics around P_0 for $D''(0) < 0$ (see text for details).

construct a positive invariant set for the system (24) under the present conditions on D. For this purpose we define a function $\rho: R \to R$ such that $\rho(0) = 0$, $\rho(1) > 0$, $\rho(\phi) > 0$ in the interval $(0, 1)$, $\rho'(\phi) \in C^1_{[0,1]}$ and $\rho'(0) > 0$. We define the region $\mathscr{R}$ as

$$\mathscr{R} = \{(\phi, v) \mid 0 < \phi < 1, \ -\rho(\phi) < v < 0\}.$$

In addition we require that in a neighbourhood of $\phi = 0$ the graph of $-\rho$ is below the graph of the centre manifold (44) (see Fig. 3.3). As the arguments are the same as in Hadeler (1981), we will check only that any trajectory of (24) which attains the curve P_0Q in Fig. 3.3 remains in the region $\mathscr{R}$. One necessary and sufficient condition for this is that the inner

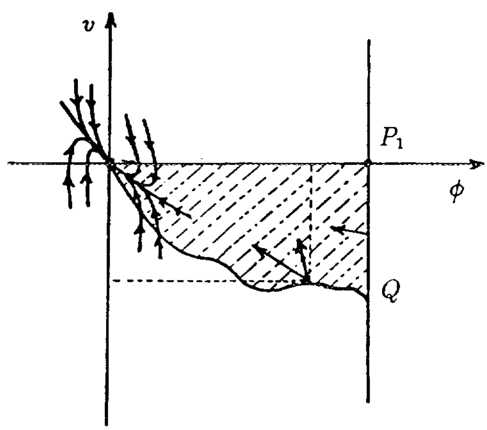


FIG. 3.3. Construction of an invariant set for the system (24) together with the conditions of this section.

product $(\rho'(\phi), 1)\cdot(\dot{\phi}, \dot{v})$ is greater than or equal to zero for all points $(\phi, -\rho(\phi))$ with $0<\phi<1$. Thus we have

$$\mathbf{n}_{\text{int}}\cdot(\dot{\phi}, \dot{v}) = (\rho'(\phi), 1)\cdot(-D(\phi)\,\rho(\phi), c\rho(\phi) - D'(\phi)\,\rho^2(\phi) - g(\phi))$$
$$= -\rho'(\phi)\,\rho(\phi)\,D(\phi) + c\rho(\phi) - D'(\phi)\,\rho^2(\phi) - g(\phi)$$

so

$$\mathbf{n}_{\text{int}}\cdot(\dot{\phi}, \dot{v}) \geqslant 0 \Leftrightarrow [-\rho'(\phi)\,\rho(\phi)\,D(\phi) + c\rho(\phi) - D'(\phi)\,\rho^2(\phi) - g(\phi)] \geqslant 0,$$

where $\mathbf{n}_{\text{int}}$ is the inward pointing normal on the boundary.

This inequality leads to the following condition on c:

$$c \geqslant \left[\rho'(\phi)\,D(\phi) + D'(\phi)\,\rho(\phi) + \frac{g(\phi)}{\rho(\phi)}\right]$$

or

$$c \geqslant \left\{\frac{d}{d\phi}[\rho(\phi)\,D(\phi)] + \frac{g(\phi)}{\rho(\phi)}\right\}.$$

Thus if we choose c such that

$$c \geqslant \sup\left\{\frac{d}{d\phi}[\rho(\phi)\,D(\phi)] + \frac{g(\phi)}{\rho(\phi)}\right\} \tag{48}$$

where the sup is taken on $\phi \in (0, 1]$, then the region $\mathscr{R}$ is a positive invariant set of (24) and so any trajectory of such a system which enters $\mathscr{R}$ remains there. This is true, in particular, for the trajectory leaving P_1 through its left unstable manifold. This trajectory enters the point P_0 through the centre manifold calculated before. This follows by a direct application of the Poincaré–Bendixon Theorem and previous analysis.

Moreover, by considering the set of functions ρ satisfying the above conditions we can characterise the lowest bound c_0 of c for which there exists a saddle (P_1)-node (P_0) connection as the variational problem

$$c_0 = \inf\sup\left\{\frac{d}{d\phi}[\rho(\phi)\,D(\rho)] + \frac{g(\phi)}{\rho(\phi)}\right\},$$

where the inf is taken on the set of functions ρ.

We summarize the above results in the following theorem:

THEOREM 3.1. *If the functions D and g in Eq. (12) satisfy the conditions (1)–(3) in Section 2, and $D \in C^2_{[0,1]}$ such that $D'(0)=0$ but $D''(0) \neq 0$, then for each $c \geqslant c_0$ satisfying (48) there exists a heteroclinic connection for the system (24) from P_1 to P_0 and hence the corresponding travelling wave solution for the reaction-diffusion Eq. (12) is of front type satisfying the boundary conditions*

$$\phi(-\infty) = 1 \qquad \textit{and} \qquad \phi(+\infty) = 0.$$

We illustrate the application of this theorem by the following example.

EXAMPLE 3.1. Here we will study the non-linear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[u^2\frac{\partial u}{\partial x}\right]+u(1-u). \tag{49}$$

As usual, suppose that $u(x,t)=\phi(x-ct)=\phi(\xi)$ is a travelling wave solution of (49). Substituting in (49), writing the corresponding singular system of ODEs and introducing the parameter τ as in (21) with $D(\phi)=\phi^2$, we get a new non-singular system

$$\begin{aligned}\dot\phi &= \phi^2 v \equiv F(\phi,v)\\ \dot v &= -cv-2\phi v^2-\phi(1-\phi)\equiv G(\phi,v)\end{aligned} \tag{50}$$

with equilibrium points $P_0=(0,0)$ and $P_1=(1,0)$. By the analysis of this section, P_0 is node-like and P_1 is a saddle point.

For this particular case it is easy to construct a positive invariant set $\mathscr{R}$ for (50) in a more elementary way instead of choosing the function ρ as mentioned above. First, we draw the vector field defined by (50). This is sketched in Fig. 3.4(a).

Let $v_i(\phi)$, $i=1,2$ be functions such that $G(\phi,v_i)=0$ for all ϕ in the interval $(0,1]$, i.e.,

$$v_1(\phi)=\frac{1}{4\phi}\left[-c+\sqrt{c^2-8\phi^2(1-\phi)}\right]$$

and

$$v_2(\phi)=\frac{1}{4\phi}\left[-c-\sqrt{c^2-8\phi^2(1-\phi)}\right];$$

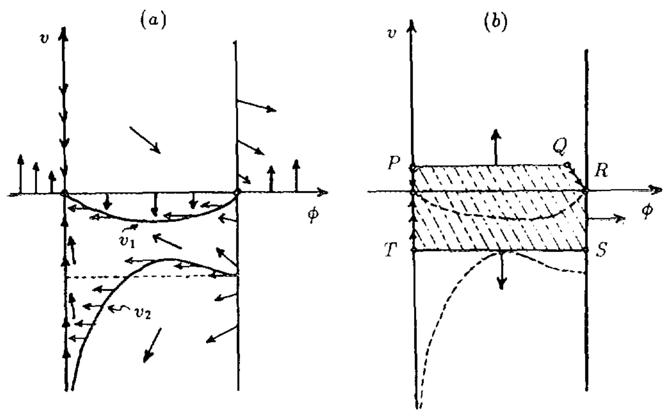


FIG. 3.4. (a) Vector field defined by (50). (b) An invariant set for the system (50).

the geometrical properties[7] of v_i will be useful in order to construct the set $\mathscr{R}$. Now we take a small neighbourhood of P_1 and we consider a closed region $\mathscr{R}$ in the ϕv-plane, as can be seen in Fig. 3.4(b), where the segments QR and ST are such that QR is on the left stable manifold of P_1 and ST is on the straight line $v \equiv \max v_2(\phi)$. Here the maximum is taken on the interval $(0, 1]$. Denote by **n** the exterior normal vector for each segment of the boundary of $\mathscr{R}$. It is easy to verify that the inner product $[\mathbf{n} \cdot (\dot{\phi}, \dot{v})]$ is negative on the segments PQ, RS, and ST. On the segments TP and QR, $[\mathbf{n} \cdot (\dot{\phi}, \dot{v})] = 0$. Therefore $\mathscr{R}$ is a positive invariant set of (50). Consequently, all trajectories which attain $\mathscr{R}$ remain there[8]. The trajectory which leaves P_1 through its left unstable manifold has this property. By a straightforward application of the Poincaré–Bendixon Theorem, such a trajectory must end in P_0 and therefore we have a heteroclinic connection between P_1 and P_0, whose corresponding travelling wave solution (of (49)) satisfies the boundary conditions $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. In Fig. 3.5(a) we show this heteroclinic trajectory for $c = \sqrt{37/24} \approx 1.08866$. Note the behaviour of the trajectory from P_1 to P_0 when the speed c varies (Figs. 3.5(b)–(d)).

Sub-Case 2. $D(0) = D'(0) = 0$ and $D''(0) < 0$. Here D has a relative maximum at $u = 0$ which implies that, at least locally, $D(u)$ must be negative. In spite of the unphysical meaning of this *negative diffusion* term, for mathematical completeness we will suppose that $D(u) < 0$ for all $u \in (0, 1]$. (In such cases, the partial differential equation problem may be ill-posed.) The local behaviour around P_1 can be characterized as follows:

1. If $[c^2 - 4g'(1)\,D(1)] \geqslant 0$, λ_1 and λ_2 are negative and so P_1 is an asympotically stable node.

2. If $[c^2 - 4g'(1)\,D(1)] < 0$, λ_1 and λ_2 are complex with negative real part. Thus P_1 is an asymptotically stable focus.

For the global behaviour in this sub-case we suspect that for certain values of the speed c the trajectory which leaves P_0 through the centre manifold

[7] Functions v_i are defined for all $\phi \in (0, 1]$ if and only if the parameter c is chosen such that $c^2 \geqslant \max \Psi(\phi) = 32/27$ where $\Psi(\phi) = 8\phi^2(1-\phi)$. For these values of c, $v_1(\phi)$ and $v_2(\phi)$ have the properties:

1. $v_i(\phi) \leqslant 0$ for all $\phi \in (0, 1]$.
2. $v_1(1) = 0$ and $v_2(1) = -c/2$.
3. $\lim_{\phi \to 0^+} v_1(\phi) = 0$ and $\lim_{\phi \to 0^+} v_2(\phi) = -\infty$.

[8] Trajectories on the segments QR and TP actually remain on them; but on the segments PQ, RS, and ST the trajectories enter $\mathscr{R}$.

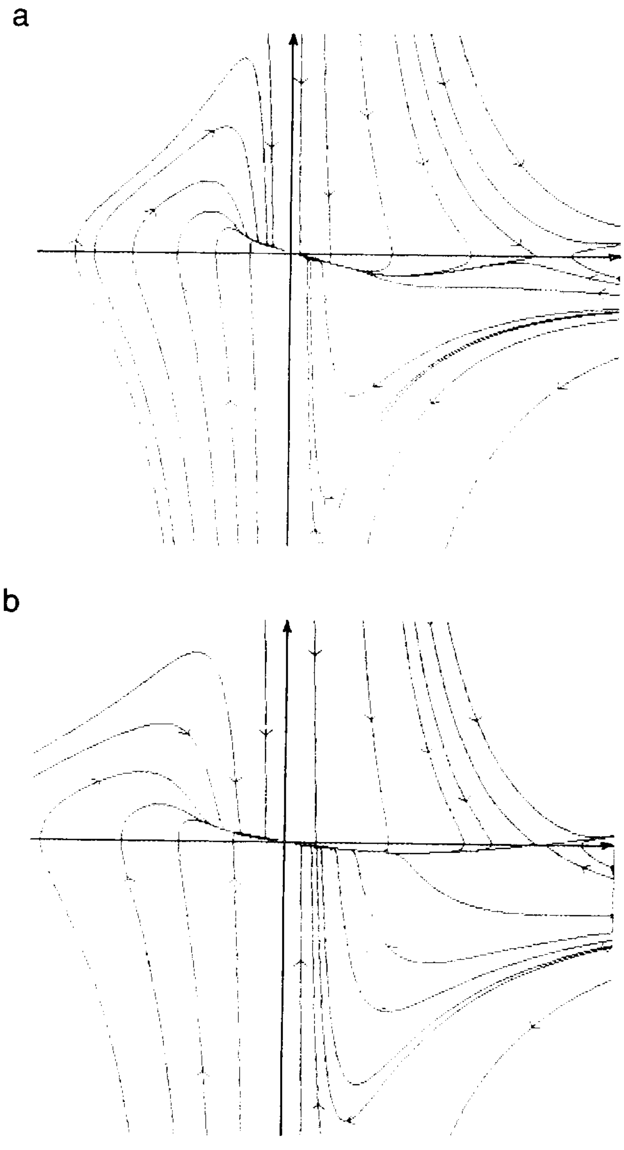


FIG. 3.5. Phase portraits of (50) for different values of c. (a) $c = 0.8$, (b) $c = 0.5$, (c) $c = 1.08866$, and (d) $c = 2$.

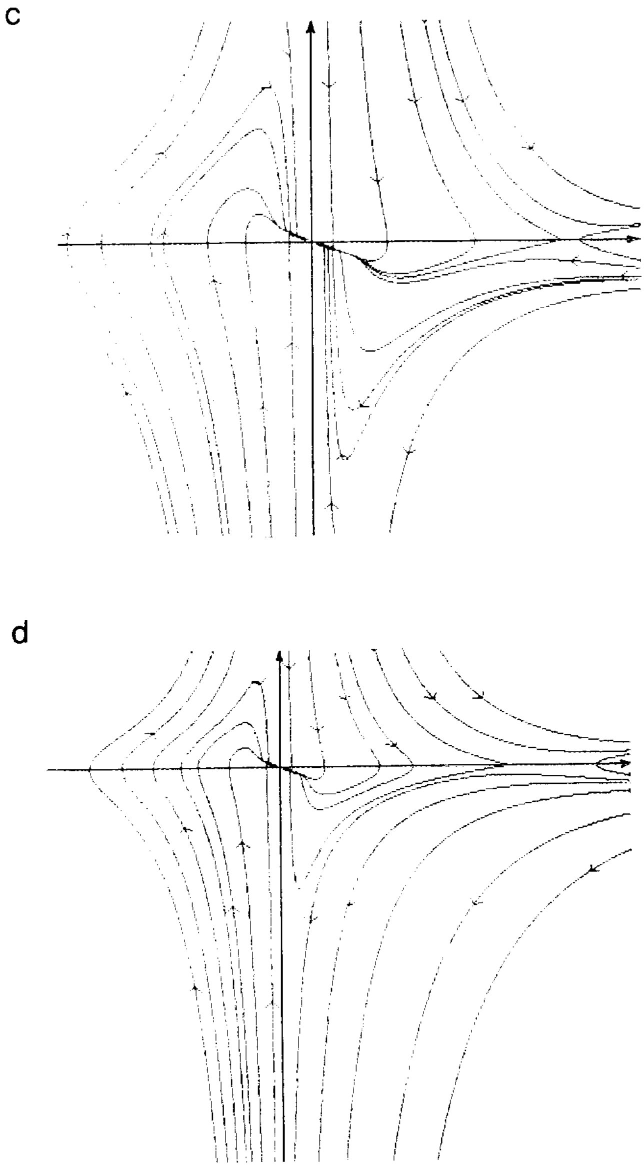


FIG. 3.5—*Continued*

enters the point P_1. In fact we have numerical evidence in some particular cases for this possibility for (1) and (2). The oscillating heteroclinic connection between P_0 and P_1 for (2) is remarkable here. Its significance, in terms of travelling wave solutions for (12), is that corresponding to this type of heteroclinic connection we have an *oscillating travelling wave solution* which satisfies the boundary conditions $\phi(-\infty)=0$ and $\phi(+\infty)=1$. The following example shows the behaviour in this sub-case.

EXAMPLE 3.2. Consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[-u^2\frac{\partial u}{\partial x}\right]+u(1-u) \tag{51}$$

whose travelling wave solution $u(x,t)=\phi(x-ct)$ satisfies a singular second order ODE. The reparametrization (21) with $D(u)=-u^2$ removes the singularity, but changes the sense of the trajectories of the corresponding system of ODEs. Thus, we have the non-singular system

$$\begin{aligned} \dot{\phi} &= -\phi^2 v \equiv F(\phi, v) \\ \dot{v} &= -cv+2\phi v^2-\phi(1-\phi)\equiv G(\phi, v) \end{aligned} \tag{52}$$

which has two equilibrium points $P_0=(0,0)$ and $P_1=(1,0)$. By the analysis in this section, P_0 is a saddle-like point, while P_1 has different features depending on the eigenvalues of the Jacobian matrix

$$J[F,G]_{(1,0)}=\begin{bmatrix} 0 & -2 \\ 1 & -c \end{bmatrix} \tag{53}$$

whose $\operatorname{tr} J[F,G]_{(1,0)}=-c$ and $\det J[F,G]_{(1,0)}=2$. The eigenvalues are

$$\lambda_1, \lambda_2=\tfrac{1}{2}[-c\pm\sqrt{c^2-4}].$$

Since $c>0$, P_1 is asymptotically stable; and

1. It is node-like if $(c^2-4)\geqslant 0$.
2. It is focus-like if $(c^2-4)<0$.

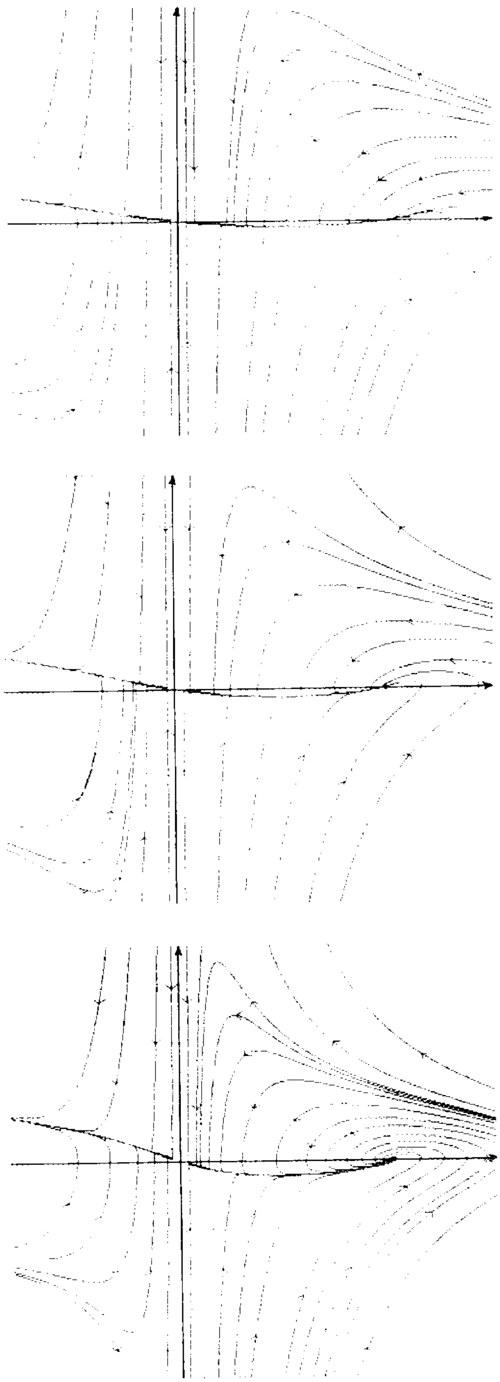
In Fig. 3.6 we show the phase portrait of the system (52) which was obtained by numerical simulation. This suggests the existence of different types of travelling wave solutions for Eq. (51) (Fig. 3.7) in particular those of oscillatory type.

FIG. 3.6. Phase portrait of the system (52). (a) For $c>2$. Here $c=3$. (b) For $c=2$. (c) For $0<c<2$. Here $c=1$.

a

b

c



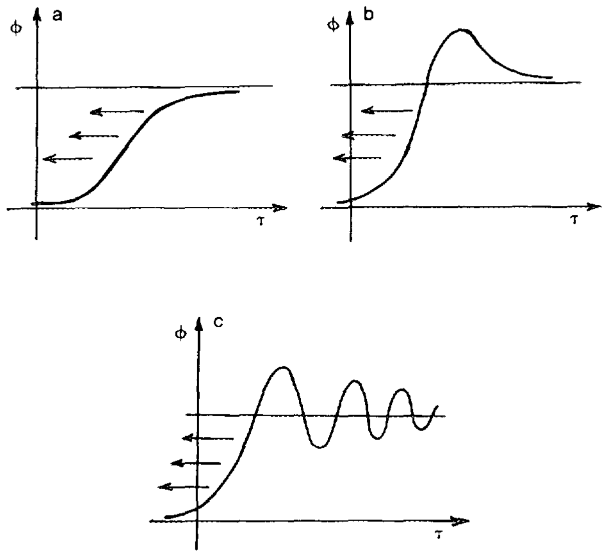


FIG. 3.7. Different travelling wave solutions for (51). (a) Monotonic front. (b) Damped front. (c) Damped oscillating front.

4. CONCLUSIONS

In this paper we have proved an uniqueness theorem for t.w.s. of sharp type for the generalized reaction–diffusion Eq. (12) with $D(0)=0$ but $D'(0)\neq 0$ and used numerical simulations to show the existence of such solutions for particular cases. As far as we know, our result on uniqueness is new in the literature, while the numerical evidence for existence extends previous work by Aronson (1980) and Murray (1989).

We have also stated sufficient conditions on the speed c for the existence of t.w.s. of front type for equation (12) with $D(0)=D'(0)=0$ and $D''(0)>0$. Here we generalized the method developed by Hadeler (1981). Furthermore we stated sufficient conditions on the diffusion coefficient D for the appearance of t.w.s. of oscillatory type for equation (12) with $D(0)=D'(0)=0$ but $D''(0)<0$ and analysed one example in particular. This phenomenon is not common in the literature.

Some natural extensions to our work are the analysis of existence of t.w.s. for (12) when both the kinetic part $g(u)$ and the diffusion coefficient $D(u)$ have different shapes. A typical example is $g(u)=u(1-u)(u-\alpha)$ for suitable values of α with changes in the sign of D corresponding to aggregation-diffusion phenomenon (Sánchez-Garduño and Maini, in preparation). Also the analysis in higher dimensions and in coupled non-linear reaction–diffusion equations could be of interest.

APPENDIX. A: Behaviour Around a Plane-Non-Hyperbolic Point

Here we will consider the system

$$\begin{aligned} \dot{x} &= ax + by + P_2(x, y) \\ \dot{y} &= cx + dy + Q_2(x, y), \end{aligned} \tag{54}$$

where P_2 and Q_2 are analytic functions in a neighbourhood of the origin and their series expansions involve only terms of higher or equal to second order. In addition we suppose that the origin is the unique isolated equilibrium point of (54) and that

$$\mathrm{tr}[A] = a + d \neq 0 \qquad \text{and} \qquad \det[A] = (ad - bc) = 0.$$

When $a = b = 0$, the non-singular linear transformation $\bar{x} = x$, $\bar{y} = (c/d)\,x + y$ reduces the system (54) to the form

$$\begin{aligned} \frac{d\bar{x}}{d\bar{t}} &= \bar{P}_2(\bar{x}, \bar{y}) \\ \frac{d\bar{y}}{d\bar{t}} &= \bar{y} + \bar{Q}_2(\bar{x}, \bar{y}), \end{aligned} \tag{55}$$

where $\bar{t} = \kappa t$ with κ a constant and $\bar{P}_2$ and $\bar{Q}_2$ satisfy the same conditions as P_2 and Q_2. For notational convenience we suppress the bar and we write

$$\begin{aligned} \dot{x} &= P_2(x, y) \\ \dot{y} &= y + Q_2(x, y). \end{aligned} \tag{56}$$

Consider the equality

$$y + Q_2(x, y) = 0. \tag{57}$$

By the Implicit Function Theorem, (57) defines a unique solution $y = \varphi(x)$ in a small neighbourhood of the origin, where φ is an analytic function such that $\varphi(0) = 0$, $\varphi'(0) = 0$.

We define the function

$$\psi(x) = P_2(x, \varphi(x)). \tag{58}$$

The function ψ does not vanish identically since if $\psi(x) \equiv 0$ then all the points of the curve $y = \varphi(x)$ are equilibrium points of (56), but we suppose that the origin is an isolated equilibrium point of (54) (and therefore of (56)). Hence the expansion series of ψ has the general form

$$\psi(x) = \Delta_m x^m + \cdots, \tag{59}$$

where $m \geqslant 2$, $\Delta_m \neq 0$. Now we will state the theorem used in Sections 3.1, 3.2 and 3.3. Its proof can be found in Andronov *et al.* (1973).

THEOREM A.1. *Let* $(0, 0)$ *be an isolated equilibrium point of the system* (56) *and let* $y = \varphi(x)$ *and* $\psi(x)$ *be as above. Then*:

1. *If* m *is odd and* $\Delta_m > 0$, *the origin is a topological node*
2. *If* m *is odd and* $\Delta_m < 0$, *the origin is a topological saddle point, two of whose separatrices tend to* $(0, 0)$ *in the directions* 0 *and* π, *the other two in the directions* $\pi/2$ *and* $3\pi/2$
3. *If* m *is even,* $(0, 0)$ *is a saddle-node, i.e., an equilibrium point whose canonical neighbourhood is the union of one parabolic and two hyperbolic sectors. If* $\Delta_m < 0$, *the hyperbolic sectors contain a segment of the positive* x*-axis bordering the origin and if* $\Delta_m > 0$ *they contain a segment of the negative* x*-axis.*

APPENDIX B: THE CENTRE MANIFOLD THEOREM

Let F: $R^n \to R^n$ be a smooth vector field with a non-hyperbolic point at the origin i.e., $F(\mathbf{0}) = \mathbf{0}$ and the Jacobian matrix $J[F]_0$ has at least one eigenvalue with zero real part. The Centre Manifold Theorem and related results:

1. ensure the existence of an invariant manifold (centre manifold) of the flow defined by the vector field F containing the point $\mathbf{0}$, and locally tangent to the eigenspace formed by the direct sum of the eigenspaces generated by the eigenvectors associated with the eigenvalues with zero real part
2. give the local dynamics (around the origin) in terms of the dynamics on the centre manifold
3. give an approximation to the centre manifold with sufficient degree of accuracy.

We will state the theorems without their proofs (see Carr, 1981, and Arrowsmith and Place, 1990 for full details). For our purpose we consider the system:

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y), \end{aligned} \tag{60}$$

where $x \in R^n$, $y \in R^m$ and A and B are constant matrices. All the eigenvalues of A have zero real parts, while all eigenvalues of B have negative real parts. The functions f and g belong to C^2 and satisfy:

1. $f(0,0)=0$, $J[f]_{(0,0)}=0$.
2. $g(0,0)=0$, $J[g]_{(0,0)}=0$.

THEOREM B.1. *There exists a centre manifold for* (60) $y=h(x)$, $|x|<\delta$ *where h is C^2.*

The flow on the centre manifold is given by the n-dimensional system

$$\dot{u}=Au+f(x,h(u)). \tag{61}$$

The next theorem tells us that the system (61) contains all the information to determine the local behaviour of the system (60) in a small neighbourhood of the origin.

THEOREM B.2. (a) *Suppose that the zero solution of* (61) *is stable (asymptotically stable) (unstable). Then the zero solution of* (60) *is stable (asymptotically stable) (unstable).* (b) *Suppose that the zero solution of* (61) *is stable. Let $(x(t), y(t))$ be a solution of* (60) *with $(x(0), y(0))$ sufficiently close to the origin. Then there exists a solution $u(t)$ of* (61) *such that as $t\to\infty$,*

$$x(t)=u(t)+o(e^{-\gamma t})$$

$$y(t)=h(u(t))+o(e^{-\gamma t}),$$

where $\gamma>0$ is a constant.

If we substitute $y(t)=h(x(t))$ into the second equation in (60) we obtain the equation for the centre manifold:

$$h'(x)[Ax+f(x,h(x))]=Bh(x)+g(x,h(x)) \tag{62}$$

which in general cannot be solved.

For functions $\phi: R^n\to R^m$ which are C^1 in a neighbourhood of the origin, if we define

$$[M\phi](x)=\phi'(x)[Ax+f(x,\phi(x))]-B\phi(x)-g(x,\phi(x)) \tag{63}$$

then, by (62), $[Mh](x)=0$.

The following theorem tells us that in principle the centre manifold can be approximated to any degree of accuracy:

THEOREM B.3. *Let ϕ be a C^1 mapping of a neighbourhood of the origin in R^n into R^m with $\phi(0)=0$ and $J[\phi]_{(0)}=0$. Suppose that as $x\to 0$, $[M\phi](x)=o(|x|^k)$ where $k>1$. Then as $x\to 0$, $|h(x)-\phi(x)|=o(|x|^k)$.*

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