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On $\frac{1}{2}$-homogeneous continua

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Dedicated to J.J. Charatonik

Abstract

A continuum is $\frac{1}{2}$-homogeneous provided there are exactly two orbits for the action of the group of homeomorphisms of the continuum onto itself. In this paper we study some relations between $\frac{1}{2}$-homogeneous continua and their set of cut points. We also prove that if X is a hereditarily decomposable continuum whose proper, nondegenerate subcontinua are arc-like, then X is $\frac{1}{2}$-homogeneous if and only if X is an arc. Suitable examples and counterexamples are given.
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1. Introduction

A *continuum* is a compact and connected metric space. For a positive integer n, a space is said to be $\frac{1}{n}$-*homogeneous* provided that the action on the space of the group of homeomorphisms of the space onto itself has exactly n orbits (an *orbit* being the action of the homeomorphism group at a given point x). Thus, 1-homogeneous spaces are the more familiar *homogeneous* spaces.

For example, and arc and the continuum with the shape of the Greek letter theta are $\frac{1}{2}$-homogeneous. Also, the Sierpiński universal curve is $\frac{1}{2}$-homogeneous [5]. A theorem about $\frac{1}{2}$-homogeneous compact absolute neighborhood retracts of dimension $\leqslant 2$ is in [12, Theorem 1, p. 25]. Moreover, results about $\frac{1}{2}$-homogeneous hyperspaces can be found in [10], and results about $\frac{1}{2}$-homogeneous cones are in [11].

The aim of this paper is to provide further results on $\frac{1}{2}$-homogeneous continua. In Section 3 we determine some relations between $\frac{1}{2}$-homogeneous continua and their set of cut points. Moreover, in Section 4 we study the structure of $\frac{1}{2}$-homogeneous hereditarily decomposable continua whose proper, nondegenerate subcontinua are arc-like. We give several examples and we also raise some open questions.

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2. Preliminaries

The symbol $\mathbb{N}$ stands for the set of all positive integers, and I will denote the unit interval. All considered spaces are assumed to be metric, and all mappings are continuous.

The symbols $\text{int}_Y(A)$, $\text{cl}_Y(A)$ and $\text{bd}_Y(A)$ stand for the interior, the closure and the boundary of the set A relative to the subspace Y of a space X, respectively. In case $X = Y$ we shall simply omit the subindex. Also, $|A|$ will denote the cardinality of A.

Let X be a space and let n be a cardinal number. A point $y \in X$ is said to be of *order less than or equal to n* provided y has a basis of open neighborhoods in X whose boundaries have at most n elements; in this case we write $\text{ord}_X(y) \leqslant n$. If n is the smallest cardinal number for which y has such neighborhoods, we will say that $\text{ord}_X(y) = n$.

A point p of a space X is a *ramification point* of X if $\text{ord}_X(p) \geqslant 3$, and it is an *end point* of X if $\text{ord}_X(p) = 1$. The set of end points of X will be denoted by $\text{End}(X)$.

A *cut point* of a connected space X is a point $p \in X$ such that $X \setminus \{p\}$ is not connected. The set of cut points of X will be denoted by $\text{Cut}(X)$. A point $x \in X$ is said to be a *local separating point of X* whenever x is a cut point of some open set in X.

We recall that a topological space is called *semilocally connected* if it has a basis of neighborhoods for which the complement of each one of its elements has only a finite number of components.

We recall some known concepts, which will be of use in the proof of Theorem 3.12.

By a *finite graph* we mean a continuum that can be expressed as the union of finitely many arcs, any two of which intersect in at most one or both of their end points. A *simple triod* is a finite graph that is the union of three arcs emanating from a single point v, and otherwise disjoint from one another.

For each $i \in \mathbb{N}$, let C_i be the circle in $\mathbb{R}^2$ with center at $(0, 1 - 2^{-i})$ and radius 2^{-i}. The *Hawaiian earring* is defined by $E = \bigcup_{i=1}^{\infty} C_i$ (see [4, p. 162]).

A continuum X is said to be *hereditarily locally connected* provided every subcontinuum of it is locally connected. Further, X is said to be *rational*, provided each point of X is contained in arbitrarily small neighborhoods whose boundaries are at most countable.

The following results are well known and will be used in proofs.

Lemma 2.1. [9, Corollary 5.9, p. 75] *Let X be a continuum and let A be a proper subcontinuum of X. If K is a component of $X \setminus A$, then $K \cup A$ is a continuum.*

Lemma 2.2. [9, 6.6, p. 89] *Every continuum has at least two noncut points.*

Lemma 2.3. [14, Theorem 9.2, p. 61] *All except a countable number of the local separating points of any locally compact separable metric space X are points of order two of X.*

The following result is commonly referred to as the Boundary Bumping Theorem.

Theorem 2.4. ([9, 5.4, p. 73] or [4, 12.10, p. 101]) *Let X be a continuum and let U be a nonempty, proper and open subset of X. Then every component of $\text{cl}(U)$ intersects $\text{bd}(U)$.*

Recall that a continuum X is said to be *irreducible* provided there exist two distinct points $a, b \in X$ such that no proper subcontinuum of X contains both a and b.

In order to prove the results of Section 4, we will make use of the layers of irreducible hereditarily decomposable continua. The reader is referred to [6] or [13] for the general theory of layers. We will use the term *layer* as defined in [6], and it will mean any element of the finest monotone upper semicontinuous decomposition of a continuum X, whose decomposition space is an arc. A mapping $\varphi : X \to I$ whose point inverses are the layers of X is called a *finest monotone map*. A *layer of continuity* is a layer $\varphi^{-1}(t)$ such that the upper semicontinuous function $\varphi^{-1} : I \to C(X)$ is continuous at t.

We next recall some known results about layers that will be used in Section 4.

Theorem 2.5. ([6, p. 216, Theorem 3] or [13, p. 15, Theorem 10]) *If X is an irreducible, hereditarily decomposable continuum, then there exists a unique finest monotone upper semicontinuous decomposition of X whose quotient space is an arc (hence a finest monotone map of X onto I).*

For uniqueness in the previous lemma see [13, p. 10, Theorem 6].

Lemma 2.6. ([6, p. 202] or [13, p. 60, Theorem 1]) *Under the assumptions of Theorem* 2.5, *the layers of continuity are dense in the quotient space of layers.*

The following result follows from [13, Theorem 8, p. 14].

Lemma 2.7. *Let X be an irreducible, hereditarily decomposable continuum and let $\varphi : X \to I$ be a finest monotone map. If $0 \leqslant s < t \leqslant 1$ and $\varphi^{-1}(s)$ and $\varphi^{-1}(t)$ are layers of continuity of X, then, $\varphi^{-1}([s, t])$ is an irreducible subcontinuum of X.*

Assume $\varphi : X \to I$ is a finest monotone map for a continuum X. If the layers of X are all degenerate, then the mapping $\varphi^{-1} : I \to X$ is well defined. Moreover, it is a homeomorphism. Therefore, we have the following observation.

Observation 2.8. *Let X be an irreducible, hereditarily decomposable continuum. If all the layers of X are degenerate, then X is an arc.*

Lemma 2.9. [6, Theorem 5, p. 217] *The layers of an irreducible, hereditarily decomposable continuum coincide with the nowhere dense saturated continua (i.e. with nowhere dense continua, which are not proper subcontinua of any other nowhere dense continuum).*

Lemma 2.10. [8] *For no compact, metric, irreducible continuum M is there a monotone interior transformation mapping M onto an arc A such that the inverse image of each point of A is an arc.*

The terms *arc-like* and *circle-like* refer to continua that admit ε-maps for each $\varepsilon > 0$ onto the interval I or the unit circle S^1, respectively. These notions are sometimes defined in terms of covers and are called *chainable* or *circularly chainable*, respectively (e.g., see [9, Theorem 12.11, p. 235]).

The following two results are well known and will be used several times.

Lemma 2.11. [9, Theorem 12.5, p. 233] *Every arc-like continuum is irreducible.*

Lemma 2.12. [6, 12, p. 225] *Every arc-like homogeneous continuum is a pseudoarc.*

3. Concerning cut points

In this section we develop some properties related to the set of cut points of a $\frac{1}{2}$-homogeneous continuum; we give characterizations of $\frac{1}{2}$-homogeneous, (semi)locally connected continua with more than one cut point and we give sufficient conditions under which $|\operatorname{Cut}(X)| = 1$ (Theorem 3.6, Corollary 3.7 and Theorem 3.11).

Definition 3.1. Let X be a semilocally connected continuum. A closed subset N of X is a *node* if either $N = \{p\}$ and $p \in \operatorname{End}(X)$ or N satisfies the following two properties:

(3.1.1) N is a maximal subset of X with respect to the property of being a connected subset of X such that $\operatorname{Cut}(N) = \emptyset$ and
(3.1.2) $N \cap \operatorname{cl}(X \setminus N)$ consists of exactly one point.

In the latter case, we denote $N \cap \operatorname{cl}(X \setminus N) = \{p_N\}$ and $N^* = N \setminus \{p_N\}$.

Observation 3.2. It follows from the definition of a node that:

(3.2.1) $p_N \in \mathrm{Cut}(X)$,
(3.2.2) N^* is an open subset of X and
(3.2.3) $N^* \cap M^* = \emptyset$ if N and M are two different nodes of X.

Also, from (3.2.2) and (3.2.3) it follows that

(3.2.4) the set of nondegenerate nodes of a continuum is at most countable.

The following results will be used in the proof of Theorem 3.6.

Lemma 3.3. *Let X be a continuum, let $k \in \mathbb{N}$ and let $N_1, \ldots, N_k$ be a subset of the set of nondegenerate nodes of X. Then the set $X \setminus \bigcup \{N_i^*: i = 1, \ldots, k\}$ is a continuum.*

Proof. We shall proceed by induction.
    Assume first that $k = 1$. By (3.2.2), $X \setminus N_1^*$ is a closed subset of X. On the other hand, $X \setminus \{p_{N_1}\} = N_1^* \cup U$, where U is an open subset of X and $U \cap N_1^* = \emptyset$. As a consequence of Lemma 2.1, we obtain that $X \setminus N_1^* = U \cup \{p_{N_1}\}$ is connected. Hence, $X \setminus N_1^*$ is a continuum.
    Let $N_1, N_2, \ldots, N_r, M$ be distinct nondegenerate nodes of X and define $T = \bigcup \{N_i^*: 1 = 1, \ldots, r\}$. We may assume $M \neq X \setminus T$.
    In order to apply the induction hypothesis, we only need to prove that M is a node of $X \setminus T$. Indeed, $M \subseteq X \setminus T$ and it satisfies condition (3.1.1). Further, note that

$$p_M \in M \cap \mathrm{cl}\big((X \setminus T) \setminus M\big) \subset M \cap \mathrm{cl}(X \setminus M) = \{p_M\}.$$

Hence, M satisfies condition (3.1.2). Therefore, we may conclude that M is a node of $X \setminus T$. The conclusion of the lemma now follows. $\quad\square$

Corollary 3.4. *Let N be a nondegenerate node of a continuum X. Then, $N \cap \mathrm{Cut}(X) = \{p_N\}$.*

Proof. Let $q \in N$, $q \neq p_N$. By (3.1.1), $N \setminus \{q\}$ is connected. Thus, since $X \setminus \{q\} = (N \setminus \{q\}) \cup (X \setminus N^*)$ and $p_N \in (N \setminus \{q\}) \cap (X \setminus N^*)$, it follows from Lemma 3.3 that $X \setminus \{q\}$ is connected. Therefore, $q \notin \mathrm{Cut}(X)$. $\quad\square$

Lemma 3.5. *If X is a $\frac{1}{2}$-homogeneous dendrite, then X is an arc.*

Proof. The lemma follows from the fact that every point in a dendrite is either a ramification point, an end point or a point of order 2, and that the set of end points and the set of points of order 2 of a dendrite are both nonempty (Lemmas 2.2 and 2.3). $\quad\square$

Theorem 3.6. *Let X be a $\frac{1}{2}$-homogeneous, semilocally connected continuum. Then, $|\mathrm{Cut}(X)| > 1$ if and only if X is an arc.*

Proof. Assume $|\mathrm{Cut}(X)| > 1$ and suppose by way of contradiction that X is not an arc. By Lemma 3.5, we have that X is not a dendrite; thus, X contains some point $p \in X \setminus (\mathrm{Cut}(X) \cup \mathrm{End}(X))$ [14, Chapter V, (1.1), p. 88]. Hence, the $\frac{1}{2}$-homogeneity of X yields $\mathrm{End}(X) = \emptyset$; thus, any node of X is nondegenerate. On the other hand, it is known that every semilocally connected continuum which has a cut point, has at least two nodes [14, Theorem 8.2, p. 77]; therefore, according to (3.2.4), we may assume that the set of nodes of X is of the form $\{N_1, N_2, \ldots\}$.
    Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the two orbits of X and let $h: X \to X$ be a homeomorphism. Then each node N_k is mapped onto some node N_j, $h(N_k^*) = N_j^*$ and $h(p_{N_k}) = p_{N_j}$. Thus, using Corollary 3.4 we may assume that $p_{N_i} \in \mathcal{O}_1$ and $N_i^* \subset \mathcal{O}_2$ for each i. Hence,

$$\mathcal{O}_1 = \{p_{N_1}, p_{N_2}, \ldots\} \quad \text{and} \quad \mathcal{O}_2 = \bigcup \{N_i^*: i \in \mathbb{N}\}. \tag{1}$$

Furthermore, for each $n \in \mathbb{N}$ define $Z_n = X \setminus \bigcup \{N_i^*: i = 1, \ldots, n\}$. Note that $Z_{n+1} \subset Z_n$ and that Z_n is a continuum (Lemma 3.3) for each n. Note also that $\mathcal{O}_1 = X \setminus \mathcal{O}_2 = X \setminus \bigcup \{N_i^*: i \in \mathbb{N}\} = \bigcap \{Z_n: n \in \mathbb{N}\}$. Hence, $\mathcal{O}_1$ is a continuum. Therefore, as a consequence of (1) we have that $|\mathcal{O}_1| = 1$.

Finally, applying Corollary 3.4 we obtain that $\mathcal{O}_2$ contains no cut points of X. Thus, $|\mathrm{Cut}(X)| = 1$, which leads us to a contradiction. Therefore, X is an arc.

The converse is immediate. $\quad\square$

As a corollary of Theorem 3.6 and [14, Corollary 13.21, p. 20] we obtain the following result.

Corollary 3.7. *Let X be a $\frac{1}{2}$-homogeneous, locally connected continuum. Then, $|\mathrm{Cut}(X)| > 1$ if and only if X is an arc.*

The hypothesis of semilocal connectedness in Theorem 3.6 is needed, as it can be seen in the following example.

Example 3.8. Let Y be the circle with a spiral, that is, $Y = S^1 \cup S$, where S^1 is the unit circle in the plane, and S is the spiral given in polar coordinates (ρ, θ) by:

$$S = \left\{ (\rho, \theta): \ \rho = 1 + \frac{1}{1+\theta} \text{ and } \theta \geqslant 0 \right\}$$

(see [4, Fig. 14, p. 51]). Thus, S approximates S^1. Let Y' be a homeomorphic copy of Y and let X be the space obtained by identifying the end points of Y and Y'. Then, it is easy to see that X is a $\frac{1}{2}$-homogeneous continuum with uncountably many cut points which is not an arc.

Before proving another result about cut points in $\frac{1}{2}$-homogeneous continua, we prove two lemmas.

Lemma 3.9. *Let X be a continuum, let S be a closed subset of X and let L be a component of $X \setminus S$. If $p \in \mathrm{Cut}(X) \cap L$, then $p \in \mathrm{Cut}(\mathrm{cl}(L))$.*

Proof. Let U_1 and U_2 be two nonempty, disjoint, open subsets of X such that $X \setminus \{p\} = U_1 \cup U_2$. Let W be an open subset of X such that $p \in W \subset \mathrm{cl}(W) \subset X \setminus S$ and $U_i \setminus \mathrm{cl}(W) \neq \emptyset$ for each i. Also for each i, let L_i be the component of $\mathrm{cl}(W \cap \mathrm{cl}(U_i))$ containing p. Then, $L_i \subset \mathrm{cl}(W) \subset X \setminus S$. Thus,

$$L_i \subset L \cap \mathrm{cl}(U_i) \subset \mathrm{cl}(L) \cap (U_i \cup \{p\}). \tag{2}$$

Since $\mathrm{cl}(U_i)$ is a continuum (Lemma 2.1), we may apply Theorem 2.4 to the open subset $W \cap \mathrm{cl}(U_i)$ of $\mathrm{cl}(U_i)$ to obtain that L_i is nondegenerate. According to this and to (2) we get that $\mathrm{cl}(L) \cap U_i \neq \emptyset$. It follows that $\mathrm{cl}(L) \setminus \{p\}$ is not connected. $\quad\square$

Lemma 3.10. *Let X be a continuum and let $F \subseteq \mathrm{Cut}(X)$ a finite set with at least two elements. Then there exist components C_1 and C_2 of $X \setminus F$ such that $|\mathrm{cl}(C_1) \cap F| = 1$ and $|\mathrm{cl}(C_2) \cap F| > 1$.*

Proof. To construct C_1, we proceed by induction.

Assume first that F is the two-point set $\{p, q\}$ and let U and V be two nonempty, disjoint, open subsets of X such that $X \setminus \{p\} = U \cup V$. Assume that $q \in U$ and let C_1 be any component of V. Then, $\mathrm{cl}(C_1) \subset X \setminus U$. Hence, $\mathrm{cl}(C_1) \cap F = \{p\}$.

Assume now that F has $m > 2$ elements, let $p \in F$ and consider the set $F^* = F \setminus \{p\}$.

By induction hypothesis, there exist $q \in F^*$ and a component L of $X \setminus F^*$ such that $\mathrm{cl}(L) \cap F^* = \{q\}$. If $p \notin L$, then $C_1 = L$ has the desired properties. Thus, we may assume that $p \in L$. Then, by Lemma 3.9, $\mathrm{cl}(L) \setminus \{p\}$ is not connected. Let C_1 be any component of $\mathrm{cl}(L) \setminus \{p\}$ which does not contain $\{q\}$. It is clear that C_1 satisfies the conditions required in the lemma.

On the other hand, given F as in our assumptions and $p \in F$, we let J be an irreducible continuum from p to $F \setminus \{p\}$. Then by [6, Chapter V, Theorem 5, p. 220] $J \setminus F$ is a connected set which is dense in J. Therefore, it is contained in a component C_2 of $X \setminus F$ and clearly C_2 has the required properties. $\quad\square$

As we saw in Example 3.8, there is a $\frac{1}{2}$-homogeneous continuum X with more than one cut point that is not an arc. In this case, however, X had infinitely many cut points; thus, one could ask if there is a $\frac{1}{2}$-homogeneous continuum with finitely many cut points (but more than one). The answer is negative, as we now show.

Theorem 3.11. *Let X be a $\frac{1}{2}$-homogeneous continuum such that $\mathrm{Cut}(X)$ is a finite, nonempty set. Then $|\mathrm{Cut}(X)| = 1$.*

Proof. Define $F = \mathrm{Cut}(X)$. Since $F \neq \emptyset$, by Lemma 2.2 we have that the two orbits of X are F and $X \setminus F$.

Suppose that $|F| > 1$ and let C_1 and C_2 be as in Lemma 3.10. Hence, if $a \in C_1$ and $b \in C_2$, then there exists a homeomorphism $h : X \to X$ such that $h(a) = b$. Thus, $h(C_1) = C_2$; hence, $h(\mathrm{cl}(C_1) \cap F) = \mathrm{cl}(C_2) \cap F$, which is impossible by the way C_1 and C_2 were chosen. Therefore, $|F| = 1$. $\square$

Finally, as an application of previous results of this section, we have the following characterization.

Theorem 3.12. *Let X be a hereditarily locally connected, $\frac{1}{2}$-homogeneous continuum which is not a finite graph. Then $\mathrm{Cut}(X) \neq \emptyset$ if and only if X is homeomorphic to the Hawaiian earring.*

Proof. Assume $\mathrm{Cut}(X) \neq \emptyset$. Then, it follows from Corollary 3.7 that X has exactly one cut point x_0. Hence, the two orbits of X are $\{x_0\}$ and $X \setminus \{x_0\}$. Further, since X is hereditarily locally connected, X is rational [14, Theorem 3.3, p. 94]; hence, the set of local separating points of X is a dense subset of X [14, Corollary 9.43, p. 63]. Thus, by the $\frac{1}{2}$-homogeneity of X, the set of local separating points of X contains $X \setminus \{x_0\}$. Moreover, applying Lemma 2.3 we obtain

$$\mathrm{ord}_X(x) = 2, \quad \text{whenever } x \neq x_0. \tag{3}$$

Note that if x_0 has finite order, then X is a finite graph [9, Theorem 9.10, p. 144]. Whence, since X is a rational continuum, we get $\mathrm{ord}_X(x_0) = \aleph_0$.

Let W be a component of $X \setminus \{x_0\}$, and suppose $\mathrm{ord}_{\mathrm{cl}(W)}(x_0) \geqslant 3$. For $i \in \{1, 2, 3\}$ take an arc $L_i \subset \mathrm{cl}(W)$, with end points x_0 and a_i such that $L_i \cap L_j = \{x_0\}$ whenever $i \neq j$ [6, p. 277]. Since X is locally connected, W is open and connected. Thus, W is arcwise connected [9, Theorem 8.26, p. 132]. Let $A, B \subset W$ be arcs whose end points are a_1 and a_2, and a_2 and a_3, respectively. Now, according to (3) it is easy to see that W does not contain simple triods, hence the set $A \cup L_1 \cup L_2$ is a simple closed curve. Nevertheless, it is impossible for the set $A \cup L_1 \cup L_2 \cup B$ not to contain a simple triod in W. This yields a contradiction with (3). Thus, we may conclude that $\mathrm{ord}_{\mathrm{cl}(W)}(x_0) \leqslant 2$; hence, $\mathrm{ord}_{\mathrm{cl}(W)}(x_0) = 2$. Therefore, by (3) and [9, Corollary 9.6, p. 142] we obtain that $\mathrm{cl}(W)$ is a simple closed curve.

Finally, if $\{W_i\}_{i=1}^{\infty}$ is the family of components of $X \setminus \{x_0\}$, then necessarily $\mathrm{diam}(W_i) \to 0$ [14, Corollary 2.2, p. 90]. Therefore, X is homeomorphic to the Hawaiian earring.

The converse is readily seen.

The assumption of hereditary local connectedness in Theorem 3.12 is essential, as can easily be seen by identifying two 2-spheres at a point. In fact, there are also infinite-dimensional examples, as we now show.

Example 3.13. We give an example of an infinite-dimensional, locally connected, $\frac{1}{2}$-homogeneous continuum, whose set of cut points is nonempty.

Let H, H' be two Hilbert cubes and let X be the space obtained by identifying a point $x \in H$ with a point $x' \in H'$. Let y, z be two distinct points in $H \setminus \{x\}$, let $A = \{x, y\}$ and let $B = \{x, z\}$. Note that both A and B are Z-sets in H since they are finite (see [4, p. 78]). Let $f : A \to B$ be given by $f(y) = z$ and $f(x) = x$. Then f can be extended to a homeomorphism $h : H \to H$ (see [4, 11.9.1, p. 93]). It follows easily that X is $\frac{1}{2}$-homogeneous.

Clearly, X is a locally connected continuum which is neither a finite graph nor the Hawaiian earring, and that has exactly one cut point.

The continuum Y described in the following example satisfies the same conditions as the continuum of Example 3.13 except that, this time, Y is one-dimensional.

Example 3.14. We give an example of a one-dimensional, locally connected, $\frac{1}{2}$-homogeneous continuum, which is neither a finite graph nor the Hawaiian earring, and whose set of cut points is nonempty.

Let M be a Menger universal curve. It is known that M is a locally connected, one-dimensional continuum which has no local cut points (see [2]). Further, in [1, Theorems II and III, pp. 320 and 322], it is shown that if a, b, c are three distinct points of M, then there exists a homeomorphism $h : M \to M$ such that $h(a) = b$ and $h(c) = c$.

Let M, M' be two Menger curves and let Y be the space obtained by identifying a point $y \in M$ with a point $y' \in M'$. As a simple consequence of the preceding paragraph, Y is $\frac{1}{2}$-homogeneous. It follows that Y has the required properties.

4. Concerning hereditary decomposability

In this section we show that if X is a hereditarily decomposable arc-like continuum, then X is $\frac{1}{2}$-homogeneous if and only if X is an arc. Using this result we prove a more general characterization, namely, Theorem 4.6. As a corollary we obtain that there is no hereditarily decomposable, $\frac{1}{2}$-homogeneous, circle-like continuum. At the end of the section we raise some open questions.

The next five results will allow us to prove Theorem 4.6, which is the main result of this section.

Lemma 4.1. *Let X be a continuum and let $h : X \to X$ be a homeomorphism. Assume A is an irreducible, hereditarily decomposable subcontinuum of X. Assume L and L' are layers of A such that $L, L' \subset \operatorname{int}(A)$ and such that $h(L)$ intersects L'. Then, $h(L) = L'$.*

Proof. Let $J = h(L) \cup L'$. Then, by Lemma 2.9, J is a nowhere dense subcontinuum of X. We shall first prove that $J \subset A$.

Suppose that $J \not\subseteq A$; then, $J \setminus \operatorname{int}(A) \neq \emptyset$. Let K be the component of $\operatorname{cl}(\operatorname{int}(A) \cap J)$ that contains L'; note that $K \subset A$.

Applying Theorem 2.4 to the open subset $\operatorname{int}(A) \cap J$ of J, we obtain that $K \cap \operatorname{bd}_J(\operatorname{int}(A) \cap J) \neq \emptyset$. This implies that K is a subcontinuum of A that properly contains L'. Now, since J is nowhere dense, K is nowhere dense. This contradicts Lemma 2.9. Thus, $J \subset A$. However, applying again Lemma 2.9, we obtain that $J = L'$. Hence, $h(L) \subset L'$.

Similarly, $h^{-1}(L') \subset L$. It follows that $L' \subset h(L)$. Therefore, $h(L) = L'$. $\quad\square$

Lemma 4.2. *Let X be a hereditarily decomposable continuum whose proper, nondegenerate subcontinua are arc-like. Assume A is an irreducible subcontinuum of X and L is a nondegenerate layer of A such that $L \subset \operatorname{int}(A)$. If X is $\frac{1}{2}$-homogeneous, then L intersects both orbits of X.*

Proof. Suppose on the contrary that L is contained in one orbit of X and let $x, w \in L$. Then, there exists a homeomorphism $h : X \to X$ such that $h(x) = w$. Thus, by Lemma 4.1, $h(L) = L$. Hence, L is a homogeneous continuum. Since L is a proper, nondegenerate subcontinuum of X, we have that L is arc-like. Hence, by Lemma 2.12, L is a pseudoarc, which yields a contradiction to the hereditary decomposability of X. Therefore, L intersects both orbits of X. $\quad\square$

The following theorem is a particular case of Theorem 4.6, but it will be used to prove it.

Theorem 4.3. *Let X be a hereditarily decomposable arc-like continuum. Then, X is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. Assume X is $\frac{1}{2}$-homogeneous.

Since X is arc-like, we have that X is an irreducible continuum (Lemma 2.11). Thus, by Theorem 2.5 we may take a finest monotone map $\varphi : X \to I$.

Note that the set of points of irreducibility of X coincides with $\varphi^{-1}(\{0, 1\})$. Thus, since X is $\frac{1}{2}$-homogeneous, the two orbits of X must be $\varphi^{-1}(\{0, 1\})$ and $\varphi^{-1}((0, 1))$. In particular, each layer of X is contained only in one orbit. Thus, as a consequence of Lemma 4.2, each layer of X is degenerate. Therefore, by Observation 2.8, X is an arc.

The converse is immediate. $\quad\square$

We will see in Example 4.8 that an arc-like, $\frac{1}{2}$-homogeneous continuum need not be an arc.

Lemma 4.4. *Let X be a $\frac{1}{2}$-homogeneous, hereditarily decomposable continuum whose proper, nondegenerate subcontinua are arc-like. Assume A is an irreducible subcontinuum of X and L is a layer of A. If $L \subset \text{int}(A)$, then L is degenerate.*

Proof. Since X is $\frac{1}{2}$-homogeneous, X has two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$.

Suppose by way of contradiction that L is nondegenerate. Then, there exist two points $l, m \in L$ that belong to the same orbit $\mathcal{O}_i$ in X. Hence, there exists a homeomorphism $g : X \to X$ taking l to m. Thus, by Lemma 4.1 we get that $g(L) = L$. It follows that $L \cap \mathcal{O}_i$ is contained in one orbit of L for each i. Therefore, L is either a homogeneous or a $\frac{1}{2}$-homogeneous continuum.

Further, since L is a proper, nondegenerate subcontinuum of X, we have that L is arc-like and hereditarily decomposable. Hence, combining Lemma 2.12 and Theorem 4.3 we obtain that

$$L \text{ is an arc.} \tag{4}$$

Next, let $\varphi : A \to I$ be a finest monotone map. Let $s, t \in [0, 1]$ such that $s < t$ and

$$L \subset \varphi^{-1}([s, t]) \subset \text{int}(A).$$

By Lemma 2.6 we may assume, without loss of generality, that $\varphi^{-1}(s)$ and $\varphi^{-1}(t)$ are layers of continuity of A.

Let $r \in [s, t]$ and let $z \in \varphi^{-1}(r)$. We may assume without loss of generality that z belongs to the orbit $\mathcal{O}_1$. By Lemma 4.2, L intersects both $\mathcal{O}_1$ and $\mathcal{O}_2$; hence, there exist $y \in L \cap \mathcal{O}_1$ and a homeomorphism $h_r : X \to X$ such that $h_r(y) = z$. Thus, using Lemma 4.1 we get

$$h_r(L) = \varphi^{-1}(r). \tag{5}$$

According to this and to (4) we obtain that

$$\text{each layer } \varphi^{-1}(r) \text{ is an arc whenever } r \in [s, t]. \tag{6}$$

Now, note that $\varphi^{-1}([s, t])$ is an irreducible subcontinuum M of A (Lemma 2.7) and that the layers of M coincide with the set $\{\varphi^{-1}(r) \colon r \in [s, t]\}$.

Furthermore, by Lemma 2.6 there exists $r' \in (s, t)$ such that $\varphi^{-1}(r')$ is a layer of continuity. Therefore, applying (5) to $r = r'$ we obtain that L is a layer of continuity of M. Hence, using (5) again, we obtain that each layer $\varphi^{-1}(r)$ is a layer of continuity of M. Thus, $\varphi|_M : M \to [s, t]$ is an open mapping (see [9, Theorem 13.10, p. 283]). However, according to this and to (6), we obtain a contradiction with Lemma 2.10.

Therefore, we may conclude that L is degenerate. $\quad\square$

Recall that an arc F in a continuum X is said to be a *free arc* provided the manifold interior of F is an open subset of X.

Corollary 4.5. *Let X be a $\frac{1}{2}$-homogeneous, hereditarily decomposable continuum whose proper, nondegenerate subcontinua are arc-like. Assume A is an irreducible subcontinuum of X and $\varphi : A \to I$ is a finest monotone map. If $s, t \in I$ are such that $s < t$ and $\varphi^{-1}([s, t]) \subset \text{int}(A)$, then $\varphi^{-1}([s, t])$ is a free arc in X.*

Proof. By Lemma 4.4, the layer $\varphi^{-1}(r)$ is degenerate for each $r \in \varphi^{-1}([s, t])$. Thus, $\varphi^{-1}([s, t])$ is an arc (Observation 2.8). Finally, since $\varphi^{-1}((s, t))$ is an open subset of $\text{int}(A)$, we conclude that $\varphi^{-1}([s, t])$ is a free arc in X. $\quad\square$

Theorem 4.6. *Let X be a hereditarily decomposable continuum whose proper, nondegenerate subcontinua are arc-like. Then, X is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. Assume X is $\frac{1}{2}$-homogeneous. Since X is decomposable, we may take two proper subcontinua A and B of X such that $X = A \cup B$.

Since A is a proper, nondegenerate subcontinuum of X, A is hereditarily decomposable and arc-like, thus, irreducible (Lemma 2.11). Hence, by Theorem 2.5 we may take a finest monotone map $\varphi : A \to I$. Let $s, t \in (0, 1)$ such that $s < t$ and

$$\varphi^{-1}([s, t]) \subset A \setminus B \subset \text{int}(A).$$

Let $F = \varphi^{-1}([s, t])$. By Corollary 4.5, F is a free arc in X. Let a, b be the end points of F.

Let M be the component of $X \setminus F$ that contains B. Then, $\emptyset \neq \mathrm{cl}(M) \cap F \subset \{a, b\}$.

In what follows we shall assume that $\mathrm{cl}(M) \cap F = \{a, b\}$ (the case in which $\mathrm{cl}(M) \cap F \subsetneq \{a, b\}$ can be handled in a very similar way).

Let F_a and F_b be two subarcs of F that contain a and b, respectively, such that $F_a \cap F_b = \emptyset$. Assume that the end points of F_a and F_b are $\{a, a'\}$ and $\{b, b'\}$, respectively.

Let $K = F_a \cup \mathrm{cl}(M) \cup F_b$. Note that K is a proper, nondegenerate subcontinuum of X. Thus, K is hereditarily decomposable and arc-like, hence, irreducible (Lemma 2.11).

By Theorem 2.5 there exists a finest monotone map $\psi : K \to I$. Since F_a and F_b are free arcs in X, and since every proper, nondegenerate subcontinuum of X is arc-like, it is easy to see that the only layers of K that intersect the boundary of K are $\{a'\}$ and $\{b'\}$. This implies that $\psi^{-1}(r) \subset \mathrm{int}(K)$ for every $r \in (0, 1)$. In particular, since $B \subset \psi^{-1}(0, 1)$, there exist $s', t' \in (0, 1)$ such that $s' < t'$ and

$$B \subset \psi^{-1}([s', t']) \subset \mathrm{int}(K).$$

However, applying Corollary 4.5 we obtain that $\psi^{-1}([s', t'])$ is a free arc in X. Hence, B is an arc.

Similarly, A is an arc.

Finally, according to our assumptions on X, it is easy to see that X is atriodic. Thus, X is the atriodic union of the arcs A and B, whence X is an arc or a simple closed curve [9, 8.40(b), p. 135]. Note however that an arc is $\frac{1}{2}$-homogeneous while a simple closed curve is not (it is homogeneous). Therefore, X is an arc.

The converse is immediate. $\quad\square$

The next corollary follows directly from Theorem 4.6.

Corollary 4.7. *There is no hereditarily decomposable, $\frac{1}{2}$-homogeneous, circle-like continuum.*

The assumption of hereditary decomposability of X is essential in Theorems 4.3, 4.6 and Corollary 4.7, as the following example shows.

Example 4.8. We give two examples, one of an arc-like, $\frac{1}{2}$-homogeneous continuum X, which is not an arc, and one of a $\frac{1}{2}$-homogeneous, circle-like continuum Y.

Let X be an arc of pseudoarcs, that is, an arc-like continuum with a continuous decomposition into pseudoarcs $\{P_x : 0 \leqslant x \leqslant 1\}$ such that the decomposition space is an arc (see [3,7]). As a consequence of [3, Theorem 10, p. 181] and [7, Theorem 5, p. 98], the set $\{P_x : 0 < x < 1\}$ is contained in some orbit $\mathcal{O}_1$ of X and the set $P_0 \cup P_1$ is contained in some orbit $\mathcal{O}_2$ of X.

According to [7, p. 98], the set P_x is terminal in X for each $x \in I$, in other words, every subcontinuum of X is either contained in a single decomposition element or is a union of decomposition elements. Thus, if $p \in P_0$, it is easy to see that no subcontinuum containing p separates X, while $P_{\frac{1}{2}}$ does. Therefore, $\mathcal{O}_1 \neq \mathcal{O}_2$ and X is $\frac{1}{2}$-homogeneous.

Next, let Y be the quotient space obtained from X by shrinking $P_0 \cup P_1$ to a point, say y_0. Then, Y is circle-like. Further, since $\{P_x : 0 < x < 1\}$ is an orbit of X, it is easy to see that $Y \setminus \{y_0\}$ is contained in an orbit of Y. Also, it is readily seen that y_0 is the only local separating point of Y. Therefore, we conclude that Y is $\frac{1}{2}$-homogeneous.

The authors do not know the answer to the following questions.

Question 4.9. Does there exist an indecomposable, $\frac{1}{2}$-homogeneous, arc-like continuum?

Question 4.10. Does there exist an indecomposable, $\frac{1}{2}$-homogeneous, circle-like continuum?

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