Asymptotics for Supersonic Soliton Propagation in the Toda Lattice Equation

By L. A. Cisneros and A. A. Minzoni

We study the problem of the adjustment of an initial condition to an exact supersonic soliton solution of the Toda latice equation. Also, we study the problem of soliton propagation in the Toda lattice with slowly varying mass impurities. In both cases we obtain the full numerical solution of the soliton evolution and we develop a modulation theory based on the averaged Lagrangian of the discrete Toda equation. Unlike previous problems with coherent subsonic solutions we need to modify the averaged Lagrangian to obtain the coupling between the supersonic soliton and the subsonic linear radiation. We show how this modified modulation theory explains qualitatively in simple terms the evolution of a supersonic soliton in the presence of impurities. The quantitative agreement between the modulation solution and the numerical result is good.

1. Introduction

The Toda lattice represents an anharmonic one-dimensional lattice system with the nearest neighbor interaction of the exponential type, therefore it covers a wide range of interaction potentials from the harmonic oscillator to the strong nonlinear one. This discrete system is known to be completely integrable through the inverse scattering method [1]. The soliton solution to the equation

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of motion was found by M. Toda [1, 2]. Several problems related to the Toda lattice have been considered such as the scattering of solitons from mass impurities, the excitation of localized mode vibration caused by the incidence of solitons on a mass impurity [3–6]. In these works experimental, numerical and perturbation techniques have been developed to obtain the evolution of the system, however these asymptotic results gave qualitative descriptions of the system and they were not compared quantitatively with the numerics.

In the present paper, we study the problem of the adjustment of an initial condition to an exact soliton. It is known that the inverse scattering method solves completely this problem but this method is complicated and it does not give a manageable description of the radiation coupled to the soliton. Thus, we propose a modulation theory to explain the evolution in simple terms. We also study the problem of a lattice with mass impurities, in this problem the inverse scattering method can not be applied due to no integrability, but the modulation theory found can be used to explain the evolution for the inhomogeneous problem. Our modulation theory is based on the averaged Lagrangian of the Toda lattice equation. However, because the soliton is supersonic we must modify the Toda soliton to couple the trial function to the radiation. We find that the asymptotic theory describes accurately the numerical evolution.

In the following section the problems are stated and the corresponding full numerical solutions are presented. In the third section the problem of the adjustment of an initial condition to an exact soliton and the lattice with adiabatic mass variation are studied. We obtain modulation solutions which explain the temporal evolution in terms of simple phase planes. The damping produced by the linear radiation is included in the asymptotics in the fourth section. Comparisons between the full numerical solution and the modulation theory of both problems are presented. Our conclusions are presented in the last section.

2. Statement of the problem and full numerical solution

The equation of motion for the infinite one-dimensional Toda lattice is given by,

$$m_n \frac{d^2 y_n}{dt^2} = V_n - V_{n+1},$$
$$V_n = a_n \left(e^{-b_n(y_n - y_{n-1})} - 1\right), n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where the parameters m_n and y_n are the mass and the displacement of the nth particle while a_n and b_n are the constants of the nth nonlinear springs, see Figure 1. These parameters represent the impurities of the lattice if they depend on the node n. The set of Equation (1) models an infinite chain of nonlinear oscillators with exponential potential. The potential covers a wide range from the harmonic oscillator limit to the strong nonlinear limit. The equation of motion (1) is obtained from the Lagrangian,

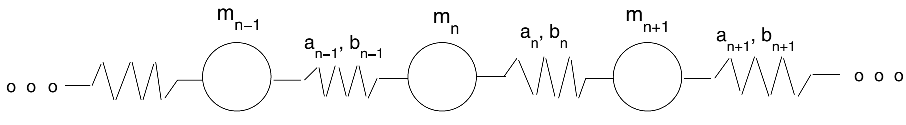

Figure 1. Array of mass and springs for the one-dimensional lattice.

$$L = \sum_{n=-\infty}^{\infty} \left[\frac{m_n}{2} \dot{y}_n^2 - \frac{a_n}{b_n} \left(e^{-b_n(y_{n+1}-y_n)} - 1 \right) - a_n(y_{n+1} - y_n) \right]. \tag{2}$$

For the uniform lattice, that is constant parameters in (1), M. Toda [1, 2] found the following exact kink solution,

$$y_n(t) = \frac{1}{b} \ln \frac{1 + e^{2(kn-k-\zeta t)}}{1 + e^{2(kn-\zeta t)}} + \text{constant}, \tag{3}$$

with the dispersion relation,

$$\zeta = \sqrt{\frac{ab}{m}} \sinh k, \, k > 0, ab > 0. \tag{4}$$

Equation (3) represents a nonlinear wave front (kink) travelling at constant velocity and constant height with position and velocity described by the relations,

$$\xi = \frac{\zeta}{k} t, \quad \dot{\xi} = \frac{\zeta}{k}, \tag{5}$$

respectively. In these new parameters the solution (3) can be rewritten as,

$$\begin{aligned} y_n(t) &= \frac{1}{b} \ln \frac{1 + e^{2k(n-1-\xi)}}{1 + e^{2k(n-\xi)}} + \text{constant} \\ &= \frac{1}{b} \ln \frac{sech(k(n-\xi))}{sech(k(n-1-\xi))} - \frac{k}{b} + \text{constant} \\ &= \frac{1}{b} \ln \frac{sech(k(n-\xi))}{sech(k(n-1-\xi))} + \frac{k}{b}, \end{aligned} \tag{6}$$

where the constant $\frac{2k}{b}$ has been added to take into account a nonzero backward mean level. The dispersion relation (4) can be transformed to,

$$\dot{\xi} = \sqrt{\frac{ab}{m}} \frac{\sinh k}{k} > \sqrt{\frac{ab}{m}}, \tag{7}$$

where $\sqrt{\frac{ab}{m}}$ corresponds to the velocity of the linear waves in the harmonic limit of the uniform lattice showing that the kink (3) is always supersonic.

Because we are interested in developing an approximated method to find the kink evolution for the Toda lattice with mass impurities (1). We need a detailed picture of the soliton evolution to obtain the suitable trial function for the averaged Lagrangian. To this end we solve the problem numerically using a fourth order Runge–Kutta method to solve the uniform lattice problem for a finite set of $2N + 1$ ordinary differential equations with the following initial conditions,

$$y_n(0) = \ln \frac{sech(k_0(n-\xi))}{sech(k_0(n-1-\xi))} + \frac{k_0}{b}, -N \leq n \leq N \tag{8}$$

$$\dot{y}_n(0) = \dot{\xi}_0 k_0 \sinh k_0 sech(k_0(n-\xi_0)) sech(k_0(n-1-\xi_0)), \tag{9}$$

where $a = b = 1$, $\xi_0 = \xi(0)$, $\dot{\xi}_0 = v_0 = \dot{\xi}(0)$ and $k_0 = k(0)$ do not satisfy the dispersion relation (7) initially.

The initial conditions (8) and (9) develop a fast initial transient to adjust to an exact soliton shedding backward radiation. This evolution is shown in Figure 2. From this figure it is clear that the leading wave is a supersonic shock adjusted to the exact travelling wave. At the back there is a disturbed region bounded by the characteristics (cone of radiation) of the linearized equation produced by the readjustment at $t = 0$, see Figure 3. For large times the leading wave separates from the linear waves as the inverse scattering results predict [1].

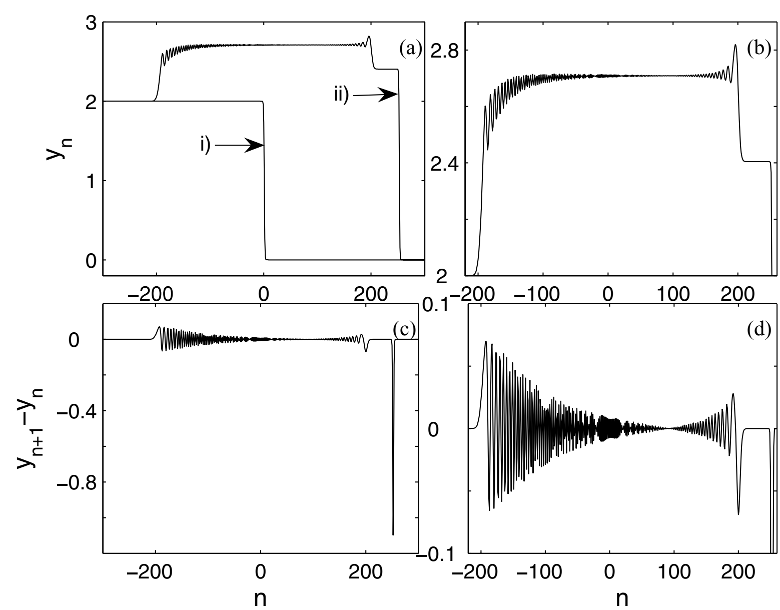

Figure 2. Adjustment to an exact soliton for the initial parameters $a = b = m = 1, \xi(0) = 0, \dot{\xi}(0) = 2, k(0) = 1, \dot{k}(0) = 0$. (a) (i) Initial condition and (ii) final solution. (b) Birth of a backward new mean level in the coherent structure and linear radiation. (c) Divided difference for the final solution and (d) zoom of (c).

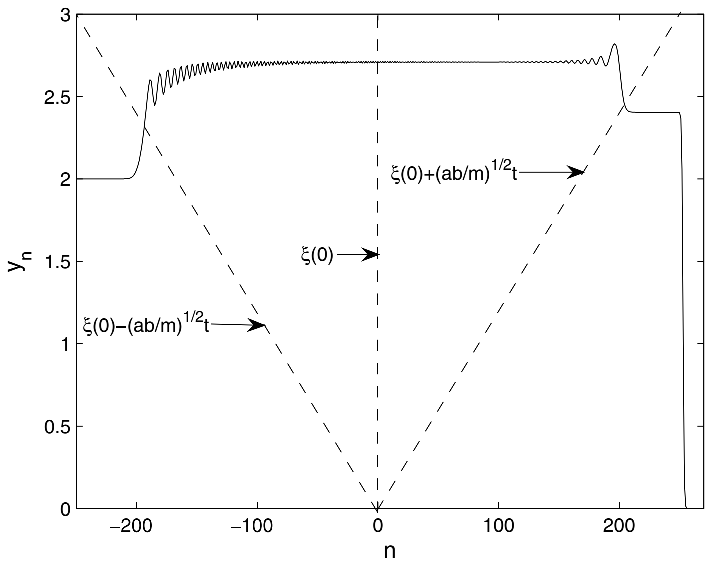

Figure 3. Cone of radiation bounded by the characteristic lines of the linear waves radiated by the coherent structure with the initial parameters $a = b = m = 1$, $\xi(0) = 0$, $\dot{\xi}(0) = 2$, $k(0) = 1$, $\dot{k}(0) = 0$.

Ahead of the shock there is, as expected, no radiation because the shock is supersonic. This situation is different from the one encountered in other hyperbolic problems where the kink solutions are subsonic (see [7] and [9]), the radiation is then shed both ahead and behind the wave.

Now, we consider the numerical solution of the inhomogeneous problem (1) when the mass distribution m_n is given by a slowly varying function and the spring distribution is homogeneous as before. The initial kink profile is set in the node n where the masses take the left asymptotic value and the initial kink parameters are given to satisfy the dispersion relation (7). The initial profile is thus given by Equations (8) and (9).

The full numerical solution for this case shows an exact kink profile travelling to the right without any disturbance until the variation in the masses is felt by the travelling coherent wave. When this occurs the travelling wave starts to emit radiation at the back due to the loss of integrability as it is shown in Figure 4. The radiation shed adjusts the kink to the appropriate one for right asymptotic value of the mass distribution. From this figure we note that we need a slowly varying dependence in the masses, in terms of the width of the initial profile, to obtain radiation with small amplitude because the amplitude of the radiation becomes larger when the variation of the masses is stronger. In this case, we can obtain a train of reflected and transmitted solitons which is complicated to study asymptotically.

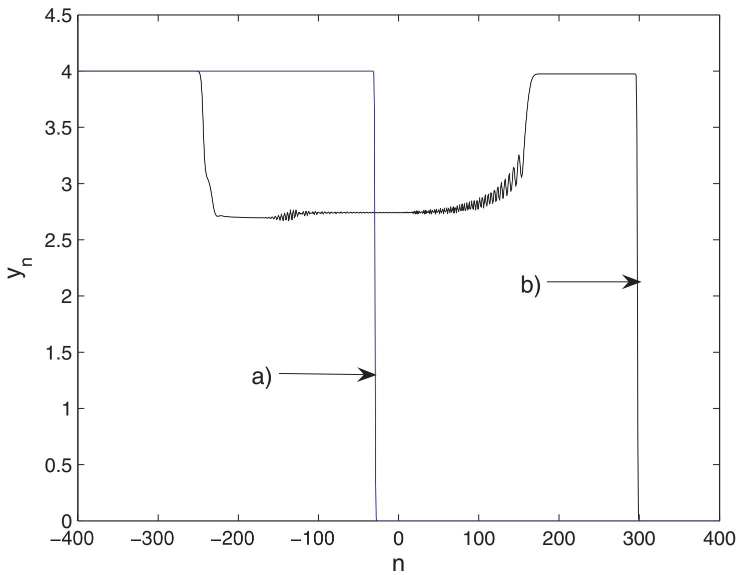

Figure 4. Numerical solution for the mass impurity $m_n = (3 + \tanh(0.5n))/2$ with the initial conditions $a = b = 1, \xi(0) = -30, \dot{\xi}(0) = \sinh 2/2, k(0) = 2, \dot{k}(0) = 0$. (a) Initial condition and (b) final solution at $t = 250$.

3. Approximate equations

To obtain the approximated equations we need to construct a trial function which includes the dynamical features shown in the last section. From Figures 2 and 3 we observe that the backward level of the kink is varying in the evolution. To take into account this we take the modulational Toda shock solution in the form,

$$y_n(t) = \frac{1}{b} \ln \frac{sech(k(n-\xi))}{sech(k(n-1-\xi))} + \frac{k}{b}, \tag{10}$$

as a trial function, where now the parameters ξ and k are functions of time. This solution matches the zero level ahead of the shock for all times. However, this solution is only valid for $n \geq \xi_0 + \sqrt{\frac{ab}{m}}t$ due to dispersive radiation, see Figure 3. From the full numerical solution we observe that the shock sheds linear waves (around a new mean level) backwards. These linear waves account for the disturbance behind the shock shown in Figure 2.

From the trial function (10) we have,

$$\dot{y}_n(t) = k\dot{\xi} G_n - \dot{k}[(n-\xi)G_n - 1 + \tanh(k(n-1-\xi))], \tag{11}$$

where

$$G_n = \tanh(k(n-\xi)) - \tanh(k(n-1-\xi))$$
$$= \sinh k \sec h(k(n-\xi)) sech(k(n-1-\xi)). \tag{12}$$

The functions (10) and (11) are used in the Lagrangian (2) for an uniform lattice. We approximate the averaged Lagrangian by using the Poisson summation formula as shown in the Appendix. From Equation (11) it follows that all the terms in $\dot{y}_n^2$ except the term $(-1 + \tanh(k(n-1-\xi)))^2$ have a finite sum. Thus, the sum up to the trailing edge is approximated by the sum on all the lattice. The nonconvergent part of the series is just summed up to the trailing edge. We thus obtain for the uniform lattice the averaged Lagrangian

$$\pounds = \frac{b^2}{m} L = \dot{\xi}^2 g_1(k,\xi) + \dot{k}\dot{\xi} g_2(k,\xi) + \dot{k}^2 g_3(k,\xi,t) - V(k,\xi), \tag{13}$$

where the coefficients g_1, g_2, g_3 are derived in the Appendix and the effective potential is given to leading order in the Poisson summation formula by:

$$V(k,\xi) = \frac{2ab}{m}\left(\frac{\sinh^2 k}{k} - k\right) + \frac{4ab}{m}\frac{\pi^2}{k^2}\sinh^2 k csch \frac{\pi^2}{k}\cos(2\pi\xi). \tag{14}$$

Notice in (14) the periodic term which is the Peierls–Nabarro (PN) potential arising from the discreteness, which appears as in other problems as a periodic potential in the solution position [7, 9]. Also the Lagrangian depends explicitly on time. This will have to be considered when the soliton motion is coupled to the shed radiation. Before taking into account the radiation we obtain the modulation equations derived from (13) in the form:

$$\delta\xi : 2\ddot{\xi} g_1 + \ddot{k} g_2 + \dot{\xi}^2 \frac{\partial g_1}{\partial \xi} + 2\dot{\xi}\dot{k}\frac{\partial g_1}{\partial k} + \dot{k}^2\left(\frac{\partial g_2}{\partial k} - \frac{\partial g_3}{\partial \xi}\right) + \frac{\partial V}{\partial \xi} = 0, \tag{15}$$

$$\delta k : \ddot{\xi} g_2 + 2\ddot{k} g_3 + \dot{\xi}^2\left(\frac{\partial g_2}{\partial \xi} - \frac{\partial g_1}{\partial k}\right) + 2\dot{\xi}\dot{k}\frac{\partial g_3}{\partial \xi} + \dot{k}^2\frac{\partial g_3}{\partial k} + 2\dot{k}\frac{\partial g_3}{\partial t} + \frac{\partial V}{\partial k} = 0, \tag{16}$$

which are written as

$$\ddot{\xi} = \frac{1}{4g_3 g_1 - g_2^2} \times \left[\begin{array}{l} \dot{\xi}^2\left(g_2\left(\frac{\partial g_2}{\partial \xi} - \frac{\partial g_1}{\partial k}\right) - 2g_3\frac{\partial g_1}{\partial \xi}\right) + \dot{\xi}\dot{k}\left(2g_2\frac{\partial g_3}{\partial \xi} - 4g_3\frac{\partial g_1}{\partial k}\right) \\ + \dot{k}^2\left(g_2\frac{\partial g_3}{\partial k} - 2g_3\left(\frac{\partial g_2}{\partial k} - \frac{\partial g_3}{\partial \xi}\right)\right) + 2\dot{k} g_2\frac{\partial g_3}{\partial t} + g_2\frac{\partial V}{\partial k} - 2g_3\frac{\partial V}{\partial \xi} \end{array}\right], \tag{17}$$

$$\ddot{k} = \frac{1}{4g_3g_1 - g_2^2} \times \left[\dot{\xi}^2 \left(g_2 \frac{\partial g_1}{\partial \xi} - 2g_1 \left(\frac{\partial g_2}{\partial \xi} - \frac{\partial g_1}{\partial k} \right) \right) + \dot{\xi}\dot{k} \left(2g_2 \frac{\partial g_1}{\partial k} - 4g_1 \frac{\partial g_3}{\partial \xi} \right) + \dot{k}^2 \left(g_2 \left(\frac{\partial g_2}{\partial k} - \frac{\partial g_3}{\partial \xi} \right) - 2g_1 \frac{\partial g_3}{\partial k} \right) - 4\dot{k}g_1 \frac{\partial g_3}{\partial t} - 2g_1 \frac{\partial V}{\partial k} + g_2 \frac{\partial V}{\partial \xi} \right]. \tag{18}$$

The second equation has a fixed point when,

$$\dot{\xi}^2 = \frac{2g_1 \frac{\partial V}{\partial k} - g_2 \frac{\partial V}{\partial \xi}}{g_2 \frac{\partial g_1}{\partial \xi} - 2g_1 \left(\frac{\partial g_2}{\partial \xi} - \frac{\partial g_1}{\partial k} \right)} = \frac{ab}{m} \frac{\sinh^2 k}{k^2}, \tag{19}$$

which as expected gives the dispersion relation (7) for an exact soliton solution. It is to be noted that the modulation theory reproduces the fact that in the integrable case the PN potential does not have any influence on the shock. This situation is very different from the one encountered in subsonic propagation [7, 9] where the PN potential is present and eventually stops the kink. This fixed point corresponds to a center and the damping term $4g_1 \frac{\partial g_3}{\partial t}$ makes to the trajectories approach it.

Again Equation (17) has $\ddot{\xi} = 0$ when $\dot{k} = \ddot{k} = 0$ and k satisfies the dispersion relation. However, when the soliton is evolving the contribution of the PN potential is not zero and there is a periodic acceleration until the soliton settles to its propagation shape. This effect is shown in Figure 2. In Figure 5, we show a comparison between the modulation solution and the full numerical solution. We observe that the modulation theory predicts very well the leading edge of the adjusted shock. The comparison of the radiating tail is off in mean because we have not yet included the effect of the radiation. Figure 6 shows the comparison at the trailing edge and again we see the same effect.

In Figure 7, we show the comparison between the full numerical and the modulational solution for mass impurities. This comparison corresponds to the nonhomogeneous masses $m_n = (1.8 - 0.2\tanh(0.25n))/2$. We also note a very good agreement for the leading edge and the same offset in the mean of the trailing edge. The modulation equations for this case are obtained as in the last problem considering a slow variation in the masses to obtain the averaged Lagrangian,

$$\pounds = b^2 L = \dot{\xi}^2 m(\xi) g_1(k,\xi) + \dot{k}\dot{\xi} m(\xi) g_2(k,\xi) + \dot{k}^2 m(\xi) g_3(k,\xi,t) - V(k,\xi), \tag{20}$$

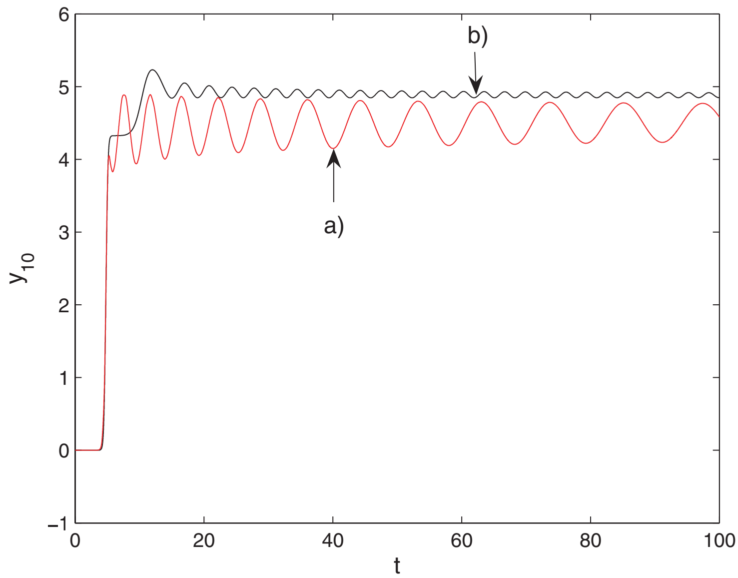

Figure 5. Numerical comparison for the initial parameters $a = b = m = 1$, $\xi(0) = 0$, $\dot{\xi}(0) = 2.5$, $k(0) = 2$, $\dot{k}(0) = 0$. (a) Approximate solution (15)–(16) and (b) full numerical solution for $y_n(t)$ both at the node $n = 10$ at $t = 100$.

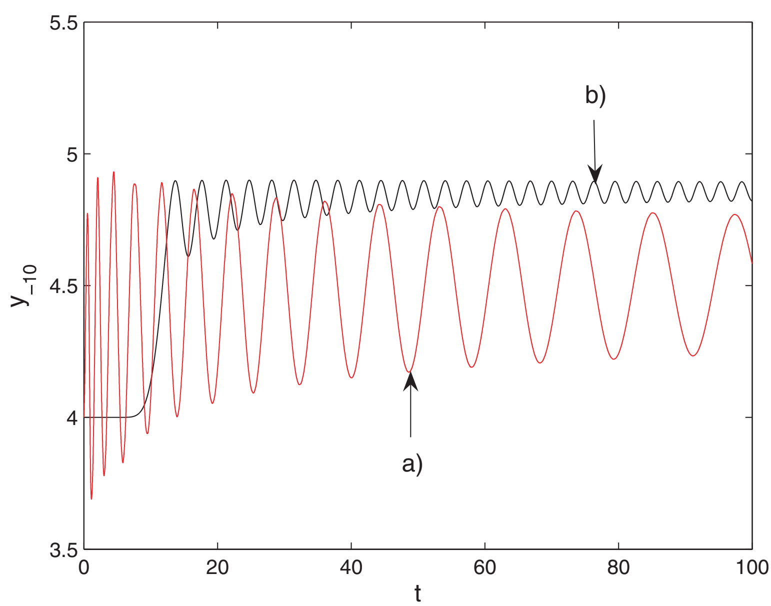

Figure 6. Numerical comparison for the initial parameters $a = b = m = 1$, $\xi(0) = 0$, $\dot{\xi}(0) = 2.5$, $k(0) = 2$, $\dot{k}(0) = 0$. (a) Approximate solution (15) and (16) and (b) full numerical solution for $y_n(t)$ both at the node $n = -10$ at $t = 100$.

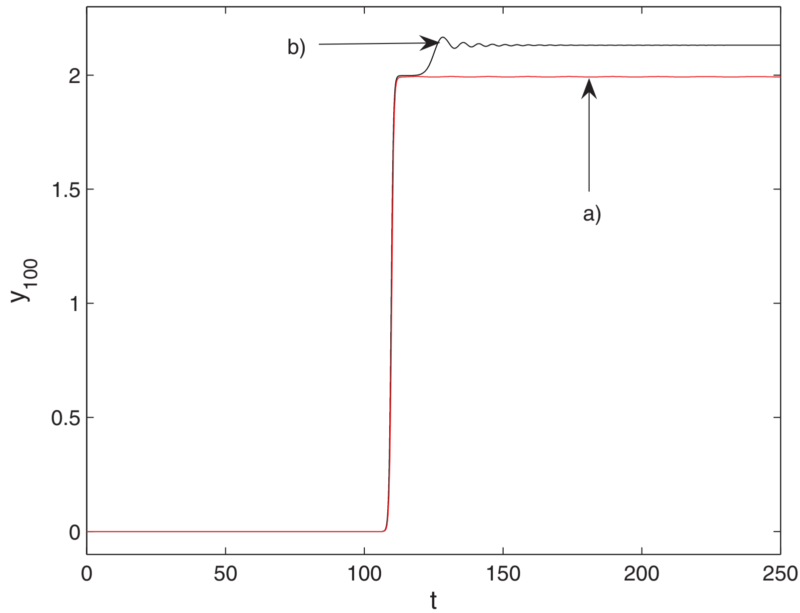

Figure 7. Numerical comparison for the initial parameters $a = b = 1, \xi(0) = -40, \dot{\xi}(0) = \sinh 1, k(0) = 1, \dot{k}(0) = 0$. (a) Approximate solution and b) full numerical solution $y_n(t)$ in the node $n = 100$ with the mass impurity $m_n = (1.8 - 0.2 \tanh(0.25n))/2$.

where the functions g_1, g_2, g_3 are the same as before because we are assuming a slow variation of the mass impurities, with respect to the width of the travelling soliton. In this way the mass is taken out from the Poisson summation formula as a dependent function of soliton's position ξ, see Appendix. The effective potential is given to leading order by

$$V(k, \xi) = 2ab \left(\frac{\sinh^2 k}{k} - k \right) + 4ab \frac{\pi^2}{k^2} \sinh^2 k csch \frac{\pi^2}{k} \cos(2\pi\xi).$$

4. Radiation damping

To consider the effect of the radiation we proceed as for the Sine–Gordon equation [8, 9]. In the first place we consider the energy loss from the shock to the radiation to obtain the damping in the k equation. Then we calculate the corresponding momentum loss.

We assume as in the Sine–Gordon case that

$$\dot{y}_n = \dot{y}_{ns} + \dot{y}_{nr}, \quad \dot{y}_{nr} \ll 1 \tag{21}$$

and that the shock and the radiation do not overlap. We thus have that the constant Hamiltonian H takes the form:

$$H = \sum_{n=\xi(0)+\sqrt{ab/m}t}^{\infty} \dot{y}_{ns}^2 + V(y_{ns}) + \sum_{n=-\infty}^{\xi(0)+\sqrt{ab/m}t} \dot{y}_{nr}^2 + V(y_{nr}) = H_{\text{shock}} + H_{\text{rad}}. \tag{22}$$

The first term is readily obtained from the averaged Lagrangian in the form:

$$H_{\text{shock}} = \dot{k}L_{\dot{k}} + \dot{\xi}L_{\dot{\xi}} - L. \tag{23}$$

To calculate H_{rad} we assume that the radiation is both linear and of long wave length around the mean level given by the trailing edge of the soliton. In this way the Hamiltonian H_{rad} is just the corresponding one for the linear wave equation in the form:

$$H_{\text{rad}} = \int_{-\infty}^{\xi(0)+\sqrt{ab/m}t} \frac{1}{2}\left(y_t^2 + y_x^2\right) dx. \tag{24}$$

To couple the radiation with the evolution of k we use the equation of energy conservation $\frac{dH}{dt} = 0$ in the form:

$$\dot{k}\frac{d}{dt}L_{\dot{k}} + \dot{\xi}\frac{d}{dt}L_{\dot{\xi}} - \dot{k}L_k - L_t + \left[\frac{1}{2}\sqrt{\frac{ab}{m}}\left(y_t^2 + y_x^2\right) + y_t y_x\right]_{x=\xi(0)+\sqrt{ab/m}t} = 0. \tag{25}$$

Assuming small momentum loss due to the PN term we have to leading order from the Euler–Lagrange equation,

$$\frac{d}{dt}L_{\dot{\xi}} = 0, \tag{26}$$

and from Equation (13) we obtain:

$$\dot{k}\frac{d}{dt}L_{\dot{k}} = \dot{k}^2\dot{g}_3, \tag{27}$$

$$L_t = \frac{1}{2}\dot{k}^2\dot{g}_3. \tag{28}$$

Then the equation for energy conservation takes the form

$$\dot{k}\left(\frac{d}{dt}L_{\dot{k}} - L_k + \frac{1}{2}\dot{k}\dot{g}_3\right) = -\left[\frac{1}{2}\sqrt{\frac{ab}{m}}\left(y_t^2 + y_x^2\right) + y_t y_x\right]_{x=\xi(0)+\sqrt{ab/m}t}. \tag{29}$$

To close the system we need to calculate the radiation shed by the soliton. This is achieved by solving a signalling problem for the wave equation on the characteristic $\xi(0) + \sqrt{ab/m}t$. This is $y_{tt} - y_{xx} = 0$, $y_x(t, \xi(0) + \sqrt{ab/m}t) = g(t)$ where $g(t)$ has to be determined. The solution is readily found in the form $y_x(t, x) = g(\sqrt{ab/m}t + x)$. The function g is determined assuming that the excess of energy between the approximate solution and the solution at the fixed point goes into the radiation, this is:

$$H_{\text{shock}} - H_{\text{shock fixed}} = H_{\text{rad}} = \int_{\xi(0)-\sqrt{ab/m}t}^{\xi(0)+\sqrt{ab/m}t} \frac{1}{2}\left(y_t^2 + y_x^2\right) dx. \tag{30}$$

The left-hand side is expanded at the fixed point $k_0, \dot{\xi}_0$ and the right-hand side is evaluated by the trapezoidal rule to obtain using $\dot{\xi} = \dot{\xi}_0 + \hat{\dot{\xi}}$, $k = k_0 + \hat{k}$

$$\frac{1}{2}g_3\hat{\dot{k}}^2 + \frac{1}{2}g_1\hat{\dot{\xi}}^2 - g_2\hat{\dot{k}}\dot{\xi}_0 = \frac{1}{4}2\sqrt{ab/m}tg^2(t) = \frac{\sqrt{ab/m}t}{2}g^2(t). \tag{31}$$

Because the equation is to be satisfied for large t we have to leading order

$$g^2(t) = \frac{1}{t}g_3\hat{\dot{k}}^2 = \sqrt{\frac{m}{ab}}\frac{1}{t}g_3\dot{k}^2. \tag{32}$$

The final energy equation becomes using (32) in (29):

$$\frac{d}{dt}£_{\dot{k}} - £_k + \dot{k}\left(\sqrt{\frac{m}{ab}}\frac{1}{t}g_3 - \frac{\partial g_3}{\partial t}\right) = 0. \tag{33}$$

Equation (33) is to be solved coupled to the momentum equation:

$$\frac{d}{dt}L_{\dot{\xi}} = 0.$$

To calculate the momentum loss we approximate to leading order the total momentum term using its continuum approximation $y_t y_x$ which is conserved. In this way we obtain:

$$\frac{d}{dt}\int_{-\infty}^{\infty} y_x y_t = \frac{d}{dt}\int_{\xi(0)+\sqrt{ab/m}t}^{\infty} y_x^{\text{shock}} y_t^{\text{shock}}\, dx + \frac{d}{dt}\int_{-\infty}^{\xi(0)+\sqrt{ab/m}t} y_x^{\text{rad}} y_t^{\text{rad}}\, dx = 0. \tag{34}$$

In (34) the first term is just the average momentum $L_{\dot{\xi}}$

$$\frac{d}{dt}L_{\dot{\xi}} + \left[\frac{1}{2}\sqrt{\frac{ab}{m}}\left(y_t^2 + y_x^2\right) + y_t y_x\right]_{x=\xi(0)+\sqrt{ab/m}t} = 0. \tag{35}$$

The momentum loss in Equation (35) after using (32) takes the form:

$$\frac{d}{dt}£_{\dot{\xi}} = -\sqrt{\frac{m}{ab}}\frac{1}{t}g_3\dot{k}^2. \tag{36}$$

This in turn modifies Equation (33) which becomes,

$$\frac{d}{dt}L_{\dot{k}} - L_k + \frac{1}{2}\dot{k}\dot{g}_3 + (1-\dot{\xi})\frac{g_3}{t}\dot{k} = 0. \tag{37}$$

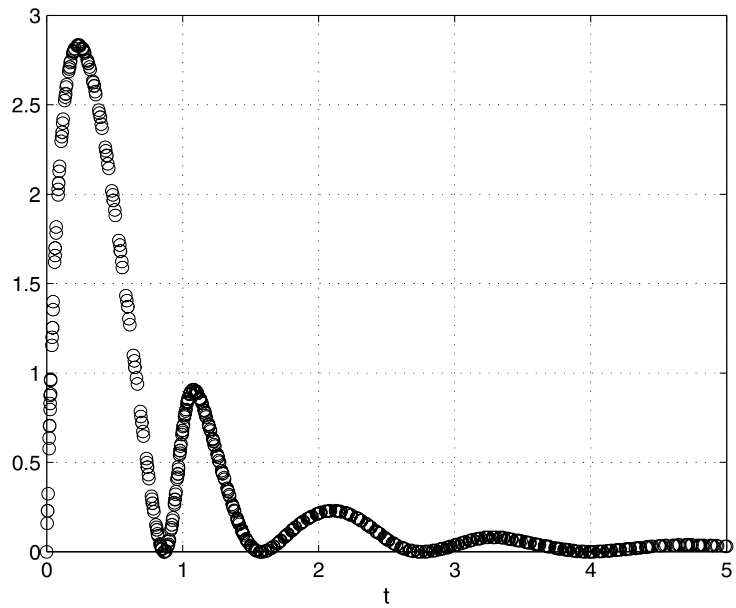

Figure 8. Graph of $\sqrt{\frac{m}{ab}}\frac{1}{t}g_3\dot{k}^2$ as a function of t in the range of the transient.

We note that the first part of the last two equations corresponds to the Euler–Lagrange equations given by (15) and (16). In this way the study of the radiation gives the damping terms $\sqrt{\frac{m}{ab}}\frac{1}{t}g_3\dot{k}^2$ and $\dot{k}\frac{\partial g_3}{\partial t} + (1 - \dot{\xi})\frac{g_3}{t}\dot{k}$ in Equations (36) and (37), respectively. The new modulational Equations (36) and (37) that include the loss of radiation have to be compared with the corresponding full numerical results. However, due to the wave is supersonic we obtain the same comparisons as before, see Figures 5–7. On the other hand, the comparisons given in Figures 5–7 show that the difference between the adjusted soliton and the mean level of the radiation is of the order of 0.3. This correlates well with the averaged value of the damping term $\sqrt{\frac{m}{ab}}\frac{1}{t}g_3\dot{k}^2$, as it can be see from Figure 8. We remark that in the supersonic case the effect of the radiation is effectively taken into account by the cut off trial function used ahead of the characteristic $\xi(0) + \sqrt{ab/mt}$.

5. Conclusions

We have shown how the modulation theory developed for subsonic kinks in discrete lattices can be used for supersonic solutions. The extension is achieved by modifying the trial function for the kink solution taking into account its validity only in the supersonic region. We have shown that this trial function takes into account the dominant contribution of the energy loss to the linear waves. The modulation solution is in excellent agreement with the numerics for the leading edge of the shock. The present results could be used to describe the evolution of supersonic solitons as the ones studied by Flytzanis et al. [11] for nonintegrable lattices. In [11] it is shown numerically that nonintegrable equations sustain supersonic solitons with a similar shape as the Toda solitons with an appropriate nonlinear dispersion relation. The present work could be used to develop the corresponding modulation theory. However, in this case a delicate point must be addressed, namely the mechanism responsible for the vanishing of the PN potential. In the present work this fact was captured to leading order by the asymptotics. In the general case this question must be carefully addressed because it may involve complicated evolutions in the long time scale such as the ones studied in the subsonic case in Refs. [9] and [12]. On the other hand, the present ideas are suitable to understand the main evolution of the supersonic kink. Some of these questions are currently under study.

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Appendix

The function $g_1(k, \xi)$ is given by,

$$g_1(k,\xi) = \frac{k^2}{2}\sum_{n=-\infty}^{\infty}(\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))^2$$
$$= 2k^2\coth k - k^2\sum_{n=-\infty}^{\infty} sech^2(k(n-\xi)),$$

which give using the Poisson summation formula [10] with the first mode retained,

$$g_1(k,\xi) = 2k^2\coth k - 2k - 4\pi^2 csch\frac{\pi^2}{k}\cos(2\pi\xi).$$

The periodic term is the contribution of the PN potential that comes from the discreteness of the lattice problem. In the same way we obtain

$$g_2(k,\xi) =$$
$$-k\left[\begin{array}{l}\displaystyle\sum_{n=-\infty}^{\infty}(n-\xi)(\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))^2\\ \displaystyle+\sum_{n=-\infty}^{\infty}(-1+\tanh(k(n-1-\xi)))(\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))\end{array}\right]$$

$$= 2k + 2k \sum_{n=-\infty}^{\infty} (n-\xi) sech^2(k(n-\xi))$$

$$+ 2k \coth k \left(1 - \sum_{n=-\infty}^{\infty} (n-\xi)(\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))\right)$$

$$= 2k + 4\pi csch \frac{\pi^2}{k} \sin(2\pi\xi) \left(\frac{1}{k} - \coth k - \frac{\pi^2}{k^2} \coth \frac{\pi^2}{k}\right).$$

The function $g_3(k, \xi, t)$ takes the form

$$g_3(k,\xi,t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (n-\xi)^2 (\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))^2$$
$$+ \sum_{n=-\infty}^{\infty} (n-\xi)(-1 + \tanh(k(n-1-\xi)))$$
$$\times (\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))$$
$$+ \frac{1}{2} \sum_{n=\xi_0+\sqrt{\frac{ab}{m}}t}^{\infty} (-1 + \tanh(k(n-1-\xi)))^2,$$

where

$$g_3(k,\xi,t) = \coth k \sum_{n=-\infty}^{\infty} (n-\xi)^2 (\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))$$
$$- \sum_{n=-\infty}^{\infty} (n-\xi)^2 sech^2(k(n-\xi))$$
$$- (1 + \coth k) \sum_{n=-\infty}^{\infty} (n-\xi)(\tanh(k(n-\xi)) - \tanh(k(n-1-\xi)))$$
$$+ \frac{1}{2} \sum_{n=-\infty}^{\infty} sech^2(k(n-\xi))$$
$$+ \frac{1}{2} \sum_{n=\xi_0+\sqrt{\frac{ab}{m}}t}^{\infty} (-1 + \tanh(k(n-1-\xi)))^2.$$

Now, we remark that the sum $\sum_{n=\xi_0+\sqrt{\frac{ab}{m}}t}^{\infty} (-1 + \tanh(k(n-1-\xi)))^2$ is calculated to leading order using the Poisson formula with zero values for $n < \xi_0 + \sqrt{\frac{ab}{m}}t$. The rest of the series have been calculated summing from minus infinity to infinity due to the exponentially small contribution of the left-hand tail. We thus obtain:

$$
\begin{aligned}
g_3(k,\xi,t) = & -1 + \frac{1}{k} - \frac{\pi^2}{6k^3} + \coth k\left(\frac{\pi^2}{6k^2} - \frac{1}{3}\right) \\
& + \frac{2\pi^2}{k^2}\coth k \coth\frac{\pi^2}{k} csch \frac{\pi^2}{k}\cos(2\pi\xi) \\
& + \frac{\pi^2}{2k^4} csch^3\frac{\pi^2}{k}\left(3\pi^2 + \pi^2\cosh\frac{2k^2}{k} - 2k\sinh\frac{2\pi^2}{k}\right)\cos(2\pi\xi) \\
& + \frac{2\pi^2}{k^2} csch\frac{\pi^2}{k}\cos(2\pi\xi) \\
& + \frac{2\pi}{k}\coth k csch\frac{\pi^2}{k}\sin(2\pi\xi) - \frac{2\pi}{k}(1+\coth k) csch\frac{\pi^2}{k}\sin(2\pi\xi) \\
& + \frac{1}{2k}\left[\begin{array}{l} -1 + \ln 4 + 4k\left(\xi - \xi_0 + 1.5 - \sqrt{\frac{ab}{m}}t\right) \\ \quad - \tanh\left(k\left(\xi - \xi_0 + 1.5 - \sqrt{\frac{ab}{m}}t\right)\right) \\ -2\ln\left(1 + \tanh\left(k\left(\xi - \xi_0 + 1.5 - \sqrt{\frac{ab}{m}}t\right)\right)\right) \end{array}\right].
\end{aligned}
$$

The contribution of the potential term is evaluated as follows,

$$
\begin{aligned}
V(k,\xi) &= -\frac{2ab}{m}k + \frac{ab}{m}\sinh^2 k\sum_{n=-\infty}^{\infty} sech^2(k(n-\xi)) \\
&= \frac{2ab}{m}\left(\frac{\sinh^2 k}{k} - k\right) + \frac{4ab}{m}\frac{\pi^2}{k^2}\sinh^2 k csch\frac{\pi^2}{k}\cos(2\pi\xi).
\end{aligned}
$$

In the case of mass impurities we have to calculate the averaged Lagrangian (2) using the trial function (8) for slowly varying masses. The relevant terms come from the dynamical part, that is,

$$
\sum_{n=-\infty}^{\infty}\frac{m_n}{2}\dot{y}_n^2 = \sum_{n=-\infty}^{\infty}\frac{m(n)}{2}\dot{y}^2(n-\xi).
$$

Using the Poisson summation formula and the assumption on the masses we obtain,

$$
\sum_{n=-\infty}^{\infty}\frac{m_n}{2}\dot{y}_n^2 = \frac{1}{2}m(\xi)\sum_{n=-\infty}^{\infty}\dot{y}_n^2,
$$

where the last sum is the same as before.

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