Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 330 (2007) 388–405
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Vector-valued Hardy spaces in non-smooth domains

Salvador Pérez-Esteva [a], Jorge Rivera-Noriega [b,*]

[a] *Instituto de Matemáticas – Unidad Cuernavaca, Universidad Nacional Autónoma de México, Apdo. Postal 273-3, Cuernavaca Mor. CP 62251, Mexico*
[b] *Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Av. Universidad 1001. Col. Chamilpa, Cuernavaca Mor. CP 62209, Mexico*

Received 10 March 2006
Available online 24 August 2006
Submitted by Steven G. Krantz

Abstract

We characterize the Radon–Nikodým property of a Banach space $\mathcal{X}$ in terms of the existence of non-tangential limits of $\mathcal{X}$-valued harmonic functions u defined in a domain $D \subset \mathbb{R}^n$, $n > 2$, with Lipschitz boundary and belonging to maximal Hardy spaces. This extends the same result previously known for the unit disk of $\mathbb{C}$. We also prove an atomic decomposition of the Borel $\mathcal{X}$-valued measures in ∂D that arise as boundary limits of $\mathcal{X}$-valued harmonic functions whose non-tangential maximal function is integrable with respect to harmonic measure of ∂D.

Keywords: Radon–Nikodým property; Vector-valued harmonic functions; Fatou theorems; Atomic decompositions of Hardy spaces

1. Introduction

In this paper we study some aspects of the theory of Hardy spaces of harmonic functions on Lipschitz domains taking values in a Banach space. We first consider the existence of boundary values of harmonic functions taking values in the Banach space $\mathcal{X}$, and relate this property with the geometry of $\mathcal{X}$. More precisely, if we consider Hardy spaces $H^p_{\mathcal{X}}(D)$ of harmonic functions $u : D \to \mathcal{X}$ defined on a starlike Lipschitz domain $D \subset \mathbb{R}^n$, $n > 2$, with center Ξ and surface measure σ (the definition will be given in the next section), we want to relate the existence of

* Corresponding author.
E-mail addresses: salvador@matcuer.unam.mx (S. Pérez-Esteva), rnoriega@servm.fc.uaem.mx (J. Rivera-Noriega).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.06.097

non-tangential pointwise limits of u with the *Radon–Nikodým property* of $\mathcal{X}$. This generalizes well-known results in [1,3,11] when D is the unit disk in $\mathbb{C}$. The second part of this paper is devoted to prove an atomic decomposition of the boundary 'distributions' of functions u in the class $H^1_{\mathcal{X}}(D)$. This decomposition is proved under no assumption on the underlying Banach space $\mathcal{X}$, and in particular is a more general result than the one in [2], where it is assumed that $\mathcal{X}$ has the Radon–Nykodým property. In fact, under this assumption on $\mathcal{X}$, the atoms we consider here are reduced to those in [2].

A technical difference is that, since we consider n-dimensional Lipschitz domains D, with $n > 2$, we rely on tools to represent harmonic functions, different to those used in the mentioned papers (e.g. Poisson integrals or Fourier series). One way to represent real-valued harmonic functions is by means of the *harmonic measure*. This is really a family of probability measures ω^X for $X \in D$, that may be obtained via the Riesz representation theorem applied to the operator $f \mapsto u_f(X)$, where u_f is the Perron–Wiener–Brelot solution of the Dirichlet problem

$$\begin{cases} \Delta u_f = 0 & \text{on } D, \\ u_f = f & \text{on } \partial D, \end{cases} \tag{1}$$

and where f is continuous on ∂D. This leads to the representation

$$u(X) = \int_{\partial D} f(Y)\, d\omega^X(Y).$$

It is well known from Harnack's principle, that all the measures ω^X, $X \in D$, are absolutely continuous with respect to $\omega \equiv \omega^{\Xi}$, where Ξ is the center of D. We will call ω *the* harmonic measure of ∂D. By fundamental results from [6] and [7], the measures ω and σ are mutually absolutely continuous in the A_∞ sense (see e.g. [10]). The above representation can be replaced by the following equivalent expression

$$u_f(X) = \int_{\partial D} f(Y) K(X, Y)\, d\omega(Y),$$

where $K(X, Y)$ is the Radon–Nikodým derivative $(d\omega^X/d\omega)(Y)$, and it is called the *kernel function*. We refer the reader to [4,13,15,16] for its basic properties, some extensions and its use in the scalar theory.

The first theorem of this paper extends a well-known characterization of the Radon–Nikodým property of the Banach space $\mathcal{X}$ (cf. [3]) which we describe now. A Banach space $\mathcal{X}$ has the *Radon–Nikodým property* ($\mathcal{X} \in RNP$), if for every probability space $(\Omega, \Sigma, \lambda)$, and every λ-continuous measure μ defined on Σ with values in $\mathcal{X}$, one can find a Bochner λ-integrable function $f : \Omega \to \mathcal{X}$, such that $\mu(E) = \int_E f\, d\lambda$ for every $E \in \Sigma$. We recall that μ is *λ-continuous* if $\mu(E) = 0$ whenever $\lambda(E) = 0$, $E \subseteq \partial D$ a Borel set. Also recall that the Radon–Nikodým property is independent of the (non-atomic) probability space $(\Omega, \Sigma, \lambda)$ [5, Theorem 2], so we may consider the Radon–Nikodým property with respect to ω in the Borel σ-algebra of ∂D. We refer the reader to [8] or [9] for the terminology of vector measures and further results on the Radon–Nikodým property.

The *non-tangential maximal function* of $u : D \to \mathcal{X}$ is defined as $u^*(Q) = \sup\{u(X)$: $x \in \Gamma_\alpha(Q)\}$. Here, the *non-tangential region* $\Gamma_\alpha(P)$ at $P \in \partial D$ is defined as the cone with vertex P, aperture $\alpha > 0$, with principal axis pointing in the radial direction, and truncated at height $|P - \Xi|$. The aperture $\alpha > 0$ depends only on the *Lipschitz character* of D, and it is chosen so that $\Gamma_\alpha(P)$ is always properly contained in D. Given a positive Borel measure λ and $p \geqslant 1$, $L^p_{\mathcal{X}}(\lambda)$ will denote the space of all Bochner integrable functions with norm

$$\|f\|_{L^p_{\mathcal{X}}(\lambda)} = \left(\int_{\partial D} \|f(Q)\|^p_{\mathcal{X}} \, d\lambda(Q) \right)^{1/p} < \infty.$$

The notation $L^p(\lambda)$ is reserved for Lebesgue spaces of scalar-valued functions. For $1 \leqslant p \leqslant \infty$, we define $H^p_{\mathcal{X}}(D)$ as the Banach space of all harmonic functions $u : D \to \mathcal{X}$ with $u^* \in L^p(\omega)$. As in the scalar theory, we endow $H^p_{\mathcal{X}}(D)$ with the norm

$$\|u\|_{H^p_{\mathcal{X}}} = \|u^*\|_{L^p_{\mathcal{X}}(\omega)}.$$

Theorem 1.1. *Let $\mathcal{X}$ be a Banach space. Then $\mathcal{X} \in RNP$ if and only if for some $1 \leqslant p \leqslant \infty$ and all $u \in H^p_{\mathcal{X}}(D)$, the limit*

$$\lim_{\substack{X \to P \\ X \in \Gamma(P)}} u(X) = u(P)$$

exists for σ-almost every $P \in \partial D$. This is equivalent to the same statement for all $1 \leqslant p \leqslant \infty$.

A consequence of the proof of this theorem is that we can solve an analogue of what is called the *L^p-Dirichlet problem* on Lipschitz domains (in the sense of [7]), when the boundary data is in the Bochner class $L^p_{\mathcal{X}}(\sigma)$.

Theorem 1.2. *Suppose $\mathcal{X} \in RNP$. Then for $2 < p < \infty$ and $f \in L^p_{\mathcal{X}}(\sigma)$ there exists a harmonic function $u : D \to \mathcal{X}$ such that*

$$\lim_{\substack{X \to P \\ X \in \Gamma(P)}} u(X) = f(P)$$

for σ-almost every $P \in \partial D$, and such that

$$\|u^*\|_{L^p(\sigma)} \leqslant C \|f\|_{L^p(\sigma)} \tag{2}$$

for an appropriate constant C not depending on f.

Remark 1.3. In this theorem the same conclusion holds for $1 < p < \infty$ if we assume the domain D has C^1 boundary. This will be apparent from the proof by the results of [7].

In Section 3 we study a characterization of functions in $H^1_{\mathcal{X}}(D)$ through its boundary limits. More precisely, we provide a one-to-one correspondence between functions in $H^1_{\mathcal{X}}(D)$ and a subspace denoted by $\mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$ of the space of all Borel $\mathcal{X}$-valued measures on ∂D of bounded variation.

We say that a Borel $\mathcal{X}$-valued measure defined on ∂D is an *atom* if the following three conditions hold:

(1) $\mu(\partial D) = 0$,
(2) there exists a ball B centered at a point in ∂D, such that $\operatorname{supp} \mu \subset \Delta \equiv B \cap \partial D$,
(3) $\|\mu\|_{V^\infty_{\mathcal{X}}} \equiv \sup\{\|\mu(E)\|_{\mathcal{X}} / \omega(E)\colon\ E \subset \partial D \text{ is a Borel set with } \omega(E) > 0\} \leqslant 1/\omega(\Delta)$.

Observe that by definition every atom is ω-continuous. We define the *atomic space* $\mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$ as the set of ω-continuous $\mathcal{X}$-valued measures μ that can be represented as $\mu = \sum_{i=1}^{\infty} \lambda_i \mu_i$, where μ_i are all atoms and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$.

Theorem 1.4. *A function u belongs to $H^1_{\mathcal{X}}(D)$ if and only if there exists $\mu \in \mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$ such that*

$$u(X) = \int_{\partial D} K(X, Q)\, d\mu(Q).$$

This result is a refinement to Lipschitz domains of $\mathbb{R}^n$ of the result in the unit disc of $\mathbb{C}$. Indeed, when $\mathcal{X} \in RNP$ and D is the unit disk of $\mathbb{C}$ one recovers results in [2].

Theorems 1.1 and 1.2 will be proved in the next section. In Section 3 we will prove Theorem 1.4 combining ideas from [15,18,20]. To accomplish this, we give a characterization of $H^1_{\mathcal{X}}(D)$ in terms of a 'grand maximal function' and prove an *ad hoc* Calderon–Zygmund decomposition for measures.

2. Non-tangential limits and the Radon–Nikodým property

We will keep the notations from the previous section and will introduce new terminology and definitions as needed. An open set $D \subset \mathbb{R}^n$ is a *starlike Lipschitz domain* centered at the origin with character M if, letting $S^{n-1} = \{x \in \mathbb{R}^n\colon |x| = 1\}$, there is a function $\varphi : S^{n-1} \to \mathbb{R}$ with $|\varphi(t) - \varphi(s)| \leqslant M|t - s|$ and $\varphi(t) \geqslant \delta > 0$, and such that in polar coordinates $D = \{(\rho, s)\colon 0 \leqslant \rho \leqslant \varphi(s),\ s \in S^{n-1}\}$. For $0 < r < 1$ set $D_r = \{(\rho, s)\colon 0 \leqslant \rho \leqslant r\varphi(s)\}$, and for $Q \in \partial D$, $Q = \varphi(s_0)$, we let $rQ \in D_r$ be the point $rQ = (r\varphi(s_0), s_0)$. We will keep the notation of Ξ for the center of D.

We will say that a countably additive $\mathcal{X}$-valued function ν defined on the Borel sets of ∂D has *bounded variation* if

$$\|\nu\|_{M_{\mathcal{X}}(\partial D)} = \sup \sum_{A \in \pi} \|\nu(A)\|_{\mathcal{X}},$$

where the supremum is taken over all the partitions π by measurable sets of ∂D. The space of Borel $\mathcal{X}$-valued measures of bounded variation is denoted by $M_{\mathcal{X}}(\partial D)$. If $\nu \in M_{\mathcal{X}}(\partial D)$ there exists a finite positive measure denoted by $|\nu|$ such that

$$\|\nu(A)\| \leqslant |\nu|(A)$$

for all Borel sets A and $|\nu|$ is minimal with this property (see [8, Chapter 1]).

The space of $\mathcal{X}$-valued continuous functions defined on ∂D is denoted by $C_{\mathcal{X}}(\partial D)$. For $1 \leqslant p < \infty$ we define $H^p_{\mathcal{X}}(D)$ as the space of $\mathcal{X}$-valued harmonic functions u such that $u^* \in L^p(\omega)$, while $H^\infty_{\mathcal{X}}(D)$ denotes the space of all bounded (with respect to the norm $\|\cdot\|_{\mathcal{X}}$ of $\mathcal{X}$) harmonic functions on D. For the scalar-valued case we will keep standard notations for the analogous spaces.

The next two basic lemmas prepare the ground for the proof of Theorem 1.1, which will be explained afterwards.

Lemma 2.1. *If $f \in C_{\mathcal{X}}(\partial D)$ then*

$$v(X) = \int_{\partial D} f(Y)\, d\omega^X(Y)$$

is a vector-valued harmonic function with

$$\lim_{\substack{X\to P\\ X\in\Gamma(P)}} v(X)=f(P)$$

for every $P\in\partial D$.

Proof. Since $\omega^{(\cdot)}(Y)$ is a harmonic function, the first part of the lemma is immediate. On the other hand, we have the representation

$$v(X)=\int_{\partial D} K(X,Y)f(Y)\,d\omega(Y). \tag{3}$$

Given $\varepsilon>0$ choose $\delta>0$ such that $|f(P)-f(Q)|<\varepsilon$ provided that $|Q-P|<\delta$ with $P,Q\in\partial D$ and $\operatorname{ess\,sup}_{\{Y\in\partial D\colon |Y-P|>\delta\}}|K(X,Y)|<\varepsilon$ whenever $|X-P|<\delta$, $X\in D$ and $P\in\partial D$ (see [13, p. 316]). Let Δ be the ball centered at P and radius δ. Then

$$\begin{aligned}\|v(X)-f(P)\|_{\mathcal{X}} &\leqslant \int_{\partial D} K(X,Y)\|f(Y)-f(P)\|_{\mathcal{X}}\,d\omega(Y)\\ &=\Bigg[\int_{\Delta}+\int_{\partial D\setminus\Delta}\Bigg]K(X,Y)\|f(Y)-f(P)\|_{\mathcal{X}}\,d\omega(Y)\\ &\leqslant \varepsilon\Bigg(1+\int_{\partial D}\|f(P)-f(Y)\|_{\mathcal{X}}\,d\omega(Y)\Bigg)\leqslant(1+2M)\varepsilon,\end{aligned}$$

where for every $X\in\overline{D}$ we have $\|f(X)\|_{\mathcal{X}}\leqslant M$. □

Lemma 2.2. *Let* $u\in H^1_{\mathcal{X}}(D)$. *Then there exists an* ω*-continuous* $\mathcal{X}$*-valued measure* $\mu\in M_{\mathcal{X}}(\partial D)$ *such that*

$$u(X)=\int_{\partial D} K(X,Q)\,d\mu(Q).$$

Proof. For $0<r<1$ the family u_r forms a bounded set on $L^1_{\mathcal{X}}(\omega)\subset M_{\mathcal{X}^{**}}(\partial D)$ and by Singer's theorem (see e.g. [12]) $M_{\mathcal{X}^{**}}(\partial D)=C_{\mathcal{X}^*}(\partial D)^*$. Lemma 2.1 applied to u_r and a standard weak* argument give us a vector measure $\mu\in M_{\mathcal{X}^{**}}(\partial D)$ such that

$$u(X)=\int_{\partial D} K(X,Q)\,d\mu(Q).$$

Consider the Lebesgue decomposition $\mu=\mu_c+\mu_s$ of μ with respect to ω, that is, μ_c is ω-continuous and μ_s is singular with respect to ω (cf. [8, Theorem 3.5.9]). Let ℓ be any continuous functional on $\mathcal{X}^{**}$. Then the function $v=\ell\circ u$ belongs to the space of scalar-valued functions $H^1(D)$ and there exists a unique measure $\nu\in M(\partial D)$ such that

$$v(X)=\int_{\partial D} K(X,Q)\,d\nu(Q)$$

(compare with [15, Theorem 5.11]). The fact that $v\in H^1(D)$ implies that ν is ω-continuous. Since we obviously have $\nu=\ell\circ\mu_c+\ell\circ\mu_s$, it follows that $\ell\circ\mu_s=0$, hence $\mu_s=0$ and

$\int_{\partial D} K(X,Q)\,d\mu_s(Q) = 0$. Now we prove that μ_c takes all its values in $\mathcal{X}$ arguing by contradiction.

Assume that there exists a Borel set A such that $\mu_c(A) \notin \mathcal{X}$. By the Hahn–Banach theorem, we can choose $\ell \in (\mathcal{X}^{**})^*$ such that $\ell = 0$ on X and $\ell(\mu_c(A)) = 1$. Since the scalar function $v = \ell \circ u$ is such that $v^* \in L^1(\omega)$, then by [15, Theorem 8.3]

$$v(X) = \int\limits_{\partial D} K(X,Q) f_\ell(Q)\,d\omega(Q)$$

where f_ℓ is the Radon–Nikodým derivative $d(\ell \circ \mu_c)/d\omega$. As usual, denoting $v_r(Q) = v(rQ)$, $Q \in \partial D$ we have the $v_r \to f_\ell$ in $L^1(\omega)$ as $r \to 1$. Hence

$$0 = \int\limits_{\partial D} v_r(Q)\,d\omega(Q) \to \int\limits_{\partial D} f_\ell(Q)\,d\omega(Q) = 1.$$

This contradiction yields the lemma. □

Now we provide the two main blocks to construct the proof of Theorem 1.1.

Lemma 2.3. *If $\mathcal{X} \in RNP$ then every function $u \in H^1_{\mathcal{X}}(D)$ has non-tangential limits for ω-almost every $Q \in \partial D$.*

Proof. According to Lemma 2.2 and by the Radon–Nikodým property of $\mathcal{X}$, we can represent $u \in H^1_{\mathcal{X}}(D)$ as

$$u(X) = \int\limits_{X} K(X,Q) f(Q)\,d\omega(Q),$$

with $f \in L^1_{\mathcal{X}}(\omega)$. Then we claim that the non-tangential limits exist in every Lebesgue point of f. To prove this assertion, and also that f is the function of non-tangential (ω almost everywhere) limits of u, let $P \in \partial D$ be a Lebesgue point of f and $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $|Q - P| < \delta$ one has

$$\frac{1}{\omega(\Delta)} \int\limits_{\Delta} \left\| f(Q) - f(P) \right\|_{\mathcal{X}} d\omega(Q) < \varepsilon,$$

where $\Delta \equiv \Delta_\delta(P)$. Choose now $\delta' > 0$ such that $\operatorname{ess\,sup}_{\{Y \in \partial D \setminus \Delta\}} |K(X,Y)| < \varepsilon$ provided that $|X - P| < \delta'$. For $X \in \Gamma(P)$, since $\int K(X,Q)\,d\omega(Q) = 1$,

$$\begin{aligned} u(X) - f(P) &= \int\limits_{\partial D} K(X,Q)\big[f(Q) - f(P)\big]\,d\omega(Q) \\ &\leqslant \Bigg[\int\limits_{\Delta} + \int\limits_{\partial D \setminus \Delta}\Bigg] K(X,Q)\big[f(Q) - f(P)\big]\,d\omega(Q). \end{aligned}$$

But it is well known (see e.g. [14] or [15]) that for $Q \in \Delta$ and X as above, $K(X,Q) \leqslant C/\omega(\Delta)$, for a constant $C > 0$. This already implies that

$$\left\| u(X) - f(P) \right\|_{\mathcal{X}} \leqslant (1 + M)\varepsilon$$

whenever $|X-P|<\min\{\delta,\delta'\}$, where again M is an upper bound for $\|f\|_{\mathcal{X}}$ on D. Since almost every $P\in\partial D$ is a Lebesgue point of f (cf. [8]), the proof is complete. □

Lemma 2.4. *If every function in $H^\infty_{\mathcal{X}}(D)$ has non-tangential limits ω-a.e. then $\mathcal{X}\in RNP$.*

Proof. We will prove that every continuous linear operator $T: L^1(\omega)\to\mathcal{X}$ is representable by a function $f\in L^\infty_{\mathcal{X}}(\omega)$, namely

$$T(g)=\int_{\partial D} f(Q)g(Q)\,d\omega(Q)$$

(cf. [8, Chapter III, Section 1, Theorem 5]). Define for $X\in D$, $v(X)=T(K(X,\cdot))$ so that v is harmonic and

$$\|v(X)\|_{\mathcal{X}}\leqslant\|T\|\,\|K(X,\cdot)\|_{L^1(\omega)}=\|T\|,$$

that is, $v\in H^\infty_{\mathcal{X}}(D)$. Let $f\in L^\infty_{\mathcal{X}}(\omega)$ be the non-tangential limit of v. We claim that f represents the operator T. By a standard density argument it suffices to prove that

$$T(\chi_A)=\int_A f\,d\omega$$

for every Borel set A in ∂D, where χ_A denotes the characteristic function of A. Now,

$$\begin{aligned}\int_A f\,d\omega=\lim_{r\to 1}\int_A v(rP)\,d\omega(P)&=\lim_{r\to 1}\int_{\partial D} T\big(K(rP,\cdot)\big)\chi_A(P)\,d\omega(P)\\ &=\lim_{r\to 1}\int_{\partial D} T\big(K(rP,\cdot)\chi_A(P)\big)\,d\omega(P),\end{aligned}\tag{4}$$

where the integral $\int_{\partial D}T(K(rP,\cdot)\chi_A(P))\,d\omega(P)$ is interpreted as a Bochner integral. The continuity of K on $D\times\partial D$ implies that

$$\int_{\partial D} T\big(K(rP,\cdot)\chi_A(P)\big)\,d\omega(P)=T\bigg(\int_{\partial D} K(rP,\cdot)\chi_A(P)\,d\omega(P)\bigg).$$

We now claim that

$$\lim_{r\to 1}\int_{\partial D} K(rP,\cdot)\chi_A(P)\,d\omega(P)=\chi_A$$

in the weak topology of $L^1_{\mathcal{X}}(\omega)$. In fact, for every $g\in L^\infty_{\mathcal{X}}(\omega)$, we have

$$\int_{\partial D}\bigg(\int_A K(rP,Q)\,d\omega(P)\bigg)g(Q)\,d\omega(Q)=\int_A\bigg(\int_{\partial D} K(rP,Q)g(Q)\,d\omega(Q)\bigg)\,d\omega(P).$$

But $\int_{\partial D}K(r(\cdot),Q)g(Q)\,d\omega(Q)$ is uniformly bounded and converges almost everywhere to g. Then

$$\lim_{r\to 1}\int_{\partial D}\bigg(\int_A K(rP,Q)\,d\omega(P)\bigg)g(P)\,d\omega(Q)=\int_A g\,d\omega$$

and our claim follows. The continuity of T implies that T is continuous when $L^1(\omega)$ and $\mathcal{X}$ are endowed with the weak topology (see e.g. [19, Theorem 2.5.11]). It follows that for every $\ell \in \mathcal{X}^*$ we have

$$\langle T(\chi_A), \ell\rangle = \lim_{r\to 1}\left\langle T\left(\int_{\partial D} K(rP,\cdot)\chi_A(P)\,d\omega(P)\right), \ell\right\rangle. \tag{5}$$

From (4) and (5) we conclude that

$$\langle T(\chi_A), \ell\rangle = \left\langle \int_A f\,d\omega, \ell\right\rangle$$

for all $\ell \in \mathcal{X}^*$ and therefore $T(\chi_A) = \int_A f\,d\omega$. The theorem follows. □

Notice that Lemmas 2.3 and 2.4 imply Theorem 1.1. We observe that the proof did not rely at all on Fourier series as the original proof of [11], since we do not have an explicit representation of the Poisson kernel, as in the unit disk of $\mathbb{C}$.

Proof of Theorem 1.2. As observed above, one may use Lemma 2.1 and the Lebesgue points argument of Lemma 2.2 to prove that the function

$$u(X) = \int_{\partial D} K(X,Q) f(Q)\,d\omega(Q)$$

has non-tangential limits equal to $f(Q)$ for ω-almost every $Q \in \partial D$, whenever $f \in L^p_{\mathcal{X}}(d\sigma)$, $2 < p$. To obtain the L^p bound for u^* recall first that the Radon–Nikodým derivative $(d\sigma/d\omega)(Q) \equiv k(Q)$ belongs to the *reverse Hölder class of weights* $B^q(\partial D)$, for all $1 < q < 2$ (cf. [7]). This means that for every surface ball $\Delta \subset \partial D$ one has

$$\left(\frac{1}{\sigma(\Delta)}\int_{\Delta} k(Q)^q\,d\omega(Q)\right)^{1/q} \leqslant b_q\left(\frac{1}{\sigma(\Delta)}\int_{\partial D} k(Q)\,d\omega(Q)\right)$$

with a uniform constant b_q which depends on n, q and the Lipschitz character of ∂D. It is well known (see e.g. [10]) that $k \in B^q(\partial D)$ implies that the *Hardy–Littlewood maximal function*

$$M_\omega g(Q) = \sup\left\{\frac{1}{\omega(\Delta)}\int_{\Delta} \left|g(Q)\right|\,d\omega(Q)\colon\ \Delta \text{ is a surface ball with } Q \in \Delta\right\}$$

satisfies the weighted inequality

$$\|M_\omega g\|_{L^p_{\mathcal{X}}(k\,d\omega)} \leqslant C\|g\|_{L^p_{\mathcal{X}}(k\,d\omega)} \tag{6}$$

with a constant C not depending on g, and with $1/p + 1/q = 1$, $2 < p < \infty$. However, notice that the norm $\|\cdot\|_{L^p(k\,d\omega)}$ is exactly the norm $\|\cdot\|_{L^p(d\sigma)}$.

Now, given $P \in \partial D$ and $X \in \Gamma_\alpha(P)$, the argument of [16, p. 14] yields

$$\left\|u(X)\right\|_{\mathcal{X}} \leqslant C M_\omega\left(\left\|f(\cdot)\right\|_{\mathcal{X}}\right)(P),$$

which by the above notes and (6) implies the theorem. □

3. Atomic decomposition for $\mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$

We start this section recalling some concepts of vector measures and referring the reader to [9] for more details. For $1 < q \leqslant \infty$, the space $V^q_{\mathcal{X}}$ consists of all the measures $\nu \in M_{\mathcal{X}}(\partial D)$, such that

$$\|\nu\|_{V^q_{\mathcal{X}}} < \infty,$$

where $\|\nu\|_{V^\infty_{\mathcal{X}}}$ was given in the definition of atoms in Section 1, and for $1 < q < \infty$,

$$\|\nu\|_{V^q_{\mathcal{X}}} = \sup\biggl(\sum_{A\in\pi} \frac{\|\nu(A)\|^q_{\mathcal{X}}}{\omega(A)^{q-1}}\biggr)^{1/q},$$

where the supremum is taken over all the partitions π by measurable sets of ∂D. For $\nu \in V^q_{\mathcal{X}}$, $q > 1$, $|\nu|$ is absolutely continuous with respect to ω, with $d|\nu|/d\omega \in L^q(\omega)$. Moreover, if $\nu \in V^q_{\mathcal{X}}$ has a density f then $f \in L^q_{\mathcal{X}}(\omega)$ and

$$\|\nu\|_{V^q_{\mathcal{X}}} = \|f\|_{L^q_{\mathcal{X}}(\omega)}.$$

In case $\varphi \in C(\partial D)$ then one has

$$\|\varphi\, d\nu\|_{V^q_{\mathcal{X}}} \leqslant \|\varphi f\|_{L^q_{\mathcal{X}}(\omega)}. \tag{7}$$

Lemma 3.1. *If $\mu \in M_{\mathcal{X}}(\partial D)$ is an atom then*

$$u(X) = \int_{\partial D} K(X, Q)\, d\mu(Q)$$

belongs to $H^1_{\mathcal{X}}(D)$ with $\|u\|_{H^1_{\mathcal{X}}}$ bounded by an absolute constant $C > 0$.

Proof. Suppose for definiteness that μ is supported in the surface ball $\Delta \equiv \Delta_r(Q_0)$ of radius r centered at Q_0. Since μ is an atom we have $\|\mu(E)\| \leqslant \frac{\omega(E)}{\omega(\Delta)}$ for every Borel set E, which implies that $|\mu|(E) \leqslant \frac{\omega(E)}{\omega(\Delta)}$ and $\frac{d|\mu|}{d\omega} \leqslant \frac{1}{\omega(\Delta)}$.

Then for all $X \in D$,

$$\bigl\|u(X)\bigr\|_{\mathcal{X}} \leqslant \int_{\partial D} K(X, Q)\, d|\mu|(Q) \leqslant \int_{\partial D} K(X, Q) \frac{d|\mu|}{d\omega}(Q)\, d\omega(Q) \leqslant \frac{1}{\omega(\Delta)}. \tag{8}$$

Estimate (8) gives an upper bound for $u^*(P)$ for $P \in \Delta$. Also notice that since $\sigma \ll \omega$, estimate (8) implies a uniform bound of $\|u(X)\|_{\mathcal{X}}$ independent of μ provided $r > r_0$ for a fixed number r_0.

According to [15, Lemma 4.11], we can choose $r_0 > 0$ such that if $r < r_0$, $\Delta' = \Delta_s(Q_0) \subset \Delta_{r/2}(Q_0)$ and $|Y - Q_0| \geqslant 2r$ then

$$\omega^A(\Delta') \approx \frac{\omega^Y(\Delta')}{\omega^Y(\Delta)}, \tag{9}$$

where $A = A^r(Q_0)$ is any point D such that $|Y - Q_0| \approx r \approx \operatorname{dist}(A, \partial D)$. Here $A \approx B$ means that the ratio A/B is bounded above and below by constants depending at most on n and the Lipschitz character of D.

Assume that $r \leqslant r_0$ and let $P \notin \Delta$, then for some $j \in \mathbb{N}$, $P \in \Delta_j \backslash \Delta_{j-1}$, with $\Delta_j = 2^j\Delta$. It is readily seen that if $X \in \Gamma_\alpha(P)$ then $|X - Q_0| \geqslant C_\alpha 2^j r$. Then by [15, Theorem 7.1] we have

$$\bigl|K(X, Q) - K(X, Q_0)\bigr| \leqslant C2^{-j} K(X, Q_0)$$

for all $Q \in \Delta$. The cancellation property of the atoms implies that

$$\begin{aligned}\|u(X)\|_{\mathcal{X}} &\leqslant \int_{\Delta} |K(X,Q) - K(X,Q_0)| \frac{d|\mu|}{d\omega}(Q)\, d\omega(Q) \\ &\leqslant \frac{1}{\omega(\Delta)} \int_{\Delta} |K(X,Q) - K(X,Q_0)|\, d\omega(Q) \\ &\leqslant \frac{C}{2^j} K(X,Q_0),\end{aligned}$$

where the constant C depends on the Lipschitz character of D.

Since for almost all $Q_0 \in \partial D$ (see [15])

$$K(X,Q_0) = \lim_{s\to 0} \frac{\omega^X(\Delta_s(Q_0))}{\omega(\Delta_s(Q_0))},$$

then for s small we apply (9) with $A = X$ and $Y = \Xi$, to obtain

$$K(X,Q_0) \leqslant \frac{C}{\omega(\Delta_j)},$$

provided $2^j r < r_0$.

Hence for these values of j we have

$$\int_{\Delta_j} u^*(Q)\, d\omega(Q) \leqslant \frac{C}{2^j}. \tag{10}$$

The set J consisting of all $j \in \mathbb{N}$ such that $2^j r \geqslant r_0$ and $\Delta_j \neq \emptyset$, has at most a finite number of elements depending on D only. Since $|X - Q_0| \geqslant C_\alpha 2^j r$, then $K(X,Q_0)$ is uniformly bounded for all $j \in J$. It follows that

$$\sum_{j\in J} \int_{\Delta_j} u^*(Q)\, d\omega(Q) \leqslant C. \tag{11}$$

Finally from (10) and (11) we conclude that there exists $C > 0$ independent of μ such that

$$\int_{\partial D} u^*(Q)\, d\omega(Q) \leqslant C. \qquad \square$$

Remark 3.2. Notice that from Lemma 3.1 it follows that $u \in H^1_{\mathcal{X}}(D)$ whenever

$$u(X) = \int_{\partial D} K(X,Q)\, d\mu(Q)$$

with $\mu \in \mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$.

To prove the converse of Theorem 1.4 we start with $u \in H^1_{\mathcal{X}}(\partial D)$. Lemma 2.2 guarantees the existence of an ω-continuous vector measure $\mu \in M_{\mathcal{X}}(\partial D)$ such that

$$u(X) = \int_{\partial D} K(X,Q)\, d\mu(Q). \tag{12}$$

To complete the proof it remains to prove that $\mu \in \mathcal{H}^1_{\mathcal{X}}(\partial D, d\omega)$. First we introduce some notation and definitions from the theory of *spaces of homogeneous type*, as it can be readily checked that ∂D endowed with ω and the Euclidean distance is indeed a space of homogeneous type. For $Q_1, Q_2 \in \partial D$ define the *measure distance*

$$m(P,Q) = \inf\{\omega(\Delta)\colon\ x, y \in \Delta,\ \text{and } \Delta \text{ is any surface ball}\}.$$

The balls with respect to the measure distance are denoted by $\Delta^m_r(P) = \{Q \in \partial D\colon m(P,Q) < r\}$ and will be called *metric balls*. An important well-known property is that we then have $\omega(\Delta^m_r) \approx r$. In [17], it is proved that m can be modified to an equivalent quasi-distance m' defined on ∂D satisfying the additional property that for some $0 < \alpha < 1$,

$$\left|m'(P,Q_1) - m'(P,Q_2)\right| \leqslant r\left(\frac{m'(Q_1,Q_2)}{r}\right)^{\alpha} \tag{13}$$

whenever $m'(P,Q_1) < r$ and $m'(P,Q_2) < r$. (A *quasi-distance* satisfies by definition the properties of a metric except for the triangle inequality which is replaced by $m'(P,Q) \leqslant \kappa(m'(P,R) + m'(R,Q))$ for all $P, Q, R \in \partial D$.) Then we may assume that the quasi-distance m satisfies (13).

According to [15], from arguments in [17] one can see that atoms with respect to Euclidean balls are the same as atoms with respect to metric balls, except for constant factors. For completeness we provide a short argument, specialized to ∂D with the harmonic measure.

Lemma 3.3. *Given an Euclidean ball Δ, there exists a metric ball Δ^m such that $\Delta \subseteq \Delta^m$ and $\omega(\Delta^m)/\omega(\Delta) \approx 1$. This statement can be reversed and the constants involved may depend on the doubling property of ω.*

Proof. We can assume that the radius of Δ is small. We recall [16, p. 11] that if $\Delta \equiv \Delta_r(Q)$ is an Euclidean ball of radius $0 < r < r_0$ centered at $Q \in \partial D$, then $\omega(\Delta) \approx r^{n-2} G(\Xi, A_r)$, where $A_r \in D$ is a point whose distance to Q is proportional to r, and $G(X,Y)$ denotes the Green's function on D. Since the Green's function is uniformly bounded far from the diagonal and A_r is always far from Ξ, this implies that $\omega(\Delta) \approx r^{n-2}$ with constants that may also depend on the diameter of D.

Now notice that from the definitions, $\Delta \subseteq \Delta^m \equiv \Delta^m_{Cr^{n-2}}(Q)$ and $\omega(\Delta^m) \approx r^{n-2}$ which by the above remark implies the first claim. On the other hand, the Euclidean diameter of $\Delta^m_r(Q)$ is always less than $Cr^{1/(n-2)}$, as it can be verified from the definition of metric distance. This already implies that $\Delta^m_r(Q) \subseteq \Delta_{Cr^{1/(n-2)}}(Q) \equiv \Delta'$, and since $\omega(\Delta') \approx r^{(n-2)/(n-2)} = r$, with constants that may depend on the doubling property of ω, the proof is complete. $\square$

For $\alpha > 0$ the class $\mathrm{Lip}_m(\alpha)$ will denote the class of functions f in ∂D for which

$$L(f,\alpha,m) \equiv \sup\left\{\frac{|f(P) - f(Q)|}{m(P,Q)^{\alpha}}\colon\ P, Q \in \partial D,\ P \neq Q\right\} < \infty.$$

Let $K(r,P,Q) \geqslant 0$ be the continuous function defined on $(0,1] \times \partial D \times \partial D$ and such that for some $0 < \gamma < 1$ and $A > 0$:

(i) $K(r,P,Q) \leqslant (1 + m(x,y)/r)^{-1-\gamma}$,
(ii) $K(r,P,P) \geqslant A^{-1}$,

(iii) $|K(r,P,Q) - K(r,P,Q_0)| \leqslant (m(Q,Q_0)/r)^\gamma((1+m(P,Q))/r)^{-1-2\gamma}$ whenever $m(Q,Q_0) \leqslant (r+m(P,Q))/4A$.

In [15, Lemma 8.11] it is given the construction of $K(r,P,Q)$ for some $\gamma > 0$. A computation shows that for $\gamma' < \gamma$, the three properties defining K remain true with γ' replacing γ, so we will fix $\gamma < \alpha$, with α as in (13). With these in mind we will refer to results and techniques from the references [18,20] with no restriction, and in particular our aim is to generalize some of their results to the vector-valued setting.

For the function $K(r,P,Q)$ and $\mu \in M_{\mathcal{X}}(\partial D)$ define

$$\mu^+(P) = \sup_{0<r\leqslant 1} \left\| \frac{1}{r} \int_{\partial D} K(r,P,Q)\, d\mu(Q) \right\|_{\mathcal{X}}.$$

This function, as in the scalar case, satisfies

$$\mu^+(P) \lesssim u^*(P) \tag{14}$$

for every $P \in \partial D$, with u as in (12).

Define now the *grand maximal function* for $P \in \partial D$

$$\mathcal{M}\mu(P) = \sup_{0<r\leqslant 1} \left\| \frac{1}{r} \int_{\partial D} \varphi(P)\, d\mu(P) \right\|_{\mathcal{X}},$$

where the supremum is taken for $r > 0$ over φ supported on $\Delta_r^m(P)$ and satisfying $L(\varphi,\gamma,m) \leqslant r^{-\gamma}$ and $\|\varphi\|_\infty \leqslant 1$, with γ as above (in this case we write $\varphi \in T(P)$).

Lemma 3.4. *There exists $p_1 < 1$, independent of γ, such that for every $\mu \in M_{\mathcal{X}}(\partial D)$ and every $p > p_1$ one has $\|\mathcal{M}\mu\|_{L^p_{\mathcal{X}}(\omega)} \leqslant C\|\mu^+\|_{L^p_{\mathcal{X}}(\omega)}$ with a constant C depending on p and ∂D.*

Proof. We can adapt arguments in [20, Theorem 1] whose proof relies on the structure of the subset $T(P)$ of the scalar space $\mathrm{Lip}_m(\gamma)$. That argument is based on the following (see [20, Lemma 3]):

There exist $p_1 < 1$ and C (only depending on ∂D) such that

$$\left\| \frac{1}{r_0} \int_{\partial D} \varphi(Q)\, d\mu(Q) \right\|_{\mathcal{X}} \leqslant C_4 \left(\frac{1}{r_0} \int_{\Delta^m_{r_0}(P_0)} \big(\mu^+(Q)\big)^{p_1}\, d\omega(Q) \right)^{1/p_1}$$

whenever φ is supported on $\Delta^m_{r_0}(P_0)$ and satisfying $L(\varphi,\gamma,m) \leqslant r_0^{-\gamma}$ and $\|\varphi\|_\infty \leqslant 1$.

With this result in hand we observe that if we define for $p > 0$ and $f \in L^1(\omega)$

$$M_p(f)(P) = \sup_{r>0} \left(\frac{1}{r} \int_{\Delta^m_r} \big|f(Q)\big|^p\, d\omega(Q) \right)^{1/p}$$

then the following holds:

$$\|\mathcal{M}\mu\|_{L^p_{\mathcal{X}}(\omega)} \leqslant M_{p_1}\big(\mu^+\big) \leqslant C \big\| M_1\big(\big[\mu^+\big]^{p_1}\big)\big\|^{1/p_1}_{L^{p/p_1}_{\mathcal{X}}(\omega)} \leqslant C(p,p_1)\big\|\mu^+\big\|_{L^p_{\mathcal{X}}(\omega)}$$

for $p > p_1$, where the last inequality is the continuity of the Hardy–Littlewood maximal function with respect to metric balls in $L^{p/p_1}_{\mathcal{X}}(\omega)$. This clearly implies the lemma. □

We conclude that $\mathcal{M}\mu \in L^1_{\mathcal{X}}(\omega)$ for every $u \in H^1_{\mathcal{X}}(\partial D, d\omega)$ provided that we choose $K(r, P, Q)$ as above. This is the basis of the atomic decomposition we describe next, which is based on arguments from [18]. For convenience we quote the following unified adaptation of Lemmas 2.9 and 2.16 of [18].

Lemma 3.5 *(Partition of unity). Let Ω be a proper open subset of ∂D and let $d(P) = \inf\{m(P, Q)\colon\ Q \in \partial D \setminus \Omega\}$ and $C = 5\kappa$, with κ the constant of the quasi-distance m. Then there exist constants $M \in \mathbb{N}$, $c_0 > 0$, $c_1 > 1$, a sequence of positive functions $\{\varphi_n\}$, a sequence $\{\Delta^m_n \equiv \Delta^m_{r_n}(P_n)\}$ of metric balls with the following properties*:

(1) *the balls $\Delta^m_{(4\kappa)^{-1}r_n}(P_n)$ are pairwise disjoints and $\bigcup \Delta^m_n = \bigcup \Delta^m_{Cr_n}(P_n) = \Omega$,*
(2) *$c_1 r_n < \operatorname{diam} D/2$, and the number of balls $\Delta^m_{Cr_k}(P_k)$ that intersect a fixed $\Delta^m_{Cr_n}(P_n)$ is at most M,*
(3) *if $P \in \Delta^m_n$ then $Cr_n \leqslant d(P) \leqslant 3C\kappa^2 r_n$, and there exists $Q_n \notin \Omega$ such that $m(P_n, Q_n) < 3C\kappa r_n$,*
(4) *φ_n is supported in $\Delta^m_{2r_n}(P_n)$ and $\varphi_n > 1/M$ over Δ^m_n,*
(5) *$\varphi_n \in \operatorname{Lip}(\alpha)$ with $\|\varphi\|_\alpha \leqslant c_0/r_n^\alpha$ and*

$$\sum_n \varphi_n(X) = \chi_\Omega(X).$$

Next we prove a Calderón–Zygmund type decomposition adapted to our situation.

Lemma 3.6 *(Calderón–Zygmund type Lemma). Let $1 < q < \infty$ and $\mu \in V^q_{\mathcal{X}}(\partial D)$ such that $\mathcal{M}\mu \in L^1_{\mathcal{X}}(\omega)$. Let $t > \int \mathcal{M}\mu(P)\, d\omega(P)$ and for $\Omega = \{P\colon\ \mathcal{M}\mu(P) > t\}$, we consider the cover $\{\Delta^m_{r_n}(P_n)\}$ of Ω and the partition of the unity $\{\varphi_n\}$ given by the previous lemma. For every $n \in \mathbb{N}$ define*

$$db_n = \varphi_n\, d\mu - m_n\varphi_n\, d\omega \quad \text{with } m_n = \Bigl(\int_{\partial D} \varphi_n\, d\omega\Bigr)^{-1} \int_{\partial D} \varphi_n\, d\mu \tag{15}$$

and

$$dG = \chi_{\Omega^c}\, d\mu + \sum m_n\varphi_n\, d\omega. \tag{16}$$

Then there exists a constant $c > 0$ such that

$$\mathcal{M}b_n(P) \leqslant ct\left[\frac{r_n}{m(P, P_n) + r_n}\right]^{1+\gamma} \chi_{(\Delta^m_{4\kappa r_n}(P_n))^c}(P) + c\mathcal{M}\mu(P)\chi_{\Delta^m_{4\kappa r_n}(P_n)}(P). \tag{17}$$

Also $dB \equiv \sum db_n$ converges in $V^q_{\mathcal{X}}$, and the following estimates hold:

$$\mathcal{M}G(P) \leqslant ct \sum_n \left[\frac{r_n}{d(P, P_n) + r_n}\right]^{1+\gamma} + c\mathcal{M}\mu(P)\chi_{\Omega^c}(P), \tag{18}$$

$$\int_{\partial D} \mathcal{M}G\, d\omega \leqslant c \int_{\Omega} \mathcal{M}\mu\, d\omega, \tag{19}$$

$$\|G\|_{V^\infty_{\mathcal{X}}} \leqslant ct, \tag{20}$$

where γ is as in the definition of $K(r, P, Q)$, and the constant c depends on n, the Lipschitz character of D and the doubling property of ω.

Proof. First notice that for $\psi \in C(\partial D)$,

$$\int_{\partial D} \psi \, db_n = \int_{\partial D} S_n \psi \, d\mu,$$

where the operator S_n is defined as

$$S_n(\psi)(P) = \phi_n(P) \frac{\int_{\partial D} [\psi(P) - \psi(z)] \phi_n(z) \, d\omega(z)}{\int_{\partial D} \phi_n(z) \, d\omega(z)}.$$

The proof of (17) is the same as that of the scalar version of this statement included in [18, Lemma 3.2]. We first note that the following estimate holds (compare with [18, Lemma 3.36])

$$\|m_n\|_{\mathcal{X}} \leqslant ct. \tag{21}$$

Then the convergence in $V^q_{\mathcal{X}}(\partial D)$ of $\sum_n db_n$ is a consequence of estimates (7) and (21), while (18) and (19) follow as in the scalar case [18, pp. 283–284]. Finally, to prove (20) we start with the estimate

$$\left\| \int_{\partial D} \psi \, d\mu \right\|_{\mathcal{X}} \leqslant C \int_{\partial D} |\psi| \mathcal{M}\mu \, d\omega \tag{22}$$

valid for $\psi \in \mathrm{Lip}(\alpha)$, $\alpha > \gamma$. This inequality can be proved following [18, Theorem 3.25] whose proof is based on the construction of a *Lipschitz approximate identity* $\rho = \rho(P, Q, s)$ on ∂D, which by definition satisfies:

- $\rho \geqslant 0$,
- for fixed $s > 0$, $\operatorname{supp} \rho \subset \{(P, Q)\colon m(P, Q) < 2s\}$,
- for Q and s fixed, $\rho(\cdot, Q, s) \in \mathrm{Lip}(\alpha)$ and $s\rho(P, Q, s)$ is uniformly bounded,
- $\int_{\partial D} \rho(P, Q, s) \, d\omega(Q) = 1$.

For a given $f \in L^1_{\mathcal{X}}(\omega)$, let

$$\psi_s(P) = \int_{\partial D} \rho(P, Q, s) f(Q) \, d\omega(Q).$$

We claim that for every measurable set $E \subset \partial D$,

$$\big\|\mu(E)\big\|_{\mathcal{X}} \leqslant C \int_E \mathcal{M}\mu \, d\omega. \tag{23}$$

To see this, notice first that the weak $(1, 1)$ estimate for Hardy–Littlewood maximal function implies that almost every $P \in \partial D$ is a Lebesgue point for $f \in L^1_{\mathcal{X}}(\omega)$. For such point P we have that $\psi_s(P)$ converges to $f(P)$ as $s \to 0$. Applying this regularization to χ_E, and using (22) we conclude (23).

To prove (20) let $E \subset \Omega^c$. We have by (23) that

$$\big\|\mu(E)\big\| \leqslant ct\omega(E),$$

in other words,

$$\|\chi_{\Omega^c} \, d\mu\|_{V^\infty_{\mathcal{X}}} \leqslant ct.$$

This together with (21) implies (20). □

From (14) and Lemma 3.4, the next theorem completes the 'only if' part of Theorem 1.4.

Theorem 3.7. *Let $\mu \in M_{\mathcal{X}}(\partial D)$ such that $\mathcal{M}\mu \in L^1(\omega)$. Then there exists a sequence of atoms A_k, and a sequence $\{\lambda_k\}$ in ℓ^1 such that*

$$\mu = \sum_{k=1}^{\infty} \lambda_k A_k$$

and

$$C \sum_{k=1}^{\infty} |\lambda_k| \leqslant \|\mathcal{M}\mu\|_{L^1_{\mathcal{X}}(\omega)} \leqslant C^{-1} \sum_{k=1}^{\infty} |\lambda_k|$$

for a constant C independent of μ.

As in the scalar case, the proof of this theorem is based on the following lemma.

Lemma 3.8. *Let $\mu \in M_{\mathcal{X}}(\partial D)$ such that $\|\mu\|_{V^\infty_{\mathcal{X}}} \leqslant 1$ and $\mathcal{M}\mu \in L^q_{\mathcal{X}}(\omega)$ for some $(1+\gamma)^{-1} < q < 1$. Then there exists a sequence of atoms $A_k \in V^\infty_{\mathcal{X}}$ and a sequence $\{\lambda_k\}$ in ℓ^1 such that*

$$\mu = \sum_{k=1}^{\infty} \lambda_k A_k$$

and

$$C \sum_{k=1}^{\infty} |\lambda_k| \leqslant \int_{\partial D} (\mathcal{M}\mu)^q \, d\omega.$$

Proof. The argument goes parallel to that of [18, Lemma 4.2], so we just outline the proof. For fixed $\varepsilon > 0$ we construct inductively a (possibly finite) sequence $\{G_k\}$ of measures such that:

(i) $G_0 = \mu$,
(ii) G_k and B_k are respectively the "good part" and "bad part" in the Calderon–Zygmund decomposition described in Lemma 3.6, for the measure G_{k-1} at $t = \varepsilon^k$,
(iii) $\int_{\partial D} \mathcal{M}G_{k-1} \, d\omega < \varepsilon$.

If condition (iii) is violated, we stop the construction and obtain a finite sequence. For each permissible k, we define measures $\{b_{k,n}\}_n$, balls $\{\Delta^m_{r_{k,n}}(P_{k,n})\}_n$ and a partition of unity$\{\phi_{k,n}\}_n$ for $E_k = \{Q \in \partial D\colon \mathcal{M}G_{k-1}(Q) > \varepsilon^k\}$, according to Lemmas 3.5 and 3.6, so we have

$$G_{k-1} - G_k = \sum_n b_{k,n},$$
$$\|G_k\|_{V^\infty_{\mathcal{X}}} \leqslant c\varepsilon^k. \tag{24}$$

As the scalar case, one can prove that

$$\mathcal{M}G_k(P) \leqslant \mathcal{M}\mu(P) + c \sum_{i=1}^{k} \varepsilon^i \sum_n \left[\frac{r_{i,n}}{m(P, P_{i,n}) + r_{i,n}} \right]^{1+\gamma}. \tag{25}$$

If the sequence $\{G_k\}_k$ is infinite, (24) implies that we have the representation

$$\mu = \sum_{k=1}^{\infty} \sum_n b_{k,n},$$

with convergence in $V_{\mathcal{X}}^{\infty}$. We let $\lambda_{k,n} = 2C\varepsilon^{k-1}\omega(\Delta_{r_{k,n}}^m(P_{k,n}))$ and $A_{k,n} = (\lambda_{k,n})^{-1}b_{k,n}$. Considering the expression

$$db_{k,n} = \phi_{k,n}(dB_{k-1} - m_{k,n}\,d\omega),$$

where $m_{k,n} = (\int_{\partial D} \phi_{k,n}\,d\omega)^{-1} \int_{\partial D} \phi_{k,n}\,d\omega$, as in Lemma 3.6, it follows that $A_{k,n}$ is an atom with respect to the quasi-distance m.

Let $E_k = \{P \in \partial D\colon \mathcal{M}G_{k-1} > \varepsilon^k\}$. Then

$$\sum_n |\lambda_{k,n}| \leqslant c\varepsilon^{-1}M\varepsilon^k\omega(E_k).$$

By (25) and [18, Lemma 2.2]

$$\begin{aligned}
\varepsilon^{kq}\omega(E_k) &\leqslant \int_{\partial D} \mathcal{M}G_{k-1}^q\,d\omega \\
&\leqslant \int_{\partial D} \mathcal{M}\mu^q\,d\omega + c^q \sum_{i=1}^{k-1} \varepsilon^{iq} \sum_n \int_{\partial D} \left[\frac{r_{i,n}}{m(P, P_{i,n}) + r_{i,n}}\right]^{(1+\gamma)q} d\omega(p) \\
&\leqslant c\left(\int_{\partial D} \mathcal{M}\mu^q\,d\omega + \sum_{i=1}^{k-1} \varepsilon^{iq}\omega(E_i)\right).
\end{aligned}$$

Then by Gronwall's inequality we have

$$\varepsilon^{kq}\omega(E_k) \leqslant (c+2)^i \int_{\partial D} \mathcal{M}\mu^q\,d\omega.$$

Choosing ε such that $\varepsilon(c+2) < 1$ we have

$$\sum_{k,n} |\lambda_{k,n}| \leqslant C \int_{\partial D} \mathcal{M}\mu^q\,d\omega.$$

The case when the sequence of G_k is finite can be obtained in a similar way. □

Proof of Theorem 3.7. For every $k \in \mathbb{N}$, consider the Calderón–Zygmund decomposition

$$\mu = G_k + B_k$$

corresponding to $\Omega_k = \{P \in \partial D\colon \mathcal{M}\mu(P) > 2^k\}$ and with covering $\Delta_{r_n}^m(P_n)$ as in Lemma 3.5. Let $H_k = G_{k+1} - G_k = B_k - B_{k+1}$. We have

$$\|H_k\|_{V_{\mathcal{X}}^{\infty}} \leqslant C2^k,$$

hence

$$\|\mathcal{M}H_k\|_{L^{\infty}(\omega)} \leqslant C2^k,$$

moreover (see [18, p. 300]),

$$\mathcal{M}H_k(P) \leqslant C2^k \sum_{j=k}^{k+1} \sum_i \bigl[r_{i,j}/\bigl(r_{i.j} + m(P, P_{i,j})\bigr)\bigr]^{1+\gamma},$$

implying

$$\int_{\partial D} \mathcal{M}H_k{}^q \, d\omega \leqslant C2^{kq} \omega(\Omega_k),$$

for $(1+\gamma)^{-1} < q \leqslant 1$.

From the identity

$$\mu - \sum_{k=-m}^{m} H_k = B_{m+1} + G_{-m}$$

we see by (19) and (20) that $\mathcal{M}(\mu - \sum_{k=-m}^{m} H_k)$ converges to zero in $L^1(\omega)$. Then by (22) we have

$$\int_{\partial D} \psi \, d\mu = \sum_{k=-\infty}^{\infty} \int_{\partial D} \psi \, dH_k, \tag{26}$$

for all $\psi \in \mathrm{Lip}(\gamma)$.

Next, $\|C^{-1}2^{-k}H_k\|_{V_{\mathcal{X}}^{\infty}} \leqslant 1$ and for any $(1+\gamma)^{-1} < q < 1$,

$$\int_{\partial D} \mathcal{M}\bigl(C^{-1}2^{-k}H_k\bigr) \, d\omega \leqslant C\omega(\Omega_k).$$

From Lemma 3.8 we have the representation

$$C^{-1}2^{-k}H_k = \sum_i \lambda_{k,i} A_{k,i},$$

where $A_{k,i}$ is an atom and $\sum_i |\lambda_{k,i}| \leqslant C\omega(\Omega_k)$. If we let $\rho_{k,i} = C2^k\lambda_{k,i}$, the series $\mu' = \sum_k \sum_i \rho_{k,i} A_{k,i}$ converges in $M_{\mathcal{X}}(\partial D)$ since

$$\sum_k \sum_i \rho_{k,i} \sim \int_{\partial D} \mathcal{M}\mu \, d\omega.$$

Finally, from (26), it follows that $\int_{\partial D} \psi \, d\mu = \int_{\partial D} \psi \, d\mu'$, for all $\psi \in \mathrm{Lip}(\gamma)$. Since $\mathrm{Lip}(\gamma)$ is dense in $C(\partial D)$ we conclude that $\mu = \mu'$. □

Acknowledgments

This work started while the first author visited the second author at the University of Illinois, where he was a post-doctoral professor. Both authors are grateful for the hospitality and partial support from the Department of Mathematics of that institution during this visit. The first author was partially supported by PAPIIT-UNAM IN105603. The second author was partially supported by PROMEP-México.

References

[1] O. Blasco, Boundary values of vector-valued functions in vector-valued Hardy spaces and geometry of Banach spaces, J. Funct. Anal. 78 (1988) 346–364.
[2] O. Blasco, J. García Cuerva, Hardy classes of Banach-space-valued distributions, Math. Nachr. 132 (1987) 57–65.
[3] A.V. Bukhvalov, A.A. Danilevich, Boundary properties of analytic and harmonic functions with values in Banach spaces, Math. Notes 32 (1982) 104–110.
[4] L. Caffarelli, E. Fabes, S. Mortola, S. Salsa, Boundary behavior of non-negative solutions of elliptic operators in divergence form, Indiana Univ. Math. J. 30 (1981) 621–640.
[5] S.D. Chatterji, Martingale convergence and the Radon–Nikodým theorem in Banach spaces, Math. Scand. 22 (1968) 21–41.
[6] B.E.J. Dahlberg, On estimates of harmonic measure, Arch. Ration. Mech. Anal. 65 (1977) 272–288.
[7] B.E.J. Dahlberg, On the Poisson integral for Lipschitz and C^1 domains, Acta Math. 66 (1979) 13–24.
[8] J. Diestel, J.J. Uhl Jr., Vector Measures, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[9] N. Dinculeanu, Vector Measures, Pergamon Press, New York, 1967.
[10] J. García Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, Elsevier Science Publishers, Amsterdam, 1985.
[11] W. Hensgen, Some remarks on boundary values of vector-valued harmonic and analytic functions, Arch. Math. 57 (1991) 88–96.
[12] W. Hensgen, A simple proof of Singer's representation theorem, Proc. Amer. Math. Soc. 124 (1996) 3211–3212.
[13] R. Hunt, R. Wheeden, On the boundary values of harmonic functions, Trans. Amer. Math. Soc. 132 (1968) 307–322.
[14] R. Hunt, R. Wheeden, Positive harmonic functions on Lipschitz domains, Trans. Amer. Math. Soc. 147 (1970) 507–527.
[15] D. Jerison, C.E. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, Adv. Math. 146 (1982) 80–147.
[16] C.E. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, CBMS Regional Conf. Ser., vol. 83, Amer. Math. Soc., Providence, RI, 1994.
[17] R.A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type, Adv. Math. 33 (1979) 257–270.
[18] R.A. Macías, C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, Adv. Math. 33 (1979) 271–309.
[19] R.E. Megginson, An Introduction to Banach Space Theory, Grad. Texts in Math., Springer-Verlag, New York, 1998.
[20] A. Uchiyama, A maximal function characterization of H^p on the space of homogeneous type, Trans. Amer. Math. Soc. 262 (1980) 579–592.