PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 4, August 1993

THE α-BOUNDIFICATION OF α

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(Communicated by Franklin D. Tall)

ABSTRACT. A space X is $< \alpha$-bounded if for all $A \subseteq X$ with $|A| < \alpha$, $\mathrm{cl}_X A$ is compact. Let $B(\alpha)$ be the smallest $< \alpha$-bounded subspace of $\beta(\alpha)$ containing α. It is shown that the following properties are equivalent: (a) α is a singular cardinal; (b) $B(\alpha)$ is not locally compact; (c) $B(\alpha)$ is α-pseudocompact; (d) $B(\alpha)$ is initially α-compact. Define $B^0(\alpha) = \alpha$ and $B^n(\alpha) = \{\mathrm{cl}_{\beta(\alpha)} A : A \subseteq B^{n-1}(\alpha),\ |A| < \alpha\}$ for $0 < n < \omega$. We also prove that $B^2(\alpha) \neq B^3(\alpha)$ when $\omega = \mathrm{cf}(\alpha) < \alpha$. Finally, we calculate the cardinality of $B(\alpha)$ and prove that, for every singular cardinal α, $|B(\alpha)| = |B(\alpha)|^\alpha = |N(\alpha)|^{\mathrm{cf}(\alpha)}$ where $N(\alpha) = \{p \in \beta(\alpha) : \text{there is } A \in p \text{ with } |A| < \alpha\}$.

0. INTRODUCTION

In [15] O'Callaghan proved the following properties of the α-boundification $B(\alpha)$ of the discrete space of cardinality α (for definitions see 1.3 and 1.4).

0.1. (a) If α is a regular cardinal, then $B(\alpha)$ is the set of nonuniform ultrafilters on α.

(b) α is a singular cardinal if and only if $B(\alpha)$ contains a uniform ultrafilter.

(c) If we assume one of the following statements:

(i) GCH,

(ii) α is a strong limit cardinal,

(iii) α is a regular cardinal,

then $B(\alpha) \neq \beta(\alpha)$. Moreover, if (i) or (ii) holds, then $|B(\alpha)| \leq 2^\alpha$.

From 0.1 it follows that if α is regular, then $B(\alpha) = N(\alpha) = B^\xi(\alpha)$ for each $0 < \xi < \alpha^+$. Hence, $B(\alpha)$ is known when α is a regular cardinal. Thus, the following question, due to Comfort, appears natural.

0.2. Is $B^2(\alpha) \neq B^3(\alpha)$ whenever α is a singular cardinal?

In this paper we are principally concerned with singular cardinals. It is shown, in §2, that $B(\alpha) \neq \beta(\alpha)$ for every cardinal α (Corollary 2.4), and we will

Received by the editors November 12, 1990 and, in revised form, November 28, 1991.

1991 *Mathematics Subject Classification*. Primary 54D35, 54D80, 54D40.

Key words and phrases. $<\alpha$-bounded space, singular cardinal, regular cardinal, α-good point, weak P_α-point, F-space, α-pseudocompact, initially α-compact.

The second author is pleased to thank the Department of Mathematics at Wesleyan University for their generous hospitality during the academic year 1989. He acknowledges support from Universidad Nacional Autónoma de México.

0002-9939/93 \$1.00 + \$.25 per page

also obtain some topological properties of $B(\alpha)$. In §3 we answer Comfort's question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$. In these two sections Kunen's α-good ultrafilters will play an important role. In the last section, the cardinality of $B(\alpha)$ is calculated and we will prove that $|B(\alpha)| = |B(\alpha)|^{\alpha} = |N(\alpha)|^{\mathrm{cf}(\alpha)}$ for every singular cardinal.

1. Preliminaries

Throughout this paper, all spaces are assumed to be completely regular and Hausdorff. If X is a space and $B \subseteq X$, $\mathrm{cl}_X B$ denotes the closure of B in X. For $x \in X$, $\mathscr{N}(x)$ is the set of neighborhoods of x in X. $\mathscr{P}(X)$ is the set of all subsets of a set X. The Greek letters stand for ordinal numbers; in particular, α, κ, θ denote infinite cardinal numbers; γ, ν, μ denote arbitrary cardinals; and δ, ξ, λ, η denote ordinal numbers. For a cardinal α, we let α^+ stand for the smallest cardinal greater than α. For κ, γ cardinals we set $[\kappa]^{\gamma} = \{M \subseteq \kappa : |M| = \gamma\}$ and $[\kappa]^{<\gamma} = \{M \subseteq \kappa : |M| < \gamma\}$.

We do not distinguish notationally between a cardinal number α and the discrete space whose underlying set is that cardinal. For a space X, βX stands for the Stone-Čech compactification of X. If $f: X \to Y$ is a continuous function, we let $\overline{f}: \beta X \to \beta Y$ stand for the Stone extension of f. The remainder of βX is $X^* = \beta X \backslash X$; in particular, $\alpha^* = \beta(\alpha)\backslash\alpha$. For $A \subseteq \alpha$ we have that (see [2, Chapter 2])

$$\widehat{A} = \{p \in \beta(\alpha) : A \in p\} = \mathrm{cl}_{\beta(\alpha)} A \quad \text{and} \quad A^* = \widehat{A}\backslash A.$$

We shall use the terminology and notation of Comfort and Negrepontis [2].

The notion of α-bounded space was introduced in [7] and modified by Comfort as follows.

1.1. **Definition.** A space X is *$<\alpha$-bounded* if for every $A \subseteq X$ of cardinality less than α, $\mathrm{cl}_X A$ is a compact set.

It is evident that every space is $<\omega$-bounded, and if X is $<\alpha$-bounded, then X is $<\gamma$-bounded for every $\gamma \leq \alpha$. The basic properties of $<\alpha$-bounded spaces are summarized in the following proposition (see, e.g., [7, 8]).

1.2. **Proposition.** *Let α be a cardinal number. Then*

(a) *every compact space is $<\alpha$-bounded;*
(b) *every closed subset of a $<\alpha$-bounded space is $<\alpha$-bounded;*
(c) *the product of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded;*
(d) *the intersection of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded;*
(e) *the continuous image of a $<\alpha$-bounded space is $<\alpha$-bounded.*

Notice that (d) is a particular case of Lemma 2 of [8], and it is a consequence of (b) and (c).

1.3. For a $<\alpha$-bounded space Z and $X \subseteq Z$, we set

$$B_\alpha(X, Z) = \bigcap\{Y : X \subseteq Y \subseteq Z \text{ and } Y \text{ is } <\alpha\text{-bounded}\}.$$

It follows from 1.2(d) that $B_\alpha(X, Z)$ is the smallest $<\alpha$-bounded space containing X and is contained in Z. If $Z = \beta X$, then $B_\alpha(X, Z)$ will be denoted by $B_\alpha(X)$. In this case $B_\alpha(X)$ has the following extension property: For each

$<\alpha$-bounded space Y and each continuous function $f\colon X\to Y$, there exists a continuous function $\hat{f}\colon B_\alpha(X)\to Y$ such that $\hat{f}|_X=f$ [5, 8]. $B_\alpha(X)$ is called the *α-boundification* of X.

1.4. *Notation.* For a space X, we define

$B^0_\alpha(X)=X$;

$B^{\xi+1}_\alpha(X)=\bigcup\{\mathrm{cl}_{\beta X}A : A\subseteq B^\xi_\alpha(X)$ and $|A|<\alpha\}$ for each ordinal $\xi<\alpha^+$; and

$B^\xi_\alpha(X)=\bigcup_{\lambda<\xi}B^\xi_\alpha(X)$ if ξ is a limit ordinal.

Notice that $B_\alpha(X)=\bigcup_{\xi<\alpha^+}B^\xi_\alpha(X)$.

Let $p\in\beta(\alpha)$. The norm of p is $\|p\|=\min\{|A| : A\in p\}$, and p is *κ-uniform* if $\|p\|\geq\kappa$. If p is α-uniform, then p is said to be *uniform.* We set $U(\alpha)=\{p\in\beta(\alpha) : \|p\|=\alpha\}$; $N(\alpha)=\{p\in\beta(\alpha) : \|p\|<\alpha\}$; and $W_\kappa(\alpha)=\{p\in\beta(\alpha):\|p\|=\kappa\}$ $(\kappa<\alpha)$. We write $B(\alpha)$ and $B^\xi(\alpha)$ instead of $B_\alpha(\alpha)$ and $B^\xi_\alpha(\alpha)$, for $\xi<\alpha^+$, respectively. Note that $N(\alpha)=B^1(\alpha)$.

2. Some topological properties of $B(\alpha)$

In this section, we observe that $B(\alpha)\neq\beta(\alpha)$ for every cardinal α (Corollary 2.4), and we prove that some topological conditions are equivalent to the singularity of α (Theorem 2.10). We will first give some definitions and preliminary results.

2.1. **Definition.** Let X be a space, $B\subseteq X$, and $p\in\mathrm{cl}_X B$.

(a) $a(p,B)=\min\{|M| : M\subseteq B$ and $p\in\mathrm{cl}_X M\}$,

(b) p is said to be a *weak P_α-point* of X if $a(p,X\backslash\{p\})\geq\alpha$. A *weak P-point* is a weak P_{ω_1}-point.

2.2. **Definition.** (a) Let γ be a cardinal. A function $h\colon[\gamma]^{<\omega}\to\mathscr{P}(X)$ is called *monotone* if $h(A)\subseteq h(B)$ for $A, B\in[\gamma]^{<\omega}$ and $B\subseteq A$, and f is said to be *multiplicative* if $h(A\cup B)=h(A)\cup h(B)$ for $A, B\in[\gamma]^{<\omega}$.

(b) $p\in X$ is *α-good* if for each $\kappa<\alpha$ and each monotone function $f\colon[\kappa]^{<\omega}\to\mathscr{N}(p)$ there is a multiplicative function $g\colon[\kappa]^{<\omega}\to\mathscr{N}(p)$ which refines f (i.e., $g(A)\subseteq f(A)$ for all $A\in[\kappa]^{<\omega}$).

2.3. **Proposition.** *Let X be a space.*

(a) *If X is $<\alpha$-bounded and p is a weak P_α-point of X, then $X\backslash\{p\}$ is $<\alpha$-bounded.*

(b) (*Kunen*) *If X is a compact 0-dimensional space and $p\in X$ is α^{++}-good, then p is a weak P_{α^+}-point* (*for a proof see* [4, 4.8]).

(c) [14] *There are 2^{2^α} countably incomplete uniform ultrafilters in $\beta(\alpha)$ which are α^+-good.*

As an immediate consequence of Proposition 2.3 we have:

2.4. **Corollary.** (a) *For every limit cardinal α, there are 2^{2^α} countably incomplete weak P_α-points in $U(\alpha)$.*

(b) *$|\beta(\alpha)\backslash B(\alpha)|=2^{2^\alpha}$ for every α. In particular, we have that $B(\alpha)\neq\beta(\alpha)$ for every cardinal α* (*see* 0.1(c)).

In the next theorem, we are going to give some topological properties of $B(\alpha)$ when α is a singular cardinal. The concepts and results that follow are needed.

2.5. **Definition** (Saks-Woods). Let $M \subseteq \beta(\alpha)$. A space X is said to be *M-compact* if for every function $f: \alpha \to X$ we have that $\overline{f}(p) \in X$ for each $p \in M$.

The definition of p-compactness for a point $p \in \beta(\alpha)$ was given initially by Bernstein [1]. For other results on spaces required to be p-compact simultaneously for various p, see Woods [18] and Saks [17].

In [10] the topological properties which are productive, closed hereditary, and surjective are characterized in terms of ultrafilters as follows.

2.6. **Proposition.** *Let P be a topological property which is productive, closed hereditary, and surjective. A space X of cardinality α has P if and only if X is $P(\alpha)$-compact, where $P(\alpha)$ is the maximal P-reflection of α. In particular, a space X is $< \alpha$-bounded iff X is $B(\alpha)$-compact.*

2.7. **Definition.** Let X be a space and $\omega \leq \alpha$.

(a) X is said to be a *α-pseudocompact* if every continuous image of X in $\mathbb{R}^\alpha$ is compact.

(b) X is *initially α-compact* if every open cover $\mathscr{U}$ of X, with $|\mathscr{U}| \leq \alpha$, has a finite subcover.

The following lemma is due to Retta [16].

2.8. **Lemma.** *Let X be a space. Then X is α-pseudocompact if and only if every cozero cover of X of cardinality $\leq \alpha$ has a finite subcover.*

2.9. **Lemma** [6]. *If α is singular, then every $< \alpha$-bounded space is initially α-compact.*

Now we will prove the main result of this section.

2.10. **Theorem.** *The following conditions are equivalent.*

(a) *α is a singular cardinal.*
(b) *$B(\alpha)$ is not locally compact.*
(c) *$B(\alpha)$ is α-pseudocompact.*
(d) *Every $< \alpha$-bounded space is α-pseudocompact.*
(e) *$B(\alpha)$ is initially α-compact.*
(f) *Every $< \alpha$-bounded space is initially α-compact.*

Proof. (d) $\Rightarrow$ (c) and (f) $\Rightarrow$ (e) are trivial, and (a) $\Rightarrow$ (f), (a) $\Rightarrow$ (d), and (e) $\Rightarrow$ (c) are direct consequences of Lemmas 2.8 and 2.9. In order to complete the proof we will show (a) $\Leftrightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose that α is singular and $B(\alpha)$ is locally compact. Then $B(\alpha) \cap U(\alpha)$ is a nonempty open subset of $U(\alpha)$. Fix an arbitrary $p \in U(\alpha)$. We will show that $p \in B(\alpha)$. Indeed, since the type $T(p) = \{q \in \beta(\alpha) :$ there is a permutation h of α such that $\overline{h}(p) = q\}$ of p is a dense subset of $U(\alpha)$ (see [2] for a proof), there is $q \in T(p) \cap B(\alpha) \cap U(\alpha)$. Choose a permutation f of α such that $\overline{f}(q) = p$. According to Proposition 2.6, we have that $B(\alpha)$ is $B(\alpha)$-compact and so $\overline{f}(q) = p \in B(\alpha)$; thus, $p \in B(\alpha)$. But this implies that $\beta(\alpha) = B(\alpha)$, a contradiction to Corollary 2.4(b).

(b) $\Rightarrow$ (a) Suppose that α is a regular cardinal. From 0.1(a) it follows that $B(\alpha) = N(\alpha)$. Since $N(\alpha)$ is open in $\beta(\alpha)$, we have that $B(\alpha)$ is locally compact.

(c) $\Rightarrow$ (a) If α is a regular cardinal and $\mathscr{C} = \{\mathrm{cl}_{\beta(\alpha)} \kappa : \kappa < \alpha\}$, then $\mathscr{C}$ is a cozero cover of $N(\alpha) = B(\alpha)$ of cardinality α without a finite subcover. Now Retta's result (Lemma 2.8) implies that $B(\alpha)$ is not α-pseudocompact.

3. The sets $B^2(\alpha)$ and $B^3(\alpha)$

Our main goal here is to give an answer to Comfort's question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$ (see 3.5, 3.7, and 3.8). Because of 0.1, we will only be concerned with singular cardinals. Thus, throughout this section, α will denote a singular cardinal.

3.1. **Definition** (Keisler [11]). For $p \in X$, let

$$G(p) = \min\{\gamma : \gamma \text{ is a cardinal number and } p \text{ is not } \gamma^+\text{-good}\}.$$

$G(p)$ is called the *degree of goodness of* p.

3.2. We point out that $B^2(\alpha)\backslash B^1(\alpha)$ is dense in $U(\alpha)$. Indeed, let $\{\kappa_\xi : \xi < \mathrm{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to α ($\kappa_\xi \nearrow \alpha$), and let $A \in [\alpha]^\alpha$. Choose $p_\xi \in W_{\kappa_\xi}(\alpha) \cap \widehat{A}$ for each $\xi < \mathrm{cf}(\alpha)$. If $p \in \beta(\alpha)$ is a complete accumulation point of $\{p_\xi : \xi < \mathrm{cf}(\alpha)\}$, then $p \in (B^2(\alpha)\backslash B^1(\alpha)) \cap \widehat{A}$.

We will prove in 3.5 that, for each $\kappa < \alpha$, the subset of $B^2(\alpha)\backslash B^1(\alpha)$ of all ultrafilters of degree of goodness equal to κ^+ is dense in $U(\alpha)$. For our purpose we need the following two lemmas (they are Theorems 10.5 and 10.6 of [2], respectively).

3.3. **Lemma** (Keisler [12]). *Let $\omega \le \gamma \le \kappa$, let r be a function from κ onto γ, and let $e\colon \beta(\gamma) \to \beta(\kappa)$ be a continuous function such that $\overline{r} \circ e$ is the identity function on $\beta(\gamma)$. If $q \in \beta(\gamma)$ is countably incomplete and $p = e(q) \in \beta(\kappa)$, then p and q have the same degree of goodness.*

Let κ be a cardinal number. A family $\mathscr{F}$ of subsets of κ is said to have *the uniform finite intersection property* if $\mathscr{F} \neq \varnothing$ and $|\bigcap_{k \le n} A_k| = \kappa$ whenever $n < \omega$ and $A_k \in \mathscr{F}$ for every $k \le n$.

3.4. **Lemma.** *Let $\omega \le \gamma \le \kappa$. Every family of subsets of κ with the uniform finite intersection property and of cardinality at most κ is contained in 2^{2^κ} distinct uniform ultrafilters each of which is countably incomplete and has degree of goodness equal to γ^+.*

By an *α-partition* of α we mean a collection $\mathscr{F}$ of subsets of α such that: (a) $\alpha = \bigcup \mathscr{F}$; (b) $|A| = \alpha$ for every $A \in \mathscr{F}$; and (c) $A \cap B = \varnothing$ whenever A and B are distinct elements of $\mathscr{F}$.

3.5. **Theorem.** *For every $\kappa < \alpha$, the set*

$$\{p \in \beta(\alpha) : p \in B^2(\alpha) \cap U(\alpha) \text{ and } G(p) = \kappa^+\}$$

is dense in $U(\alpha)$.

Proof. Let $A \in [\alpha]^\alpha$, $\{A_\xi : \xi < \kappa\}$ be an α-partition of A, and $\{\alpha_\eta : \eta < \mathrm{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to α. For every $(\xi, \eta) \in \kappa \times \mathrm{cf}(\alpha)$, pick $p(\xi, \eta) \in \widehat{A_\xi} \cap W_{\alpha_\eta}(\alpha)$. For every $\xi < \kappa$ we choose a complete accumulation point p_ξ of $\{p(\xi, \eta) : \eta < \mathrm{cf}(\alpha)\}$. It is not difficult to see that $p_\xi \in \widehat{A_\xi} \cap U(\alpha) \cap B^2(\alpha)$ for each $\xi < \kappa$. Let $f\colon \kappa \to \beta(\alpha)$ be defined by $f(\xi) = p_\xi$ for $\xi < \kappa$. According to Lemma 3.4, we can take a countably incomplete ultrafilter $q \in \beta(\kappa)$ with $G(q) = \kappa^+$. Then $\overline{f}(q) \in B^2(\alpha) \cap U(\alpha) \cap \widehat{A}$ and, by Lemma 3.3, $G(\overline{f}(q)) = \kappa^+$.

The following theorem answers question 0.2 in the affirmative when $\omega = \mathrm{cf}(\alpha) < \alpha$ (see Corollary 3.8). We need the following lemma; its proof is standard in showing that regular Lindelöf spaces are normal.

3.6. **Lemma.** *Let X be a normal space. Let $E = \bigcup_{n<\omega} E_n$ and $D = \bigcup_{n<\omega} D_n$ be subsets of X such that $\mathrm{cl}_X(E_n) \cap \mathrm{cl}_X(D) = \mathrm{cl}_X(D_n) \cap \mathrm{cl}_X(E) = \varnothing$ for every $n < \omega$. Then there are two disjoint cozero sets $S, T \subseteq X$ satisfying $E \subseteq S$ and $D \subseteq T$.*

3.7. **Theorem.** *Assume that $\mathrm{cf}(\alpha) = \omega$. For each $n < \omega$, let $p_n \in U(\alpha)$ with $G(p_n) = \kappa_n^+$ where $\kappa_n \nearrow \alpha$. If p is an accumulation point of $D = \{p_n : n < \omega\}$, then $a(p, N(\alpha)) = \alpha$.*

Proof. Let $A \subseteq N(\alpha)$ be of cardinality $\gamma < \alpha$, and let $A_n = \{x \in A : \kappa_n < \|x\| \le \kappa_{n+1}\}$ for $n < \omega$. By 2.3(b), there is $N < \omega$ such that $p_n \notin \mathrm{cl}_{\beta(\alpha)} A$ for every $n > N$. Hence, without loss of generality, we may suppose that $D \cap \mathrm{cl}_{\beta(\alpha)} A = \varnothing$. If $p \in U(\alpha)$, $M \subseteq W_\kappa(\alpha)$ with $\kappa < \alpha$, and $p \in \mathrm{cl}_{\beta(\alpha)} M$, then $|M| = \alpha$; hence, $\mathrm{cl}_{\beta(\alpha)} D \cap \mathrm{cl}_{\beta(\alpha)} A_n = \varnothing$ for all $n < \omega$. By Lemma 3.6, we can find two disjoint cozero sets S and T of $\beta(\alpha)$ such that $A \subseteq S$, $D \subseteq T$. Since α^* is an F-space [2, 14.9], $\mathrm{cl}_{\beta(\alpha)} D \cap \mathrm{cl}_{\beta(\alpha)} A = \varnothing$.

The next corollary is an immediate consequence of 3.5 and 3.7.

3.8. **Corollary.** *If $\mathrm{cf}(\alpha) = \omega$, then $B^3(\alpha) - B^2(\alpha) \neq \varnothing$.*

4. The cardinality of $B(\alpha)$

We have mentioned (0.1(c)) that $|B(\alpha)| \le 2^\alpha$ when α satisfies some additional properties. In this section we improve this result by calculating $|B(\alpha)|$ for every α (Theorems 4.9, 4.13, and 4.18). We will also establish the relations among $|B(\alpha)|$, $|\beta(\alpha)|$, and $|N(\alpha)|$.

The following concept is basic in this section. For other properties of ultraproducts not considered here and historical notes see [2, Chapter 12].

4.1. **Definition.** Let $p \in \beta(\alpha)$, and let κ be a cardinal. We define the binary relation $\equiv$ on κ^α by

$$f \equiv g \quad \text{if } \{\xi < \alpha : f(\xi) = g(\xi)\} \in p.$$

It is easy to see that $\equiv$ is an equivalence relation on κ^α. We let κ^α/p be the set of $\equiv$-equivalence classes. κ^α/p is called the *ultraproduct of κ^α modulo p*.

The next theorem follows from Lemma 2 of [13] (see [2, 12.22]).

4.2. **Theorem.** *Let $p \in \beta(\alpha)$ be countably incomplete with $G(p) = \alpha^+$. If κ is an infinite cardinal, then $|\kappa^\alpha/p| = \kappa^\alpha$.*

The proof of the following lemma is straightforward.

4.3. **Lemma** [6]. *Let $\omega \le \kappa \le \alpha$ be cardinals, $p \in U(\kappa)$, and $\{A_\xi : \xi < \kappa\}$ be a partition of α. If $f, g : \kappa \to \alpha^*$ are functions such that $f(\xi), g(\xi) \in \widehat{A_\xi}$ for every $\xi < \kappa$, then $\overline{f}(p) = \overline{g}(p)$ if and only if $\{\xi < \kappa : f(\xi) = g(\xi)\} \in p$.*

In the following two results we calculate the cardinality of $W_\kappa(\alpha)$ and $N(\alpha)$ that will allow us to estimate $|B(\alpha)|$.

4.4. **Lemma.** *For $\omega \le \kappa \le \alpha$ we have that $|W_\kappa(\alpha)| = \alpha^\kappa \cdot 2^{2^\kappa}$.*

Proof. It is evident that $2^{2^\kappa} \le |W_\kappa(\alpha)| \le 2^{2^\kappa} \cdot \alpha^\kappa$, so we only need to show the inequality $\alpha^\kappa \le |W_\kappa(\alpha)|$. Let $\{A_\xi : \xi < \kappa\}$ be an α-partition of α. For each $\xi < \kappa$, let $\{p_{\xi,\zeta} : \zeta < \alpha\}$ be a strongly discrete subset of ultrafilters contained in $W_\kappa(\alpha) \cap \widehat{A_\xi}$. Fix $q \in U(\kappa)$ countably incomplete κ^+-good and, for each $f \in \alpha^\kappa$, we define $\phi_f : \kappa \to \alpha^*$ by $\phi_f(\xi) = p_{\xi, f(\xi)}$ for $\xi < \kappa$. Let $p_f = \overline{\phi}_f(q)$. Clearly, for every $f \in \alpha^\kappa$, $p_f \in W_\kappa(\alpha)$. Since $\{\xi < \kappa : \phi_f(\xi) = \phi_g(\xi)\} = \{\xi < \kappa : f(\xi) = g(\xi)\}$, and by Lemma 4.3, we have that $p_f = \overline{\phi}_f(p) = \overline{\phi}_g(p) = p_g$ if and only if $f \equiv g$. Using Theorem 4.2, we have that $\alpha^\kappa = |\alpha^\kappa / q| \le |W_\kappa(\alpha)|$.

The next result appears in [3] without proof.

4.5. **Corollary.** *For every α, the following equality holds:*

$$|N(\alpha)| = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma}.$$

Proof. We have that $N(\alpha) = \bigcup\{W_\gamma(\alpha) : \gamma < \alpha\}$. By virtue of 4.4, it follows that $|N(\alpha)| = \sum_{\gamma<\alpha} 2^{2^\gamma} \cdot \alpha^\gamma = \alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma}$.

4.6. Observe that $|N(\alpha)| = \alpha^{<\alpha}$ if α is a strong limit cardinal; otherwise, $|N(\alpha)| = \sum_{\gamma<\alpha} 2^{2^\gamma} = \sup_{\gamma<\alpha} 2^{2^\gamma}$.

The following two lemmas are needed in order to calculate the cardinality of $B(\alpha)$. For a proof of Lemma 4.7 see [9, Lemma 6.5 and exercise 6.14].

4.7. **Lemma.** *If α is a strong limit singular original, then*

$$\alpha^{<\alpha} = \alpha^{\mathrm{cf}(\alpha)} = 2^\alpha.$$

4.8. **Lemma.** *Let α be a nonstrong limit cardinal such that, for some cardinal $\theta < \alpha$, $\sum_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$. Then*

(a) $\mathrm{cf}(|N(\alpha)|) \ge \alpha^+$, *and*
(b) $|N(\alpha)| = |N(\alpha)|^\gamma$ *for every* $\gamma < \alpha$.

Proof. Set $\theta < \alpha$ such that $|N(\alpha)| = 2^{2^\theta}$ and $2^\theta \ge \alpha$.

(a) $\mathrm{cf}(|N(\alpha)|) = \mathrm{cf}(2^{2^\theta}) > 2^\theta \ge \alpha$.
(b) If γ is a cardinal less than α, then

$$|N(\alpha)|^\gamma = 2^{(2^\theta)\cdot\gamma} = 2^{2^\theta} = |N(\alpha)|.$$

4.9. **Theorem.** *Let $\omega \le \alpha$. Then $|B(\alpha)| = |N(\alpha)|$ whenever α satisfies one of the following properties:*

(a) *α is a regular cardinal.*
(b) *α is a singular cardinal which is not a strong limit and $\sup_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$ for some $\theta < \alpha$. In this case, $|B(\alpha)| = 2^{2^\theta}$.*
(c) *α is a singular strong limit. In this case we have $|B(\alpha)| = 2^\alpha$.*

Proof. When α is regular, the conclusion is a consequence of 0.1(a). Let α be a singular cardinal. Suppose that $|B^\xi(\alpha)| = |N(\alpha)|$ for every $\xi < \eta < \alpha^+$. If η is a limit ordinal, then

$$|N(\alpha)| \leq |B^\eta(\alpha)| = \left|\bigcup_{\xi<\eta} B^\xi(\alpha)\right| \leq \sum_{\xi<\eta} |B^\xi(\alpha)| = |\eta| \cdot |N(\alpha)| = |N(\alpha)|.$$

If $\eta = \xi + 1$, then

$$|B^\eta(\alpha)| \leq \sum\{|B^\xi(\alpha)|^\gamma \cdot 2^{2^\gamma} : \gamma < \alpha\} = \sum\{|N(\alpha)|^\gamma \cdot 2^{2^\gamma} : \gamma < \alpha\}.$$

Hence, if α satisfies (b) (resp. (c)), then we obtain the equality $|B^\eta(\alpha)| = |N(\alpha)|$ because of Lemma 4.8 (resp. Lemma 4.7). Therefore, in these two cases, $|N(\alpha)| \leq |B(\alpha)| \leq \alpha^+ \cdot |N(\alpha)| = |N(\alpha)|$.

Note that if α is a strong limit cardinal, then $|B(\alpha)| = |N(\alpha)| = \alpha^{<\alpha}$.

In 4.13 we shall have $|B(\alpha)|$ for those cardinals not considered in the previous theorem. We need the following definition and lemma.

4.10. **Definition.** Let $\omega \leq \kappa \leq \alpha$. A collection $\mathcal{G}$ of subsets of α is *κ-almost disjoint* if $|G| \geq \kappa$ for $G \in \mathcal{G}$ and $|G_0 \cap G_1| < \kappa$ for $G_0, G_1 \in \mathcal{G}$ and $G_0 \neq G_1$.

A proof of the following lemma can be found in [2, 12.2].

4.11. **Lemma.** *Let κ, γ be two cardinal numbers with $\omega \leq \kappa$ and $2 \leq \gamma$. Then there is a κ-almost disjoint family $\mathcal{G} \subseteq \mathcal{P}(\gamma^{<\kappa})$ on $\gamma^{<\kappa}$ of cardinality γ^κ.*

4.12. We will denote by L the set of cardinals that do not satisfy any properties considered in 4.9; that is, $L = \{\alpha : \alpha$ is a singular nonstrong limit cardinal such that $\sup_{\gamma<\alpha} 2^{2^\gamma} > 2^{2^\nu}$ for every $\nu < \alpha\}$. Observe that (see 4.6) if α is not a strong limit and $\{2^{2^\gamma}\}_{\gamma<\alpha}$ is not eventually constant (in particular, if $\alpha \in L$), then $\mathrm{cf}(\alpha) = \mathrm{cf}(|N(\alpha)|)$.

4.13. **Theorem.** *If $\alpha \in L$, then $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = 2^\kappa$ where $\kappa = 2^{<\alpha}$. Moreover, if $\omega \leq \gamma < \mathrm{cf}(\alpha)$, then $|N(\alpha)| = |N(\alpha)|^\gamma < |B(\alpha)|$.*

Proof. Let μ be a cardinal less than $|N(\alpha)|$. We choose $\gamma < \alpha$ such that $2^\gamma \geq \alpha$ and $\mu < 2^{2^\gamma}$. If $\nu < \alpha$, then $\mu^\nu \leq (2^{2^\gamma})^\nu = 2^{(2^\gamma)\cdot\nu} = 2^{2^\gamma} < |N(\alpha)|$; therefore (see Theorem 19 in [9]),

(*) for every $\mathrm{cf}(\alpha) \leq \nu < \alpha$ we obtain $|N(\alpha)|^\nu = |N(\alpha)|^{\mathrm{cf}(\alpha)}$, and

(**) $|N(\alpha)|^{<\mathrm{cf}(\alpha)} = |N(\alpha)|$.

By using inductively the equality in (*) we obtain that $|B^\xi(\alpha)| \leq |N(\alpha)|^{\mathrm{cf}(\alpha)}$ for every $\xi < \alpha^+$. Hence,

$$|B(\alpha)| \leq |N(\alpha)|^{\mathrm{cf}(\alpha)} \cdot \alpha^+ = |N(\alpha)|^{\mathrm{cf}(\alpha)}.$$

We are now going to prove that $|N(\alpha)|^{\mathrm{cf}(\alpha)} \leq |B^2(\alpha) \backslash B^1(\alpha)|$. Let $\mathcal{G} = \{G_f : f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}\}$ be a $\mathrm{cf}(\alpha)$-almost disjoint family on $|N(\alpha)|$ of cardinality $|N(\alpha)|^{\mathrm{cf}(\alpha)}$ (see Lemma 4.11 and (**)); let $G_f = \{\lambda_{f,\xi} : \xi < \mathrm{cf}(\alpha)\}$ be a faithful indexing of G_f for each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$; and let $\mathcal{A} = \{A_\delta : \delta < \mathrm{cf}(\alpha)\}$ be an α-partition of α and $\alpha_\delta \nearrow \alpha$.

Since $|\widehat{A_\delta} \cap B^1(\alpha)| = |N(\alpha)|$ for each $\delta < \mathrm{cf}(\alpha)$, we can take $B_\delta = \{p_{\delta,\xi} : \xi < |N(\alpha)|\} \subseteq \widehat{A_\delta} \cap B^1(\alpha)$ such that $\|p_{\delta,\xi}\| = \alpha_\delta$ for $\xi < \mathrm{cf}(\alpha)$ and $p_{\delta,\xi} \neq p_{\delta,\zeta}$ for

$\xi < \zeta < \mathrm{cf}(\alpha)$. For each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$, we consider the function ϕ_f: $\mathrm{cf}(\alpha) \to B^1(\alpha)$ defined by $\phi_f(\xi) = p_{\xi, \lambda_{f,\xi}}$ for $\xi < \mathrm{cf}(\alpha)$. Fix $q \in U(\mathrm{cf}(\alpha))$. Then $\overline{\phi}_f(q) \in \mathrm{cl}_{\beta(\alpha)} \phi_f(\mathrm{cf}(\alpha)) \subseteq B^2(\alpha)$. It suffices to prove that the relation $f \to \overline{\phi}_f(q)$ from $|N(\alpha)|^{\mathrm{cf}(\alpha)}$ to $B^2(\alpha)$ is one-to-one. Indeed, let f, $g \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$ such that $f \neq g$. It is evident that $\phi_f(\xi) = \phi_g(\xi)$ iff $\lambda_{f,\xi} = \lambda_{g,\xi}$ and so $|\{\xi < \mathrm{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi)\}| = |\{\xi < \mathrm{cf}(\alpha) : \lambda_{f,\xi} = \lambda_{g,\xi}\}| \leq |G_f \cap G_g| < \mathrm{cf}(\alpha)$. Hence, $\{\xi < \mathrm{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi)\} \notin q$. From Lemma 4.3, it follows that $\overline{\phi}_f(q) \neq \overline{\phi}_g(q)$. Reasoning as in 3.2, we can prove that $\overline{\phi}_f(q) \in B^2(\alpha) \backslash B^1(\alpha)$ for each $f \in |N(\alpha)|^{\mathrm{cf}(\alpha)}$; therefore, $|N(\alpha)|^{\mathrm{cf}(\alpha)} \leq |B^2(\alpha) \backslash B^1(\alpha)| \leq |B(\alpha)|$. Thus, we have that $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)}$.

It remains to show that $|B(\alpha)| = 2^\kappa$. Let $\theta = \sup_{\gamma<\alpha} 2^{2^\gamma} = |N(\alpha)|$. Since $\alpha \in L$, κ is a limit cardinal, $\mathrm{cf}(\alpha) = \mathrm{cf}(\theta) = \mathrm{cf}(\kappa)$, and $\theta = \sup_{\mu<\kappa} 2^\mu = 2^{<\kappa}$; therefore, $|B(\alpha)| = \theta^{\mathrm{cf}(\theta)} = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$. Because of Lemma 6.5 in [9], we conclude that $|B(\alpha)| = 2^\kappa$.

The last assertion of Theorem 4.13 is implied from the following inequality which is a consequence of $(**)$:

$$|N(\alpha)|^\gamma = |N(\alpha)| < |N(\alpha)|^{\mathrm{cf}(|N(\alpha)|)} = |N(\alpha)|^{\mathrm{cf}(\alpha)} = |B(\alpha)|.$$

We have finished the proof of Theorem 4.13. □

The following result was already shown in [6]. Here we give an alternative proof (see the definition of L in 4.12).

4.14. **Corollary.** *If $\omega < \alpha$, then*

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} \leq |B(\alpha)| \leq \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)}.$$

Proof. If α is a strong limit, then (see 4.5, 4.7, and 4.9)

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} = |B(\alpha)| = 2^\alpha = \alpha^{\mathrm{cf}(\alpha)} = \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)}.$$

If α is not a strong limit and either α is a regular cardinal or $\sum_{\gamma<\alpha} 2^{2^\gamma} = 2^{2^\theta}$ for some $\theta < \alpha$, then

$$\alpha^{<\alpha} \cdot \sum_{\gamma<\alpha} 2^{2^\gamma} = \sum_{\gamma<\alpha} 2^{2^\gamma} = |B(\alpha)| \leq \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)} \quad \text{(see 4.9)}.$$

Finally, when $\alpha \in L$, we have

$$|N(\alpha)| < |B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = \left(\sum_{\gamma<\alpha} 2^{2^\gamma}\right)^{\mathrm{cf}(\alpha)} \quad \text{(see 4.13)}.$$

The next corollary improves Theorem 3.5 in [6].

4.15. **Corollary.** *For every singular cardinal α, we have*

(#) $$|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} = |N(\alpha)|^\alpha = |B(\alpha)|^{\mathrm{cf}(\alpha)} = |B(\alpha)|^\alpha.$$

Proof. If $\alpha \notin L$, then the result follows from 4.9. Suppose that $\alpha \in L$. In this case we have $\mathrm{cf}(|N(\alpha)|) = \mathrm{cf}(\alpha) < \alpha < |N(\alpha)|$ and $\gamma^{\mathrm{cf}(\alpha)} < |N(\alpha)|$ for every $\gamma < |N(\alpha)| = \sup_{\gamma<\alpha} 2^{2^\gamma}$. Then $|N(\alpha)|^\alpha = |N(\alpha)|^{\mathrm{cf}(\alpha)}$. Now all the equalities in (#) follow from Theorem 4.13.

4.16. **Corollary.** *Let* $\omega \leq \alpha$. *Then* $|N(\alpha)| = |B(\alpha)|$ *if and only if* $\alpha \notin L$.

It is possible to construct a model M of ZFC in which $|N(\aleph_\omega)| < |B(\aleph_\omega)|$ (see [6]). In this model, $\aleph_\omega \in L$.

4.17. **Corollary.** *Let* $\omega \leq \alpha$. *Then, for every* $1 < \xi < \alpha^+$, *we have that* $|B^\xi(\alpha)| = |B(\alpha)|$.

Proof. We have that $|N(\alpha)| \leq |B^\xi(\alpha)| \leq |B(\alpha)|$. If $\alpha \notin L$, then $|N(\alpha)| = |B(\alpha)|$ (Corollary 4.16). We have proved in 4.13 that $|B(\alpha)| = |N(\alpha)|^{\mathrm{cf}(\alpha)} \leq |B^2(\alpha)|$ whenever $\alpha \in L$. This completes the proof.

In the following theorem we summarize the results regarding all the possible values of $|B(\alpha)|$.

4.18. **Theorem.** *Let* $\omega \leq \alpha$, $\kappa = 2^{<\alpha}$, *and* $\theta = \sup_{\gamma<\alpha} 2^{2^\gamma}$.

(a) *If* α *is a strong limit, then*

(i) $|B(\alpha)| = \alpha$ *if and only if* α *is regular*;

(ii) $|B(\alpha)| = 2^\alpha$ *if and only if* α *is singular.*

(b) *If* α *is not a strong limit, then*

(i) $|B(\alpha)| = 2^{2^\mu}$ *for some* $\mu < \alpha$ *if and only if either* α *is a successor cardinal or* $\{2^{2^\gamma}\}_{\gamma<\alpha}$ *is eventually constant*;

(ii) $|B(\alpha)| = 2^\kappa$ *whenever* $\alpha \in L$;

(iii) $2^{2^\mu} < |B(\alpha)| = \theta = 2^{<\kappa} < 2^\kappa \leq 2^{2^\alpha}$ *for every* $\mu < \alpha$ *whenever* α *is a regular limit and* $\{2^{2^\gamma}\}_{\gamma<\alpha}$ *is not eventually constant.*

Proof. We obtain (a) as a consequence of 4.6 and 4.9(c) and 1.27 in [2]. The necessity in (b)(i) is trivial (see 4.9), and (b)(ii) is proved in 4.13. We only have to prove (b)(i)($\Rightarrow$) and (b)(iii).

(b)(i)($\Rightarrow$) In this case, α does not belong to L because $\theta^{\mathrm{cf}(\theta)} > \theta \geq 2^{2^\gamma}$ for every $\gamma < \alpha$ (if $\alpha \in L$, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\theta)$ and $|B(\alpha)| = \theta^{\mathrm{cf}(\theta)}$; see 4.13). Thus, $|B(\alpha)| = |N(\alpha)| = \theta$. So, if $|B(\alpha)| = 2^{2^\mu}$ for some $\mu < \alpha$ and $\{2^{2^\gamma}\}_{\gamma<\alpha}$ is not eventually constant, then $\mu^+ = \alpha$.

(b)(iii) Since α is a regular nonstrong limit and $\{2^{2^\gamma}\}_{\gamma<\alpha}$ is not eventually constant, $2^{2^\mu} < \theta = |B(\alpha)|$ for every $\mu < \alpha$. It is also clear that $\{2^\gamma\}_{\gamma<\alpha}$ is not eventually constant, hence, neither is $\{2^\nu\}_{\nu<\kappa}$ and so $\sup_{\nu<\kappa} 2^\nu < 2^\kappa$. The inequality $2^\kappa \leq 2^{2^\alpha}$ always holds, and $\theta = \sup_{\nu<\kappa} 2^\nu$ follows from the properties of α.

As immediate consequences of the previous theorem we have the following corollaries (in the first one we determine the conditions under which $B(\alpha)$ has the same cardinality as $\beta(\alpha)$).

4.19. **Corollary.** (a) *If* $\alpha \in L$, *then* $|B(\alpha)| = 2^{2^\alpha}$ *if and only if* $2^{2^\alpha} = 2^\kappa$ *where* $\kappa = 2^{<\alpha}$.

(b) *If* $\alpha \notin L$, *then* $|B(\alpha)| = 2^{2^\alpha}$ *if and only if* $2^{2^\alpha} = 2^{2^\mu}$ *for some* $\mu < \alpha$.

4.20. **Corollary.** *If GCH holds, then for every infinite cardinal* α *we have that* $|N(\alpha)| = |B(\alpha)| < |\beta(\alpha)|$.

4.21. **Corollary.** *If* α *is a singular cardinal, then* $|B(\alpha)| = 2^{\mu}$ *for some cardinal* μ.

ACKNOWLEDGMENTS

We are grateful to Professor Comfort for his useful comments. Also, the authors would like to thank the referee for a generous and helpful report.

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