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SHORE POINTS AND DENDRITES

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ABSTRACT. A point x in a dendroid $\mathbf{X}$ is called a shore point if there is a sequence of subdendroids of $\mathbf{X}$ not containing x and converging to $\mathbf{X}$ in the Hausdorff metric. We give necessary and sufficient conditions for a dendroid to be a dendrite, in terms of shore points and Kelley's property.

INTRODUCTION

A *dendroid* is an arcwise connected, hereditarily unicoherent metric continuum. A locally connected dendroid is called a *dendrite.* It is well known that every pair of points u and w in a dendroid are joined by a unique arc $[u, w]$ and that the subcontinua of a dendroid are themselves dendroids. If $\mathbf{X}$ is a dendroid and $x \in \mathbf{X}$, then x is an *end point* of $\mathbf{X}$ if it is an end point of every arc containing it, and x is a *shore point* of $\mathbf{X}$ [5] if there exists a sequence $\{X_n\}$ of subdendroids of $\mathbf{X}$ not containing x such that $\lim X_n = \mathbf{X}$.

It is not difficult to prove that every end point is a shore point. The shore points of $\mathbf{X}$ that are not end points will be called the *improper shore points of* $\mathbf{X}$. The following example shows that a dendroid without improper shore points is not necessarily a dendrite: Let $\mathbf{X} \subseteq \mathbb{R}^2$ be the union of the rectilinear segments $[(0, 0), (1, 1/n)]$, $n = 1, 2, 3, \ldots$ and $[(0, 0), (2, 0]$.

A dendroid will be called *neat* whenever each one of its subdendroids has no improper shore points. Obviously every subdendroid of a neat dendroid is neat.

In Theorem 2.1 we give necessary and sufficient conditions for a dendroid $\mathbf{X}$ to be a dendrite in terms of shore points and Kelley's property. In particular, it is proved that $\mathbf{X}$ is neat iff $\mathbf{X}$ is a dendrite.

1. PRELIMINARIES

A dendroid $\mathbf{X}$ is *smooth* at $\mathbf{p}$ if $[\mathbf{p}, \mathbf{a}_n]$ converges to $[\mathbf{p}, \mathbf{a}]$ in the Hausdorff metric, provided $\mathbf{a}_n$ converges to $\mathbf{a}$ in $\mathbf{X}$ (see [2]). A continuum $\mathbf{X}$ has Kelley's property if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every pair of points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{X}$ whose distance is less than δ and each subcontinuum

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A of **X** containing **a**, there is a subcontinuum **B** of **X** containing **b** whose Hausdorff distance from **A** is less than ε [4]. Recently, Czuba [1] has proved the following result:

1.1. **Theorem** (Czuba). *If a dendroid has Kelley's property then it is smooth.*

For general terminology we refer the reader to [4, 6]. A weaker version of the following lemma was proved in [5]. The proof is not difficult and is actually identical to the previous one.

1.2. **Lemma.** *If U is an arcwise connected subset of a dendroid $\mathbf{X}$ then $\mathrm{Cl}(U)$ is the limit of a sequence of subdendroids of $\mathbf{X}$ contained in U.*

1.3. **Lemma.** *Let $\mathbf{X}$ be a dendroid that has Kelley's property. Then for every $p \in \mathbf{X}$ and every arc-component U of $X\backslash\{p\}$, either U is open or $\mathrm{Int}(U) = \varnothing$.*

Proof. Suppose that $\mathrm{Int}(U) \neq \varnothing$ and let $v \in \mathrm{Int}(U)$. If $u \in U\backslash \mathrm{Int}(U)$ then the arc $[u, v] \subseteq U$. Let $0 < \varepsilon < \min\{\mathbf{d}(p, [v, u]), \alpha\}$ where $\mathbf{d}$ denotes the distance in $\mathbf{X}$ and the ball of radius α centered at v is contained in $\mathrm{Int}(U)$. For each $\delta > 0$, there exists $w \notin U$ such that $\mathbf{d}(w, u) < \delta$.

Let $\mathbf{K}$ be a subcontinuum of $\mathbf{X}$ containing w: If $p \notin \mathbf{K}$ then $\mathbf{K}$ is contained in an arc-component of $\mathbf{X}\backslash\{p\}$ different from U, so that $\mathbf{d}(v, \mathbf{K}) \geq \alpha > \varepsilon$, which implies $\mathbf{D}(\mathbf{K}, [u, v]) > \varepsilon$, where $\mathbf{D}$ denotes the distance in the Hausdorff metric. If $p \in \mathbf{K}$ then $\mathbf{d}(p, [v, u]) > \varepsilon$ and again $\mathbf{D}(\mathbf{K}, [v, u]) > \varepsilon$. Therefore, Kelley's property is not satisfied. □

1.4. **Lemma.** *A shore point in a dendroid $\mathbf{X}$ is not a cut point of $\mathbf{X}$.*

Proof. Suppose that for some $q \in \mathbf{X}$, $\mathbf{X}\backslash\{q\} = H \cup K$ is a decomposition of $\mathbf{X}\backslash\{q\}$ into disjoint, relatively closed sets H and K and, let $\varepsilon = \mathbf{D}(H, K)$. If for a subcontinuum $\mathbf{A}$ of $\mathbf{X}$, $\mathbf{D}(\mathbf{A}, \mathbf{X}) < \varepsilon$, then the sets $\mathbf{A} \cap H$ and $\mathbf{A} \cap K$ are nonempty, so that $q \in \mathbf{A}$. Therefore, q is not a shore point of $\mathbf{X}$. □

2. Main result

2.1. **Theorem.** *For a dendroid $\mathbf{X}$, the following conditions are equivalent:*

(i) *$\mathbf{X}$ is neat.*
(ii) *For every $q \in \mathbf{X}$, the arc components of $\mathbf{X}\backslash\{q\}$ are all open.*
(iii) *$\mathbf{X}$ is a dendrite.*
(iv) *$\mathbf{X}$ has Kelley's property and has no improper shore points.*
(v) *Every subcontinuum of $\mathbf{X}$ has Kelley's property.*

Proof. (i) $\Rightarrow$ (ii). Suppose that an arc component α of $\mathbf{X}\backslash\{q\}$ is not open. If for some arc component β of $\mathbf{X}\backslash\{q\}$ different from α, $\mathrm{Cl}(\beta)\cap\alpha \neq \varnothing$, we take $x \in \mathrm{Cl}(\beta) \cap \alpha$ and note that the arc $(q, x] \subseteq \mathrm{Cl}(\beta) \cap \alpha$. If $\mathbf{y} \in (q, x)$, then $\mathbf{y} \in \mathrm{Cl}\,\beta\backslash\beta$, so that there exists a sequence $\{X_n\}$ of subdendroids contained in β such that $X_n \to \mathrm{Cl}(\beta)$ (Lemma 1.2).

Clearly $\mathbf{y}$ is an improper shore point of the subdendroid $\mathrm{Cl}(\beta)$. Let Γ be the set of arc components of $\mathbf{X}\backslash\{q\}$ different from α. We suppose now that $\mathrm{Cl}(\beta)\cap\alpha = \varnothing$ for every $\beta \in \Gamma$ and denote by B the union of the members of Γ. By assumption $\mathrm{Cl}(B) \cap \alpha \neq \varnothing$, take $x \in \mathrm{Cl}(B) \cap \alpha$ and $\mathbf{y} \in (q, x)$. Notice that $\mathbf{y} \in \mathrm{Cl}(B)\backslash B$. Let $\{\mathbf{y}_n\}$ be a sequence of points such that $\mathbf{y}_n \in \beta_n \in \Gamma$ and $\{\mathbf{y}_n\}$ converges to $\mathbf{y}$ in $\mathrm{Cl}(B) \cap \alpha$. We can assume that $\beta_n \neq \beta_m$ for $m \neq n$. The sequence of dendroids $M_n = \bigcup_{j=1}^{n} \mathrm{Cl}(\beta_j)$ is increasing and satisfies $M_n \cap \alpha = \varnothing$

for each n. Moreover, $\{M_n\}$ converges to a subdendroid $Y \subseteq \mathrm{Cl}(B)$ and hence $\mathbf{y}$ is an improper shore point of Y.

(ii) $\Rightarrow$ (i). Suppose that $\mathbf{X}$ is not neat. Let X_0 be a subdendroid of $\mathbf{X}$ and q an improper shore point of X_0. Then $X_0\backslash\{q\}$ has at least two arc components.

We shall prove that every arc component α of $X_0\backslash\{q\}$ is open in $X_0\backslash\{q\}$. Since this fact contradicts the connectivity of $X_0\backslash\{q\}$, our assertion follows from 1.4. Indeed if $C(\alpha)$ is the arc component in $X\backslash\{q\}$ containing α, then $C(\alpha) \cap (X_0\backslash\{q\}) = \alpha$.

(i) $\Rightarrow$ (iii). It was proved by Charatonik and Eberhart [2, Corollaries 4 and 5] that a dendroid $\mathbf{X}$ is a dendrite iff $\mathbf{X}$ is smooth at each of its points. Suppose that $\mathbf{X}$ is not smooth at q, and let $\{x_n\}$ be a sequence that converges to x such that $[q, x_n]$ is convergent but $L = \lim[q, x_n] \neq [q, x]$. Let $z \in L\backslash[q, x]$ be a point that is not an end point of L. If $z \notin [q, x_n]$ for an infinite set J of indices, it will be clear that z is an improper shore point of $\mathrm{Cl}(\bigcup_{j\in J}[q, x_j])$.

Therefore we can assume that $z \in [q, x_n]$ for all n. In $\mathbf{X}\backslash\{z\}$, the arcs $[q, z)$ and $[x_n, z)$ belong to different arc components $\alpha([q, z))$ and $\alpha([x_n, z))$, respectively. Since (i) implies (ii), it follows that $\alpha([q, z))$ and $\bigcup_n \alpha([x_n, z))$ are open. Moreover, they are disjoint, which is impossible since $x \in \alpha([q, z))$ and $x_n \to x$.

(i) $\Rightarrow$ (iv). This follows from (i) $\Rightarrow$ (iii) since every locally connected continuum has Kelley's property.

(iv) $\Rightarrow$ (ii). Suppose that for some $p \in \mathbf{X}$, $\mathbf{X}\backslash\{p\}$ has a nonopen arc component U. Let u be a non end point of $\mathbf{X}$ contained in U and for each $n \in \mathbb{N}$, let C_n be the component of $\mathbf{X}\backslash B_{1/n}(u)$ containing p. If $x \in \mathbf{X}\backslash U$ then $[p, x] \cap U = \varnothing$. In particular, $u \notin [p, x]$, so that for n large enough $[p, x] \cap B_{1/n}(u) = \varnothing$. This implies that $[p, x] \subseteq C_n$. By Lemma 1.3, $\mathrm{Int}(U) = \varnothing$, so that $\lim C_n = \mathbf{X}$. Since $u \notin C_n$ for every n, it follows that u is an improper shore point of $\mathbf{X}$.

(v) $\Rightarrow$ (ii) By Theorem 1.1 X is smooth. By [3, Theorem 1, p. 194] X contains no subdendroid of Type 1. Next we show that X is smooth at each of its points. Let $p \in X$ and suppose X is not smooth at p. By [3, Lemma 1, p. 193], X contains a subdendroid of Type 3. But a Type 3 dendroid contains a subdendroid that does not have Kelley's property, a contradiction. By [2] X is a dendrite and a dendrite clearly satisfies (ii).

(iii) $\Rightarrow$ (v). This follows since every subdendroid of a dendrite is a dendrite.

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