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Applications of stratifying systems to the finitistic dimension

Eduardo do N. Marcos[a], Octavio Mendoza[b,*], Corina Sáenz[c]

[a] *Departamento de Matemática, Universidade de São Paulo, Caixa Postal 66.281, São Paulo, SP 05315-970, Brasil*
[b] *Instituto de Matemáticas, U.N.A.M., Circuito Exterior, Ciudad Universitaria, C.P. 04510, Mexico, D.F. México*
[c] *Departamento de Matemáticas, Facultad de Ciencias, U.N.A.M., Circuito Exterior, Ciudad Universitaria, C.P. 04510, México, D.F. Mexico*

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Abstract

Given an Ext-injective stratifying system of Λ-modules $(\theta, \underline{Y}, \preccurlyeq)$ satisfying that the projective dimension of Y is finite, we prove that the finitistic dimension of the algebra Λ is equal to the finitistic dimension of the category $\mathscr{I}(\theta) = \{X \in \operatorname{mod} \Lambda : \operatorname{Ext}^1_\Lambda(-, X)|_{\mathscr{F}(\theta)} = 0\}$. Moreover, using the theory of stratifying systems we obtain bounds for the finitistic dimension of Λ. In particular, we get the optimal bound $2n - 2$ for the finitistic dimension of a standardly stratified algebra with n simples.

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0. Introduction

In this paper algebra means finite dimensional basic algebra over an algebraically closed field k, mod Λ denotes the category of finitely generated left Λ-modules over an algebra Λ,

* Corresponding author.
E-mail addresses: enmarcos@ime.usp.br (E. do N. Marcos), omendoza@matem.unam.mx (O. Mendoza), ecsv@lya.fciencias.unam.mx (C. Sáenz).

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and $D : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{op}$ is the usual duality $\operatorname{Hom}_k(-, k)$. All subcategories considered will be full subcategories. Given morphisms $f : M \to N$ and $g : N \to L$ in $\operatorname{mod} \Lambda$ we denote the composition of f and g by gf which is a morphism from M to L.

Given a class $\mathscr{C}$ of Λ-modules, we denote by $\mathscr{F}(\mathscr{C})$ the subcategory of $\operatorname{mod} \Lambda$ whose objects are the zero module and all modules which are filtered by modules in $\mathscr{C}$. That is, a non-zero Λ-module M belongs to $\mathscr{F}(\mathscr{C})$ if there is a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in $\mathscr{C}$ for all $i=1, 2, \ldots, m$. In particular, if $\mathscr{C}=\emptyset$ then $\mathscr{F}(\mathscr{C})=\{0\}$. It is easy to see that $\mathscr{F}(\mathscr{C})$ is closed under extensions. In general, $\mathscr{F}(\mathscr{C})$ fails to be closed under direct summands, see “the remarks concerning the definition of $\mathscr{F}(\theta)$ and $\mathscr{X}(0)$” in [13, p. 210]. We recall the following notation used in [11]:

$$\mathscr{I}(\mathscr{C}) = \{X \in \operatorname{mod} \Lambda : \operatorname{Ext}^1_\Lambda(-, X)|_{\mathscr{F}(\mathscr{C})} = 0\},$$

$$\mathscr{P}(\mathscr{C}) = \{X \in \operatorname{mod} \Lambda : \operatorname{Ext}^1_\Lambda(X, -)|_{\mathscr{F}(\mathscr{C})} = 0\}.$$

Let Λ be a quasi-hereditary algebra. In the fundamental paper [13] Ringel investigated homological properties of the category of the good-modules $\mathscr{F}({}_\Lambda\Delta)$, that is the category of those modules that are filtered by the standard modules, and of the dually defined category of the cogood-modules $\mathscr{F}({}_\Lambda\nabla)$, consisting of modules that are filtered by the so-called costandard modules. Moreover, he constructed the characteristic tilting module T associated to the quasi-hereditary algebra Λ and showed that it is also cotilting. He also proved that the Ringel dual of Λ is again quasi-hereditary. Later on, there appeared several papers studying the subcategories of the good-modules and of the cogood-modules of a standardly stratified algebra, among which we mention the paper “Standardly Stratified Algebras and Tilting” by Ágoston et al., see [2]. Under this context, they showed that there is always a characteristic tilting module T such that the endomorphism algebra $\operatorname{End}_\Lambda(T)$ is standardly stratified. As an application of their results, they got that the projective finitistic dimension of a standardly stratified algebra is bounded by $2n - 1$, where n is the number of non-isomorphic simple Λ-modules. Later on, they even got the bound $2n - 2$ in [3] using different methods. Note that the bound $2n-2$ is optimal, since in [7] Dlab and Ringel showed that the best possible bound for the global dimension of a quasi-hereditary algebra is $2n - 2$.

The aim of this paper is to investigate the results in [2] from the point of view of stratifying systems. Let Λ be an algebra and $(\theta, \preccurlyeq)$ a stratifying system of size t. Associated to $(\theta, \preccurlyeq)$ there exists a uniquely determined Ext-injective stratifying system $(\theta, \underline{Y}, \preccurlyeq)$ and also a uniquely determined Ext-projective stratifying system $(\theta, \underline{Q}, \preccurlyeq)$, where the set $\underline{Y} = \{Y(1), \ldots, Y(t)\}$ (resp. $\underline{Q} = \{Q(1), \ldots, Q(t)\}$) consists of pairwise non-isomorphic indecomposable Λ-modules. Moreover, it is known that $\mathscr{F}(\theta) \cap \mathscr{I}(\theta) = \operatorname{add} Y$ and $\mathscr{F}(\theta) \cap \mathscr{P}(\theta) = \operatorname{add} Q$, where $Y = \coprod_{i=1}^t Y(i)$ and $Q = \coprod_{i=1}^t Q(i)$, see [10,11]. One of our main results is Theorem 2.6, which states that $\operatorname{pdf} \Lambda = \operatorname{pfd} \mathscr{I}(\theta) \leqslant \sup\{\operatorname{pfd} \mathscr{P}(\theta), \operatorname{pd} Y + 1\}$, if $\operatorname{pd} Y < \infty$. This result relates the projective finitistic dimension of the algebra Λ with the projective dimension of the Ext-injective Λ-module Y associated to the stratifying system

$(\theta, \preccurlyeq)$ and the projective finitistic dimension of the category $\mathscr{P}(\theta)$. We also state the dual result, that is, if id $Q < \infty$ then ifd $\Lambda =$ ifd $\mathscr{P}(\theta) \leqslant \sup\{\text{ifd}\,\mathscr{I}(\theta), \text{id}\, Q + 1\}$.

Furthermore, given a stratifying system $(\theta, \preccurlyeq)$ and a generalized tilting Λ-module ${}_\Lambda T$, we give necessary and sufficient conditions for the equality $\mathscr{I}(\theta) = T^{\perp}$, where $T^{\perp}$ is the category whose objects are the Λ-modules X satisfying $\text{Ext}^i_\Lambda(T, X) = 0$ for all $i > 0$, see Theorem 3.6.

Finally, by applying ours results, we get some of the results in [2] and [3]. In particular, we get the optimal bound $2n-2$ for the finitistic dimension of a standardly stratified algebra Λ, see Theorem 3.3. We also prove that for a standardly stratified algebra Λ, the projective dimension of the characteristic tilting Λ-module T is equal to the projective dimension of the category of the good-modules and is bounded by $n - 1$.

For the historical background of the finitistic dimension conjecture we refer to [16].

1. Preliminaries

Throughout the paper we denote by $[1, t]$ the set $\{1, 2, \ldots, t\}$ and by $\preccurlyeq$ a total order on $[1, t]$. We reserve the notation $\leqslant$ (resp. $\leqslant^{op}$) for the natural (resp. opposite natural) total order on $[1, t]$. It is known that there is a unique isomorphism of ordered sets $\omega_t : ([1, t], \preccurlyeq) \to ([1, t], \leqslant)$. We will also make use of the isomorphism of ordered sets $\sigma_t : ([1, t], \leqslant) \to ([1, t], \leqslant^{op})$ given by $\sigma_t(i) = t - i + 1$.

Let R be an algebra. We start this section by recalling the definition of stratifying system, Ext-injective stratifying system and Ext-projective stratifying system given in [10,11]. Then we recall the notions of standard, proper standard, costandard and proper costandard R-modules and also the definition of standardly stratified algebra and of quasi-hereditary algebra. Finally we state results from [9] and [11] which will be used in the following sections.

Definition 1.1 (*Marcos et al. [10]*). A stratifying system $(\theta, \preccurlyeq)$ of size t consists of a set $\theta = \{\theta(i)\}_{i=1}^{t}$ of indecomposable R-modules and a total order $\preccurlyeq$ on $[1, t]$ satisfying the following conditions:

(a) $\text{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(b) $\text{Ext}^1_R(\theta(j), \theta(i)) = 0$ for $j \succeq i$.

In the theory of stratifying systems there are three equivalent notions. One of them is the notion of stratifying system given in Definition 1.1. The second one, which is the original one, is called Ext-injective stratifying system (eiss), see [9], where it appears under the name of stratifying system, and finally there is the notion of Ext-projective stratifying system (epss), see [10,11].

The equivalence of these notions implies, in particular, that given a stratifying system $(\theta, \preccurlyeq)$ of size t, we can associate to it a uniquely determined Ext-injective stratifying system $(\theta, \underline{Y}, \preccurlyeq)$ and a uniquely determined Ext-projective stratifying system $(\theta, \underline{Q}, \preccurlyeq)$, where the set $\underline{Y}=\{Y(1), \ldots, Y(t)\}$ (resp. $\underline{Q}=\{Q(1), \ldots, Q(t)\}$) consists of pairwise non-isomorphic indecomposable R-modules. We recall now the definitions of eiss and epss.

Definition 1.2 (*Erdmann and Sáenz [9]*). Let $\theta=\{\theta(i)\}_{i=1}^t$ be a set of non-zero R-modules, $\underline{Y}=\{Y(i)\}_{i=1}^t$ be a set of indecomposable R-modules and $\preccurlyeq$ be a total order on the set $[1,t]$. The system $(\theta, \underline{Y}, \preccurlyeq)$ is an eiss of size t, if the following three conditions hold:

(a) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(b) for each $i \in [1,t]$, there is an exact sequence $0 \to \theta(i) \xrightarrow{\alpha_i} Y(i) \to Z(i) \to 0$ such that $Z(i) \in \mathscr{F}(\{\theta(j) : j \prec i\})$,
(c) $\mathrm{Ext}^1_R(-, Y)|_{\mathscr{F}(\theta)} = 0$.

Definition 1.3 (*Marcos et al. [11]*). Let $\theta = \{\theta(i)\}_{i=1}^t$ be a set of non-zero R-modules, $\underline{Q} = \{Q(i)\}_{i=1}^t$ be a set of indecomposable R-modules and $\preccurlyeq$ be a total order on $[1,t]$. The system $(\theta, \underline{Q}, \preccurlyeq)$ is an epss of size t, if the following three conditions hold:

(a) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(b) for each $i \in [1,t]$, there is an exact sequence $0 \to K(i) \to Q(i) \xrightarrow{\beta_i} \theta(i) \to 0$ such that $K(i) \in \mathscr{F}(\{\theta(j) : j \succ i\})$,
(c) $\mathrm{Ext}^1_R(Q, -)|_{\mathscr{F}(\theta)} = 0$.

Given a stratifying system $(\theta, \preccurlyeq)$ of size t, we say that it is *standard* if ${}_RR \in \mathscr{F}(\theta)$, and we say that it is *costandard* if $D(R_R) \in \mathscr{F}(\theta)$. Moreover, if $M \in \mathscr{F}(\theta)$, the filtration multiplicities $[M : \theta(i)]$ do not depend on the filtration of M, see [9, Lemma 1.4]. The θ-support of M is the set $\mathrm{Supp}_\theta(M) = \{i \in [1,t] : [M : \theta(i)] \neq 0\}$. It is clear that $M = 0$ if and only if $\mathrm{Supp}_\theta(M)$ is empty. So, if $M \neq 0$ we define $\min(M) := \min(\mathrm{Supp}_\theta(M), \preccurlyeq)$ and $\max(M) := \max(\mathrm{Supp}_\theta(M), \preccurlyeq)$. For $M=0$ we define $\min(0) := +\infty$ and $\max(0) := -\infty$.

Let R be an algebra and $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be a complete set of primitive orthogonal idempotents, where we fix the natural order on the set $[1,s] = \{1, \ldots, s\}$ of indexes. For $1 \leqslant i \leqslant s$ let $P(i) = R\varepsilon_i$ be the indecomposable projective R-module and let $S(i)$ be the simple top of $P(i)$. The *standard* module ${}_R\Delta(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $S(j)$ for $j > i$. We will also denote by ${}_R\overline{\Delta}(i)$ the *proper standard* module, which is the maximal factor module of ${}_R\Delta(i)$ such that the multiplicity condition

$$[{}_R\overline{\Delta}(i) : S(i)] = \dim_k \mathrm{Hom}_R(P(i), {}_R\overline{\Delta}(i)) = 1$$

holds. We define dually by ${}_R\nabla(i)$ the *costandard* modules and by ${}_R\overline{\nabla}(i)$ the *proper costandard* modules. Thus, ${}_R\nabla(i)$ is the maximal submodule of the injective envelope $I(i)$ of $S(i)$ without composition factors $S(j)$ for $j > i$, while ${}_R\overline{\nabla}(i)$ is the maximal submodule of ${}_R\nabla(i)$ that satisfies the multiplicity condition

$$[{}_R\overline{\nabla}(i) : S(i)] = \dim_k \mathrm{Hom}_R({}_R\overline{\nabla}(i), I(i)) = 1.$$

We use the notation ${}_R\Delta = \{{}_R\Delta(i)\}_{i\in[1,s]}$, and we define the sets ${}_R\overline{\Delta}$, ${}_R\nabla$ and ${}_R\overline{\nabla}$ similarly. We recall that $({}_R\Delta, \leqslant)$ is always a stratifying system (the *canonical* one) of size s, where s is the number of iso-classes of simple modules. Moreover, for each $i \in [1,s]$ we obtain a stratifying system $({}_R\Delta_{\leqslant i}, \leqslant)$ of size i, where ${}_R\Delta_{\leqslant i} = \{{}_R\Delta(1), \ldots, {}_R\Delta(i)\}$, see [10]. Similarly, we have the *co-canonical* stratifying system $({}_R\nabla, \leqslant^{op})$ of size s, where ${}_R\nabla = \{{}_R\nabla(i)\}_{i\in[1,s]}$.

The algebra R is called *standardly stratified* if and only if ${}_R R \in \mathscr{F}({}_R\Delta)$. A standardly stratified algebra is *quasi-hereditary* if ${}_R\Delta(i) = {}_R\overline{\Delta}(i)$ for all $1 \leqslant i \leqslant s$.

The following example shows that the size of a stratifying system can be larger than the number of iso-classes of simple modules.

Example 1.4. Consider the quotient path algebra $R = kQ/I$, where Q is the following quiver:

$$3 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2 \xleftarrow{\gamma} 4,$$

and I is the ideal generated by $\beta\gamma$. Taking $\theta(1) = S(1) = {}_R\Delta(1) = P(1)$, $\theta(2) = {}_R\Delta(2) = P(2)$, $\theta(3) = {}_R\Delta(3) = P(3)$, $\theta(4) = {}_R\Delta(4) = P(4)$ and $\theta(5) = S(4)$, we get that the stratifying system $(\theta, \leqslant)$ is standard of size 5, whereas the canonical stratifying system $({}_R\Delta, \leqslant)$ is standard of size 4.

The following statement implies that the category $\mathscr{F}(\theta)$ has enough Ext-injective and Ext-projective objects.

Lemma 1.5 (*Erdmann and Sáenz [9] and Marcos et al. [11]*). *Let $(\theta, \preccurlyeq)$ be a stratifying system of size t and $0 \neq M \in \mathscr{F}(\theta)$. Then*

(a) *there is an exact sequence* $0 \to M \to Y_0 \to M' \to 0$ *with* $M' \in \mathscr{F}(\theta)$, $Y_0 \in \operatorname{add} Y$ *and* $\max(M') \prec \max(M)$,
(b) *there is an exact sequence* $0 \to N' \to Q_0 \to M \to 0$ *with* $N' \in \mathscr{F}(\theta)$, $Q_0 \in \operatorname{add} Q$ *and* $\min(M) \prec \min(N')$.

Let R be an algebra and $X \in \operatorname{mod} R$. Associated to X we shall consider the following subcategories of $\operatorname{mod} R$: the right (resp. left) perpendicular category $X^{\perp}$ (resp. ${}^{\perp}X$) with objects X' satisfying $\operatorname{Ext}^i_R(X, X') = 0$ (resp. $\operatorname{Ext}^i_R(X', X) = 0$) for all $i > 0$, and $\operatorname{fac} X$ whose objects are those modules X' which are epimorphic images of modules in $\operatorname{add} X$.

Finally, if $\mathscr{X}$ is a class of R-modules, we denote by $\mathscr{X}^{\wedge}$ the subcategory of $\operatorname{mod} R$ whose objects are those R-modules X for which there exists a finite $\mathscr{X}$-resolution, that is, a long exact sequence $0 \to X_u \to \cdots \to X_1 \to X_0 \to X \to 0$ with $X_i \in \mathscr{X}$ for all $0 \leqslant i \leqslant u$. Dually, $\mathscr{X}^{\vee}$ is the subcategory of R-modules which have a finite $\mathscr{X}$-coresolution.

2. Finitistic dimension and stratifying systems

Let R be an algebra. For a given $X \in \operatorname{mod} R$ we denote by $\operatorname{pd} X$ the projective dimension of X and by $\operatorname{id} X$ the injective dimension of X.

Given a subcategory $\mathscr{C}$ of $\operatorname{mod} R$, we denote by $\operatorname{pd} \mathscr{C}$ the projective dimension of $\mathscr{C}$, that is, $\operatorname{pd} \mathscr{C} = \sup\{\operatorname{pd} X : X \in \mathscr{C}\}$. Dually, $\operatorname{id} \mathscr{C} = \sup\{\operatorname{id} X : X \in \mathscr{C}\}$ is the injective dimension of $\mathscr{C}$. We also consider the subcategories, $\mathscr{P}^{<\infty}(\mathscr{C}) = \{X \in \mathscr{C} : \operatorname{pd} X < \infty\}$ and $\mathscr{I}^{<\infty}(\mathscr{C}) = \{X \in \mathscr{C} : \operatorname{id} X < \infty\}$. The *projective finitistic dimension* of the category $\mathscr{C}$, denoted by $\operatorname{pfd} \mathscr{C}$, is equal to $\operatorname{pd} \mathscr{P}^{<\infty}(\mathscr{C})$. Dually, $\operatorname{ifd} \mathscr{C} = \operatorname{id} \mathscr{I}^{<\infty}(\mathscr{C})$ is the *injective finitistic dimension* of $\mathscr{C}$. We abuse notation and use $\operatorname{ifd} R$ and $\operatorname{pfd} R$ for the $\operatorname{ifd}(\operatorname{mod} R)$

and pfd (mod R), respectively, and also we shall write $\mathscr{P}^{<\infty}(R)$ (resp. $\mathscr{I}^{<\infty}(R)$) instead of $\mathscr{P}^{<\infty}(\text{mod } R)$ (resp. $\mathscr{I}^{<\infty}(\text{mod } R)$). Recall that gl dim R, the *global dimension* of R, is equal to the projective dimension of mod R and also is equal to the injective dimension of mod R.

Let $(\theta, \underline{Y}, \preccurlyeq)$ be an eiss. In this section we prove that the projective finitistic dimension of mod R is bounded by $\sup\{\text{pfd } \mathscr{P}(\theta), \text{pd } Y + 1\}$, and moreover that pdf R is equal to pfd $\mathscr{I}(\theta)$ if pd Y is finite.

Lemma 2.1. *Let $(\theta, \preccurlyeq)$ be a stratifying system of R-modules of size t. Then*

(a) *the system $(D(\theta), \preccurlyeq^{op})$ is a stratifying system of R^{op}-modules of size t,*
(b) *the category $D(\mathscr{F}(\theta))$ is equal to $\mathscr{F}(D(\theta))$,*
(c) *$D(\mathscr{P}(\theta)) = \mathscr{I}(D(\theta))$ and $D(\mathscr{I}(\theta)) = \mathscr{P}(D(\theta))$.*

Proof. The proof of (a) is straightforward, and (c) follows from (b). It remains to prove (b) and for this, it is enough to see that $D(\mathscr{F}(\theta)) \subseteq \mathscr{F}(D(\theta))$.

Let $0 \neq M \in \mathscr{F}(\theta)$, we proceed by reverse induction on min M.

Let $t_1 := \max([1, t], \preccurlyeq)$. If $\min(M) = t_1$ then $M \in \text{add } \theta(t_1)$ and so $D(M)$ belongs to $\mathscr{F}(D(\theta))$.

Assume that $i := \min(M) \prec t_1$. Then by Proposition 2.9 in [11] we have an exact sequence

$$0 \to N \to M \to \theta(i)^{m_i} \to 0 \quad \text{with } \min(M) \prec \min(N). \tag{1}$$

Applying D to (1), using induction and the fact that $\mathscr{F}(D(\theta))$ is closed under extensions, we get that $D(M) \in \mathscr{F}(D(\theta))$, proving the result. □

Proposition 2.2. *Let $(\theta, \preccurlyeq)$ be a stratifying system of R-modules of size t and let $(\theta, \underline{Y}, \preccurlyeq)$ and $(\theta, \underline{Q}, \preccurlyeq)$ be the eiss and the epss associated, respectively, to $(\theta, \preccurlyeq)$. Then $(D(\theta), \preccurlyeq^{op})$ is a stratifying system of R^{op}-modules of size t, $(D(\theta), D(\underline{Q}), \preccurlyeq^{op})$ and $(D(\theta), D(\underline{Y}), \preccurlyeq)$ are the eiss and the epss associated, respectively, to $(D(\theta), \preccurlyeq^{op})$.*

Proof. The proof follows from the previous lemma. □

Proposition 2.3. *Let $(\theta, \preccurlyeq)$ be a stratifying system of size t and $M \in \mathscr{F}(\theta)$. Then*

(a) $\text{pd } M \leqslant \text{pd } Y \leqslant \text{pd } Q + t - 1$,
(b) $\text{id } M \leqslant \text{id } Q \leqslant \text{id } Y + t - 1$.

Proof. We prove (a), since the proof of (b) is dual. We may assume that $M \neq 0$. Let $t_0 := \min([1, t], \preccurlyeq)$ and $t_1 := \max([1, t], \preccurlyeq)$.

We start by proving that $\text{pd } M \leqslant \text{pd } Y$. In order to do that we proceed by induction on $\max(M)$. If $\max(M) = t_0$, then $M \simeq Y(t_0)^m$ and so $\text{pd } M = \text{pd } Y(t_0) \leqslant \text{pd } Y$. Let $\max(M) \succ t_0$ and assume that $\text{pd } M' \leqslant \text{pd } Y$ for all $M' \in \mathscr{F}(\theta)$ with $\max(M') \prec \max(M)$. By 1.5(a) we have an exact sequence

$$0 \to M \to Y_0 \to M' \to 0 \tag{2}$$

with $M' \in \mathscr{F}(\theta)$, $Y_0 \in \operatorname{add} Y$ and $\max(M') \prec \max(M)$. From (2) we get that $\operatorname{pd} M \leqslant \sup\{\operatorname{pd} Y_0, \operatorname{pd} M' - 1\} \leqslant \sup\{\operatorname{pd} Y, \operatorname{pd} M'\} \leqslant \operatorname{pd} Y$.

We prove now that $\operatorname{pd} Y \leqslant \operatorname{pd} Q + t - 1$. Using 1.5(b), we get the exact sequence

$$0 \to M_1 \to Q_1 \to Y \to 0 \tag{3}$$

with $M_1 \in \mathscr{F}(\theta)$, $Q_1 \in \operatorname{add} Q$ and $t_0 \prec \min(M_1)$. So $\operatorname{pd} Y \leqslant \sup\{\operatorname{pd} Q, \operatorname{pd} M_1 + 1\}$. If $M_1 \neq 0$ (see (3)) then using 1.5 (b), we get the exact sequence $0 \to M_2 \to Q_2 \to M_1 \to 0$ with $M_2 \in \mathscr{F}(\theta)$, $Q_2 \in \operatorname{add} Q$ and $\min(M_1) \prec \min(M_2)$. So $\operatorname{pd} M_1 \leqslant \sup\{\operatorname{pd} Q, \operatorname{pd} M_2 + 1\}$. Therefore, $\operatorname{pd} Y \leqslant \sup\{\operatorname{pd} Q, \operatorname{pd} M_1 + 1\} \leqslant \sup\{\operatorname{pd} Q, \operatorname{pd} M_2 + 2\}$. Iterating this process at most $t - 1$ steps we get that $\min(M_{t-1}) = t_1$ and hence $M_{t-1} \simeq Q(t_1)^m$. Then $\operatorname{pd} Y \leqslant \sup\{\operatorname{pd} Q, \operatorname{pd} M_{t-1} + t - 1\} \leqslant \operatorname{pd} Q + t - 1$. □

Corollary 2.4. *Let $(\theta, \preccurlyeq)$ be a stratifying system of size t. Then*

(a) $\operatorname{pd} Y = \operatorname{pd} \mathscr{F}(\theta) = \operatorname{pd} \theta \leqslant \operatorname{pd} Q + t - 1$,
(b) $\operatorname{id} Q = \operatorname{id} \mathscr{F}(\theta) = \operatorname{id} \theta \leqslant \operatorname{id} Y + t - 1$.
(c) $\mathscr{F}(\theta) \subseteq \mathscr{P}^{<\infty}(R)$ *if and only if* $\operatorname{pd} Y < \infty$ *if and only if* $\operatorname{pd} Q < \infty$ *if and only if* $\operatorname{pd} \theta < \infty$,
(d) $\mathscr{F}(\theta) \subseteq \mathscr{I}^{<\infty}(R)$ *if and only if* $\operatorname{id} Y < \infty$ *if and only if* $\operatorname{id} Q < \infty$ *if and only if* $\operatorname{id} \theta < \infty$.

Proof. It follows from 2.3. □

The item (a) of the following corollary is also in [2]. However, the proof given here is different from that given in [2].

Corollary 2.5. *Let $({}_R\Delta, \leqslant)$ be standard of size s and ${}_RT$ be the characteristic tilting R-module associated to $({}_R\Delta, \leqslant)$. Then*:

(a) $\operatorname{pd} {}_RT = \operatorname{pd} \mathscr{F}({}_R\Delta) \leqslant s - 1$,
(b) $\operatorname{id} {}_RR = \operatorname{id} \mathscr{F}({}_R\Delta) \leqslant \operatorname{id} {}_RT + s - 1$.

Proof. Let $({}_R\Delta, \underline{Y}, \leqslant)$ and $({}_R\Delta, \underline{Q}, \leqslant)$ be, respectively, the eiss and the epss associated to $({}_R\Delta, \leqslant)$. Then by [10, Propositions 1.4 and 2.1] we have that $Y \simeq {}_RT$. On the other hand, using that R is basic and Corollary 2.16 in [11] we obtain that $Q \simeq {}_RR$. Hence the result follows now from 2.4. □

Theorem 2.6. *Let $(\theta, \preccurlyeq)$ be a stratifying system.*

(a) *If* $\operatorname{pd} Y < \infty$ *then* $\operatorname{pfd} R = \operatorname{pfd} \mathscr{I}(\theta) \leqslant \sup\{\operatorname{pfd} \mathscr{P}(\theta), \operatorname{pd} Y + 1\}$.
(b) *If* $\operatorname{id} Q < \infty$ *then* $\operatorname{ifd} R = \operatorname{ifd} \mathscr{P}(\theta) \leqslant \sup\{\operatorname{ifd} \mathscr{I}(\theta), \operatorname{id} Q + 1\}$.

Proof. We will only prove (a), since the proof of (b) is dual. Assume that $\operatorname{pd} Y < \infty$ and let X be an R-module. By [13, Lemma 4′] there is an exact sequence

$$0 \to X \to Y_X \to M_X \to 0 \quad \text{with } Y_X \in \mathscr{I}(\theta) \text{ and } M_X \in \mathscr{F}(\theta). \tag{4}$$

Suppose that $\operatorname{pd} X < \infty$. Then from (4) we get $\operatorname{pd} Y_X \leqslant \infty$ because by 2.4 we know that $\operatorname{pd} \mathscr{F}(\theta) < \infty$. Hence $\operatorname{pd} Y_X \leqslant \operatorname{pfd} \mathscr{I}(\theta)$. Therefore $\operatorname{pd} X \leqslant \sup\{\operatorname{pd} Y_X, \operatorname{pd} M_X - 1\} \leqslant \operatorname{pfd} \mathscr{I}(\theta)$, since $\operatorname{pd} M_X \leqslant \operatorname{pd} Y$ (see 2.4) and $Y \in \mathscr{I}(\theta)$. So $\operatorname{pfd} R \leqslant \operatorname{pfd} \mathscr{I}(\theta)$ and because of the fact that $\mathscr{I}(\theta) \subseteq \operatorname{mod} R$ we get $\operatorname{pfd} R = \operatorname{pfd} \mathscr{I}(\theta)$.

We prove now that $\operatorname{pfd} R \leqslant \sup\{\operatorname{pfd} \mathscr{P}(\theta), \operatorname{pd} Y + 1\}$. Using the dual version of Lemma $4'$ in [13] there is an exact sequence

$$0 \to N_X \to Q_X \to X \to 0 \quad \text{with } Q_X \in \mathscr{P}(\theta) \text{ and } N_X \in \mathscr{F}(\theta). \tag{5}$$

Assume that $\operatorname{pd} X < \infty$. Then we get from (5) that $\operatorname{pd} Q_X < \infty$. Therefore, $\operatorname{pd} X \leqslant \sup\{\operatorname{pd} Q_X, \operatorname{pd} N_X + 1\} \leqslant \sup\{\operatorname{pfd} \mathscr{P}(\theta), \operatorname{pd} Y + 1\}$, since by 2.4 we have that $\operatorname{pd} N_X \leqslant \operatorname{pd} Y$. □

Corollary 2.7. *Let $(\theta, \preccurlyeq)$ be a stratifying system. If* $\operatorname{pd} Y < \infty$ *then the finiteness of one of the following dimensions* $\operatorname{pfd} R$, $\operatorname{pfd} \mathscr{I}(\theta)$, *and* $\operatorname{pfd} \mathscr{P}(\theta)$ *implies the finiteness of all of them. A dual result holds in case that* $\operatorname{id} Q < \infty$.

Proof. It follows from 2.6. □

Let T be an R-module. We recall that T is said to be *self-orthogonal* if $\operatorname{Ext}^1_R(T, T) = 0$. An indecomposable self-orthogonal R-module T is said to be a *stone*. Given a self-orthogonal module T, we consider the following subcategories of $\operatorname{mod} R$:

$$\mathscr{Y}(T) = \{X : \operatorname{Ext}^1_R(T, X) = 0\} \quad \text{and} \quad \mathscr{X}(T) = \{X : \operatorname{Ext}^1_R(X, T) = 0\}.$$

Corollary 2.8. *Let T be a stone in* $\operatorname{mod} R$.

(a) *If* $\operatorname{pd} T < \infty$ *then* $\operatorname{pfd} R = \operatorname{pfd} \mathscr{Y}(T) \leqslant \sup\{\operatorname{pfd} \mathscr{X}(T), 1 + \operatorname{pd} T\}$.
(b) *If* $\operatorname{id} T < \infty$ *then* $\operatorname{ifd} R = \operatorname{ifd} \mathscr{X}(T) \leqslant \sup\{\operatorname{ifd} \mathscr{Y}(T), 1 + \operatorname{id} T\}$.

Proof. Let $\theta := \{T\}$. Then $(\theta, \leqslant)$ is a stratifying system of size 1 with $\underline{Y} = \underline{Q} = \{T\}$. Hence $\mathscr{F}(\theta) = \operatorname{add} T$, $\mathscr{I}(\theta) = \mathscr{Y}(T)$ and $\mathscr{P}(\theta) = \mathscr{X}(T)$. Therefore the result follows from 2.6. □

3. Applications

We give some applications of the previous theorem and its corollary by linking tilting theory and finitistic projective dimension. In order to do that, we start by recalling the definition of a generalized tilting module and defining some numerical invariants associated to a generalized tilting module.

Let R be an algebra, we say that T is a *generalized tilting* R-module if the following three conditions hold: (a) T has finite projective dimension, (b) $\operatorname{Ext}^i_R(T, T) = 0$ for all $i > 0$, and (c) there exists an exact sequence

$0 \to {}_R R \to T_0 \to T_1 \to \cdots \to T_m \to 0$ with $T_j \in \operatorname{add} T$ for all j, where $\operatorname{add} T$ is the full subcategory of $\operatorname{mod} R$ whose objects are direct sums of direct summands of T.

Let T be a generalized tilting R-module and X be an R-module. We define $\alpha_T(X) := -\infty$ if $X = 0$, $\alpha_T(X) := +\infty$ if $X \notin (\operatorname{add} T)^\wedge$, and $\alpha_T(X) := \min\{r :$ there is an exact sequence $0 \to T_r \to \cdots \to T_0 \to X \to 0$, with $T_i \in \operatorname{add} T\}$ if $X \in (\operatorname{add} T)^\wedge$. Finally, we define $\alpha_T := \sup\{\alpha_T(X) : X \in \mathscr{P}^{<\infty}(T^\perp)\}$. We recall that $T^\perp$ is a subcategory of fac T, see [2].

Lemma 3.1. *Let $_RT$ be a generalized tilting R-module. Then*

(a) *for any $M \in T^\perp$ there exists an exact sequence $0 \to K \to T_0 \xrightarrow{f} M \to 0$ such that $T_0 \in \operatorname{add} T$, $K \in T^\perp$ and $f : T_0 \to M$ is the right minimal add T-approximation of M,*
(b) $\alpha_T(X) \leqslant \operatorname{pd} X$ *for any $X \in \mathscr{P}^{<\infty}(T^\perp)$,*
(c) $\operatorname{pd} X \leqslant \operatorname{pd} T + \alpha_T(X)$ *for any $X \in (\operatorname{add} T)^\wedge$,*
(d) *let $X \in T^\perp$. Then* $\operatorname{pd} X < \infty$ *if and only if* $\alpha_T(X) < \infty$,
(e) $\mathscr{P}^{<\infty}(T^\perp) = (\operatorname{add} T)^\wedge$.

Proof.

(a) Let $M \in T^\perp$. Using that add T is a functorially finite subcategory of mod R (see [5]) we get a right minimal add T-approximation $f : T_0 \to M$ of M. Since $T^\perp$ is contained in fac T we have that $\operatorname{Im} f = M$. Also, by Wakamatsu's Lemma (see [14]) we obtain that $\operatorname{Ext}^1_R(T, \operatorname{Ker} f) = 0$.
Consider the exact sequence $0 \to K = \operatorname{Ker} f \to T_0 \to M \to 0$. Applying the functor $\operatorname{Hom}_R(T, -)$ to it, we get an exact sequence $\operatorname{Ext}^i_R(T, M) \to \operatorname{Ext}^{i+1}_R(T, K) \to \operatorname{Ext}^{i+1}_R(T, T_0)$ for any $i \geqslant 1$, proving that $K \in T^\perp$.
(b) Let $X \in T^\perp$ and $r = \operatorname{pd} X < \infty$. Using inductively the previous item for $m = -1, 0, 1, \ldots, r$ we get the exact sequences

$$\varepsilon_m : 0 \to K_{m+1} \to T_{m+1} \to K_m \to 0,$$

where $K_{-1} = X$. Applying the functor $\operatorname{Hom}_R(-, K_{r+1})$ to the exact sequence ε_{r-i} : $0 \to K_{r-i+1} \to T_{r-i+1} \to K_{r-i} \to 0$ and by setting $r' := r - i + 1$, we get the exact sequence
$\operatorname{Ext}^i_R(T_{r'}, K_{r+1}) \to \operatorname{Ext}^i_R(K_{r'}, K_{r+1}) \to \operatorname{Ext}^{i+1}_R(K_{r-i}, K_{r+1}) \to \operatorname{Ext}^{i+1}_R(T_{r'}, K_{r+1})$.
Using that $\operatorname{Ext}^i_R(T_{r'}, K_{r+1}) = \operatorname{Ext}^{i+1}_R(T_{r'}, K_{r+1}) = 0$ for $1 \leqslant i \leqslant r+1$ and $\operatorname{pd} X = r$, we get that $\operatorname{Ext}^1_R(K_r, K_{r+1}) \simeq \operatorname{Ext}^2_R(K_{r-1}, K_{r+1}) \simeq \cdots \simeq \operatorname{Ext}^{r+2}_R(X, K_{r+1}) = 0$. Hence, the exact sequence ε_r splits and so $K_r \in \operatorname{add} T$. Therefore X has a resolution in add T of length r. Then $\alpha_T(X) \leqslant r = \operatorname{pd} X$.
(c) Let $X \in (\operatorname{add} T)^\wedge$. We can assume that $X \neq 0$ (otherwise we have nothing to prove since $\operatorname{pd} 0 = -\infty$). Let

$$0 \to T_r \xrightarrow{f_r} T_{r-1} \xrightarrow{f_{r-1}} T_{r-2} \xrightarrow{f_{r-2}} \cdots \xrightarrow{f_1} T_0 \xrightarrow{f_0} X \to 0$$

be an exact sequence with $T_i \in \operatorname{add} T$ for any i, and $r = \alpha_T(X)$. Set $K_i := \operatorname{Ker} f_i$. Then, $\operatorname{pd} X \leqslant \sup\{\operatorname{pd} T, \operatorname{pd} K_0 + 1\} \leqslant \sup\{\operatorname{pd} T, \operatorname{pd} K_1 + 2\} \leqslant \cdots \leqslant \sup\{\operatorname{pd} T, \operatorname{pd} K_{r-2} + r - 1\} \leqslant \operatorname{pd} T + r$, proving that $\operatorname{pd} X \leqslant \operatorname{pd} T + \alpha_T(X)$.
(d) Follows from (b) and (c).

(e) By (b) we have to prove only that $(\operatorname{add} T)^\wedge \subseteq \mathscr{P}^{<\infty}(T^\perp)$. Let $M \in (\operatorname{add} T)^\wedge$ we prove by induction on $d = \alpha_T(M)$ that $M \in T^\perp$.
If $d = 0$ then $M \in \operatorname{add} T$ and so $M \in T^\perp$. In case $d = 1$ we have an exact sequence $0 \to T_1 \to T_0 \to M \to 0$ with $T_0, T_1 \in \operatorname{add} T$. Applying the functor $\operatorname{Hom}_R(T, -)$ to this exact sequence we get $M \in T^\perp$.
Assume $d > 1$ and consider the exact sequence $0 \to T_d \to \cdots \to T_1 \xrightarrow{f} T_0 \to M \to 0$ with $T_i \in \operatorname{add} T$ for all i. So we get that $\alpha_T(\operatorname{Im} f) = d - 1$. Then by induction we have $\operatorname{Im} f \in T^\perp$. Applying the functor $\operatorname{Hom}_R(T, -)$ to the exact sequence $0 \to \operatorname{Im} f \to T_0 \to M \to 0$ we obtain that $M \in T^\perp$. $\square$

Proposition 3.2. *Let T be a generalized tilting R-module,* $\Gamma = \operatorname{End}_R(T)^{op}$, *and let F be the functor* $\operatorname{Hom}_R(T, -) : \operatorname{mod} R \to \operatorname{mod} \Gamma$. *Then*

(a) $\alpha_T \leqslant \operatorname{pfd} T^\perp \leqslant \operatorname{pd} T + \alpha_T$,
(b) $\alpha_T < \infty$ *if and only if* $\operatorname{pfd} T^\perp < \infty$,
(c) *the functor F induces by restriction exact equivalences of categories* $T^\perp \xrightarrow{\sim} \operatorname{Im} F|_{T^\perp}$ *and* $\operatorname{add} T \xrightarrow{\sim} \mathscr{P}_\Gamma$, *where* $\mathscr{P}_\Gamma$ *is the subcategory of* $\operatorname{mod} \Gamma$ *whose objects are the projective Γ-modules*,
(d) $\alpha_T(X) \leqslant \operatorname{pd} F(X) \leqslant \operatorname{pd} X$ *for any* $X \in \mathscr{P}^{<\infty}(T^\perp)$,
(e) $\alpha_T \leqslant \operatorname{pfd} \operatorname{Im} F|_{T^\perp} \leqslant \operatorname{pfd} T^\perp$,
(f) $\operatorname{pfd} T^\perp \leqslant \operatorname{pd} T + \operatorname{pfd} \Gamma$.

Proof. (a) Follows from 3.1, and as a consequence of (a) we get (b).

(c) The first equivalence follows from the fact that for any $M \in T^\perp$ there is an exact sequence $T_1 \xrightarrow{f} T_0 \to M \to 0$ with $T_0, T_1 \in \operatorname{add} T$ and $\operatorname{Ker} f, \operatorname{Im} f \in T^\perp$, see 3.1(a). The second equivalence is well known ([6, Section 2, Chapter II]).

(d) Let $X \in \mathscr{P}^{<\infty}(T^\perp)$. We can assume that $X \notin \operatorname{add} T$ (otherwise we have nothing to prove). Using 3.1(a) it can be seen that there exist some r with $0 < r \leqslant \operatorname{pd} X$ such that there is an exact sequence

$$0 \to K_r \to T_{r-1} \xrightarrow{f_{r-1}} T_{r-2} \xrightarrow{f_{r-2}} \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} X \xrightarrow{f_{-1}} 0 \tag{6}$$

with K_r and T_i in $\operatorname{add} T$ for any i, $\operatorname{Ker} f_i \in T^\perp \backslash \operatorname{add} T$ for every $-1 \leqslant i \leqslant r-2$ and $f_i : T_i \to \operatorname{Ker} f_{i-1}$ is the right minimal $\operatorname{add} T$-approximation of $\operatorname{Ker} f_{i-1}$ for all $0 \leqslant i \leqslant r-1$. Therefore $\alpha_T(X) \leqslant r \leqslant \operatorname{pd} X$. Applying the functor F to (6) and using (c) we obtain a finite minimal projective resolution of $F(X)$, proving that $r = \operatorname{pd} F(X)$.

(e) The fact $\alpha_T \leqslant \operatorname{pfd} \operatorname{Im} F|_{T^\perp}$ follows easily from the first inequality in (d). In order to prove $\operatorname{pfd} \operatorname{Im} F|_{T^\perp} \leqslant \operatorname{pfd} T^\perp$, by using (d), it is enough to see that: if $X \in T^\perp$ and $\operatorname{pd} F(X)$ is finite then $\operatorname{pd} X$ is finite.

Assume that $X \in T^\perp$ and $\operatorname{pd} F(X)$ is finite. Let $0 \to K' \to F(T_0) \xrightarrow{F(f)} F(X) \to 0$ be the exact sequence with $F(f)$ the right minimal $\mathscr{P}_\Gamma$-approximation of $F(X)$. Since $F : T^\perp \to \operatorname{Im} F|_{T^\perp}$ is an equivalence as exact categories, we get that $f : T_0 \to X$ is the right minimal $\operatorname{add} T$-approximation of X. So using 3.1(a) we conclude that $K' \simeq F(\operatorname{Ker} f)$

with Ker $f \in T^{\perp}$. Therefore, using the fact that there is a finite minimal projective resolution of $F(X)$, we get that $\alpha_T(X)$ is finite. Hence by 3.1(d) we have pd $X < \infty$.

(f) Follows from (a) and (e). □

Theorem 3.3. *Let $({}_R\Delta, \leqslant)$ be standard of size s and T be the characteristic tilting R module associated to $({}_R\Delta, \leqslant)$.*

(a) *If $\Gamma = \mathrm{End}({}_R T)^{op}$ then $\alpha_T \leqslant \mathrm{pfd}\, \mathscr{F}({}_\Gamma\overline{\Delta}) \leqslant s-1$.*
(b) $\mathrm{pfd}\, R = \mathrm{pfd}\, T^{\perp} = \mathrm{pd}\,(\mathrm{add}\, T)^{\wedge} \leqslant \mathrm{pd}\, T + \alpha_T \leqslant 2s-2$.

Proof. (a) By 2.1 in [2], we know that $\mathscr{F}({}_R\overline{\nabla}) = T^{\perp}$ Hence by 2.6(iii) in [2] we have that $\mathrm{Im}\, F|_{T^{\perp}} = \mathscr{F}({}_\Gamma\overline{\Delta})$. So by 3.2(d) we obtain $\alpha_T \leqslant \mathrm{pfd}\, \mathscr{F}({}_\Gamma\overline{\Delta})$. We have that ${}_\Gamma\Gamma$ $\mathrm{Im}\, F|_{T^{\perp}} = \mathscr{F}({}_\Gamma\overline{\Delta})$ (see 3.2(c)). Then by Lemma 3.4 in [3] we conclude that $\mathrm{pfd}\, \mathscr{F}({}_\Gamma\overline{\Delta}) \leqslant s-1$.

(b) We have that $\mathscr{I}({}_R\Delta) = T^{\perp}$ (see [2, 1.6 and 2.1]). Hence by 2.6 we get $\mathrm{pfd}\, R = \mathrm{pfd}\, T^{\perp}$. On the other hand, 3.1(e) implies that $\mathrm{pfd}\, T^{\perp} = \mathrm{pd}\,(\mathrm{add}\, T)^{\wedge}$.

Finally, 3.2(a) yields $\mathrm{pfd}\, T^{\perp} \leqslant \mathrm{pd}\, T + \alpha_T \leqslant 2s-2$, since by the previous item $\alpha_T \leqslant s-1$ and by 2.5(a) $\mathrm{pd}\, T \leqslant s-1$. □

For the convenience of the reader, we will state and prove the dual version of Theorem 3.3. Let T be a generalized cotilting R-module and X be an R-module. We define $\beta_T(X) := -\infty$ if $X=0$, $\beta_T(X) := +\infty$ if $X \notin (\mathrm{add}\, T)^{\vee}$, and $\beta_T(X) := \min\{r$: there is an exact sequence $0 \to X \to T_0 \to T_1 \to \cdots \to T_r \to 0$, with $T_i \in \mathrm{add}\, T\}$ if $X \in (\mathrm{add}\, T)^{\vee}$. Finally, we define $\beta_T := \sup\{\beta_T(X) : X \in \mathscr{I}^{<\infty}({}^{\perp}T)\}$.

Lemma 3.4. *Let ${}_{R^{op}}T$ be a generalized tilting R^{op}-module and X be an R-module. Then*

(a) $D((\mathrm{add}\, {}_{R^{op}}T)^{\wedge}) = (\mathrm{add}\, D({}_{R^{op}}T))^{\vee}$,
(b) $D(X^{\perp}) = {}^{\perp}D(X)$,
(c) $\beta_{D({}_{R^{op}}T)}(X) = \alpha_{{}_{R^{op}}T}(D(X))$, *and so we have* $\beta_{D({}_{R^{op}}T)} = \alpha_{{}_{R^{op}}T}$.

Proof. It is straightforward. □

Theorem 3.5. *Let $({}_{R^{op}}\Delta, \leqslant)$ be standard of size s, T be the characteristic tilting R^{op}-module associated to $({}_{R^{op}}\Delta, \leqslant)$, and $T' := D({}_{R^{op}}T)$.*

(a) *If $\Lambda = \mathrm{End}({}_R T')^{op}$ then $\beta_{T'} \leqslant \mathrm{ifd}\, \mathscr{F}({}_\Lambda\overline{\nabla}) \leqslant s-1$.*
(b) $\mathrm{ifd}\, R = \mathrm{ifd}\, {}^{\perp}T' =: \mathrm{id}(\mathrm{add}\, T')^{\vee} \leqslant \mathrm{id}\, T' + \beta_{T'} \leqslant 2s-2$.

Proof. Applying 3.4 the result follows by duality from 3.3. □

Theorem 3.6. *Let $(\theta, \underline{Y}, \preceq)$ be an eiss.*

(a) *If $\mathscr{I}(\theta) = T^{\perp}$ with T a generalized tilting R-module, then Y is a direct summand of T.*
(b) *There exists a generalized tilting R-module T such that $\mathscr{I}(\theta) = T^{\perp}$ if and only if* $\mathrm{pd}\, Y < \infty$ *and* $\mathrm{Ext}^2_R(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$.

Proof. (a) Suppose that $\mathscr{I}(\theta) = T^{\perp}$ for some generalized tilting R-module T. Since $Y \in \mathscr{I}(\theta) = T^{\perp}$ we have, by 3.1(a), that there is a short exact sequence $\varepsilon : 0 \to K \to T_0 \to Y \to 0$ with $T_0 \in \operatorname{add} T$ and $K \in T^{\perp} = \mathscr{I}(\theta)$. Since $Y \in \mathscr{F}(\theta)$ and $K \in \mathscr{I}(\theta)$ the exact sequence ε splits and so $Y \in \operatorname{add} T$. Then Y is a direct summand of T, since Y is a basic R-module.

(b) Assume that $\mathscr{I}(\theta) = T^{\perp}$ with T a generalized tilting R-module. Using (a) we get that $\operatorname{pd} Y \leqslant \operatorname{pd} T < \infty$. Since $T^{\perp}$ is a coresolving subcategory of $\operatorname{mod} R$, see [5], and $\mathscr{I}(\theta) = T^{\perp}$ we get that $\mathscr{I}(\theta)$ is so. Hence by Proposition 3.3(a) in [10] we have that $\operatorname{Ext}^2_R(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$.

Assume that $\operatorname{pd} Y < \infty$ and $\operatorname{Ext}^2_R(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$. The last condition implies, by Proposition 3.3(a) in [10], that $\mathscr{I}(\theta)$ is a coresolving subcategory of $\operatorname{mod} R$. On the other hand, from [13] we get that $\mathscr{I}(\theta)$ is also a covariantly finite subcategory of $\operatorname{mod} R$. So, to get that $\mathscr{I}(\theta) = T^{\perp}$ for some generalized tilting R-module T, it is enough to prove that $\mathscr{I}(\theta)^{\vee} = \operatorname{mod} R$, see Proposition 5.5 in [5]. We prove now that $\mathscr{I}(\theta)^{\vee} = \operatorname{mod} R$. Let $X \in \operatorname{mod} R$. Since $\mathscr{I}(\theta)$ is coresolving, we have for each $d > 0$ a long exact sequence

$$0 \to X \to I_0(X) \to I_1(X) \to \cdots \to I_{d-1}(X) \to \Omega^{-d}(X) \to 0$$

with $I_i(X)$ injective for all $i = 0, 1, \ldots, d-1$. We shall see that $\Omega^{-d}(X) \in \mathscr{I}(\theta)$, if $d = \operatorname{pd} Y < \infty$. In fact, by 2.4(a), we have that $\operatorname{pd} \mathscr{F}(\theta) = d = \operatorname{pd} Y$. Hence $\operatorname{Ext}^1_R(M, \Omega^{-d}(X)) \simeq \operatorname{Ext}^{d+1}_R(M, X) = 0$ for any $M \in \mathscr{F}(\theta)$, proving that $\Omega^{-d}(X) \in \mathscr{I}(\theta)$ and so $X \in \mathscr{I}(\theta)^{\vee}$. □

Corollary 3.7. *Let* $(\theta, \preccurlyeq)$ *be a standard stratifying system of size* t. *If* $\mathscr{I}(\theta) = T^{\perp}$ *with* T *a basic generalized tilting* R*-module*, *then*

(a) *the* R*-module* Y *is isomorphic to* T *and* t *is equal to the number* s *of iso-classes of simple modules*,
(b) *there is a complete set of primitive orthogonal idempotents* $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s\}$ of R, *such that* ${}_R\Delta(i) \simeq \theta(\omega_t^{-1}(i))$ *for any* $i = 1, 2, \ldots, s$.

Proof. (a) From the previous theorem we have that the module Y is a direct summand of the generalized tilting R-module T and also that $\operatorname{Ext}^2_R(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$. Then by Proposition 3.3(c) in [10] we get that Y is a generalized tilting R-module. Hence Y is isomorphic to T.

(b) Follows from (a) and Theorem 3.1 in [10]. □

The following example shows that the condition, given in 3.7(a), of $(\theta, \preccurlyeq)$ being standard is not a necessary condition. It also shows that the mentioned condition cannot be omitted in 3.7(b).

Example 3.8. Consider the path algebra $R = kQ$, where Q is the following quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

We set $\theta(1) = Y(1) = I(2)$, $\theta(2) = Y(2) = I(1)$, $\theta(3) = S(3)$ and $Y(3) = I(3) = P(1)$. We have that $P(1) = Y(3) \in \mathscr{F}(\theta)$, $P(3) = \theta(3) \in \mathscr{F}(\theta)$ and $P(2) \notin \mathscr{F}(\theta)$. So the stratifying system $(\theta, \leqslant)$ is not the canonical one and it is not standard. On the other hand, ${}_R R \in (\mathscr{F}(\theta) \cap \mathscr{I}(\theta))^{\vee} = (\operatorname{add} Y)^{\vee}$, since, we have the exact sequence

$$0 \to {}_R R \to Y(3)^3 \to Y(2) \coprod Y(1) \to 0.$$

Moreover, using that R is hereditary and $Y = Y(1) \coprod Y(2) \coprod Y(3)$ is injective we get that Y is a generalized tilting R-module. Note that $\mathscr{I}(\theta) = Y^{\perp}$, see 3.6.

Proposition 3.9. *Let R be an algebra, s be the number of iso-classes of simple R-modules and $(\theta, \preccurlyeq)$ be a stratifying system of size t.*

(a) *If $\mathscr{I}(\theta) = T^{\perp}$ for some generalized tilting R-module T, then $t \leqslant s$ and $\mathscr{I}(\theta)$ is a coresolving subcategory of* mod R.
(b) *The following statements are equivalent*:

  (i) *$\mathscr{I}(\theta)$ is a coresolving subcategory of* mod R,
  (ii) $\operatorname{Ext}_R^i(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$ *for any $i > 0$,*
  (iii) $\operatorname{Ext}_R^2(\mathscr{F}(\theta), \mathscr{I}(\theta)) = 0$.

Proof. (a) From the previous theorem, we know that the R-module Y is a direct summand of the generalized tilting R-module T. Therefore $t \leqslant s$.

(b) (i) $\Rightarrow$ (ii) Let $M \in \mathscr{F}(\theta)$ and $N \in \mathscr{I}(\theta)$. Since $\mathscr{I}(\theta)$ is coresolving, we get for any $i \geqslant 2$ an exact sequence

$$0 \to N \to I_0(N) \to I_1(N) \to \cdots \to I_{i-2}(N) \to \Omega^{-i+1}(N) \to 0$$

with $I_m(N)$ injective for all $m=0, 1, \ldots, i-2$ and $\Omega^{-i+1}(N) \in \mathscr{I}(\theta)$. Hence $\operatorname{Ext}_R^i(M, N) \simeq \operatorname{Ext}_R^1(M, \Omega^{-i+1}(N)) = 0$.

(iii) $\Rightarrow$ (i) See [10, Proposition 3.3]. $\square$

Corollary 3.10. *Let R be a quasi-hereditary algebra, s be the number of iso-classes of simple R-modules and $(\theta, \preccurlyeq)$ be a stratifying system of size t.*

(a) *If $\mathscr{I}(\theta)$ is coresolving then $t \leqslant s$.*
(b) *If R is hereditary then $t \leqslant s$.*

Proof. (a) Since R is quasi-hereditary then $\operatorname{pd} Y \leqslant \operatorname{gl\,dim} R < \infty$. Assume that $\mathscr{I}(\theta)$ is coresolving. Then, by 3.9(b) and 3.6(b) we get $\mathscr{I}(\theta) = T^{\perp}$ with T a generalized tilting R-module. Hence, the result follows from 3.9(a).

(b) It follows from (a), since R hereditary implies that $\mathscr{I}(\theta)$ is coresolving. $\square$

Remark 3.11. Consider the stratifying system $(\theta, \leqslant)$, of size 5, given in Example 1.4. It can be seen that $\theta(i) = Y(i)$ for $i = 1, 2, \ldots, 5$. On the other hand, we have that R is a quasi-hereditary algebra but $\mathscr{I}(\theta)$ is not coresolving, since in the exact sequence $0 \to Y(1) \to Y(2) \to S(2) \to 0$ we have that $S(2) \notin \mathscr{I}(\theta)$.

Another corollary of 3.9 is the following fact which appears also in [1, Theorem 3.1(iii)].

Corollary 3.12. *If* $({}_R\Delta, \leqslant)$ *is standard then* $\mathrm{Ext}^i_R(\mathscr{F}({}_R\Delta), \mathscr{I}({}_R\Delta)) = 0$ *for any* $i > 0$.

Proof. Assume that $({}_R\Delta, \leqslant)$ is standard. Then by [5] Lemma 3.2 we have that $\mathscr{I}({}_R\Delta)$ is coresolving. Hence the result follows from 3.9(b). □

Next we enunciate the dual version of the previous theorem, for doing so, we will make use of the following notation: given a stratifying system $(\theta, \preccurlyeq)$ of size t, we have the epss $(\theta, \underline{Q}, \preccurlyeq)$ associated to $(\theta, \preccurlyeq)$ and then we denote by Q the R-module $\coprod_{i=1}^t Q(i)$, where $Q(i) \in \underline{Q}$.

Theorem 3.13. *Let* $(\theta, \preccurlyeq)$ *be a stratifying system.*

(a) *If* $\mathscr{P}(\theta) = {}^{\perp}T$ *for some generalized cotilting R-module T, then Q is a direct summand of T.*
(b) *There is a cotilting R-module T such that* $\mathscr{P}(\theta) = {}^{\perp}T$ *if and only if* $\mathrm{id}\, Q$ *is finite and* $\mathrm{Ext}^2_R(\mathscr{P}(\theta), \mathscr{F}(\theta)) = 0$.

The following result appears also in [2, Theorem 2.1].

Corollary 3.14. *Let R be an algebra. Then*

(a) *If* $({}_R\Delta, \leqslant)$ *is standard and* ${}_RT$ *is the characteristic tilting module associated to* $({}_R\Delta, \leqslant)$, *then* $\mathscr{I}({}_R\Delta) = T^{\perp}$.
(b) *If* $({}_R\nabla, \leqslant^{op})$ *is costandard and* ${}_{R^{op}}T$ *is the characteristic tilting module associated to* $({}_{R^{op}}\Delta, \leqslant)$, *then* $\mathscr{P}({}_R\nabla) = {}^{\perp}(D({}_{R^{op}}T))$.

Proof. It is enough to prove (a) since (b) is dual. We apply 3.6 to the eiss $({}_R\Delta, \underline{{}_RT}, \leqslant)$. By 3.9 we know that $\mathrm{Ext}^2_R(\mathscr{F}({}_R\Delta), \mathscr{I}({}_R\Delta)) = 0$. Then by 3.6 there is a basic generalized tilting R-module T' such that $\mathscr{I}({}_R\Delta) = T'^{\perp}$ and T is a direct summand of T'. Hence $T \simeq T'$, since T and T' are basic and they have the same number of direct summands (because both of them are generalized tilting R-modules). □

We have the following consequences for algebras R such that $\mathrm{id}\, {}_RR < \infty$ and $\mathrm{id}\, R_R < \infty$. Such algebras are called *Gorenstein* algebras in [15].

Proposition 3.15. *Let* $({}_R\Delta, \leqslant)$ *be standard of size s and T be the characteristic tilting R-module associated to it. Then*

(a) *R is Gorenstein if and only if* $\mathrm{id}\, T < \infty$,
(b) *if* $\mathrm{id}\, T < \infty$ *then* $\mathrm{ifd}\, R = \mathrm{ifd}\, \mathscr{P}({}_R\Delta) \leqslant \mathrm{id}\, T + s$,
(c) *if R is quasi-hereditary then* $\mathrm{gl\ dim}\ R = \mathrm{id}\, \mathscr{P}({}_R\Delta) \leqslant \mathrm{id}\, T + s$.

Proof. (a) If R is Gorenstein then $\mathrm{id}\, {}_RR < \infty$. So by 2.5 $\mathrm{id}\, T < \infty$.

Assume that $\mathrm{id}\, T < \infty$. Then from 2.5 we have that $\mathrm{id}\, _R R < \infty$. On the other hand, by Proposition 6.10 in [5] we have that $\mathrm{id}\, _{R^{op}} R^{op} < \infty$ if and only if $\mathrm{pfd}\, R^{op} < \infty$. The result now follows from Theorem 3.1 in [3], which states that if $(_R\Delta, \leqslant)$ is standard then $\mathrm{pfd}\, R^{op} < \infty$.

(b) Assume that $\mathrm{id}\, T < \infty$. Then by (a) we know that $\mathrm{id}\, _R R < \infty$. Hence by 2.6 we get $\mathrm{ifd}\, R = \mathrm{ifd}\, \mathscr{P}(_R\Delta) \leqslant \sup\{\mathrm{ifd}\, \mathscr{I}(_R\Delta), \mathrm{id}\, _R R + 1\}$. Using Proposition 1.8 in [2] we obtain that $\mathrm{ifd}\, \mathscr{I}(_R\Delta) \leqslant s-1$, and by 2.5 $\mathrm{id}\, _R R + 1 \leqslant \mathrm{id}\, T + s$, so the result follows.

(c) Follows from (b) by the fact that quasi hereditary algebras have finite global dimension. □

In the next proposition we give a new condition for a standardly stratified algebra to be quasi-hereditary.

Proposition 3.16. *R is quasi-hereditary if and only if it is standardly stratified and the injective dimension of $_R\overline{\nabla}$ is finite.*

Proof. Since quasi hereditary algebras have finite global dimension then the injective dimension of $_R\overline{\nabla}$ is finite.

Assume now that $(_R\Delta, \leqslant)$ is standard and $\mathrm{id}\, _R\overline{\nabla} = d < \infty$. Then $\mathrm{id}\, \mathscr{F}(_R\overline{\nabla}) \leqslant d$. By Theorem 1.6 in [2] we have that $\mathscr{F}(_R\overline{\nabla}) = \mathscr{I}(_R\Delta)$ and so $\mathrm{id}\, \mathscr{I}(_R\Delta) \leqslant d < \infty$. From 3.14 we have that $\mathscr{I}(_R\Delta) = T^{\perp}$, where T is the characteristic tilting R module. Then by the dual version of Theorem 5.5 in [5] we get that $\mathscr{I}(_R\Delta)^{\vee} = \mathrm{mod}\, R$.

But the facts $\mathrm{id}\, \mathscr{I}(_R\Delta) < \infty$ and $\mathscr{I}(_R\Delta)^{\vee} = \mathrm{mod}\, R$ imply that $\mathscr{I}^{<\infty}(R) = \mathrm{mod}\, R$ and also that R is Gorenstein (see 3.15(a)). Since R is Gorenstein we have by Lemma 6.9 in [5] that $\mathscr{P}^{<\infty}(R) = \mathscr{I}^{<\infty}(R)$. Hence $\mathscr{P}^{<\infty}(R) = \mathrm{mod}\, R$ and therefore $\mathrm{gl\ dim}\ R = \mathrm{pfd}\, R < \infty$, since R is standardly stratified (see 3.3). Now the result follows from the well-known fact that standardly stratified algebras of finite global dimension are quasi-hereditary. □

Proposition 3.17. *Let R be an algebra.*

(a) *Let $(_R\Delta, \leqslant)$ be standard of size s and $_RT$ be the characteristic tilting R-module. If $\mathscr{F}(_R\Delta) = \mathscr{P}^{<\infty}(R)$ then* $\mathrm{pfd}\, R = \mathrm{pd}\, _RT \leqslant s-1$.
(b) *Let $(_R\nabla, \leqslant^{op})$ be costandard of size s, $_{R^{op}}T$ be the characteristic tilting R^{op}-module associated to the standard stratifying system $(_{R^{op}}\Delta, \leqslant)$, and $_RT' = D(_{R^{op}}T)$. If $\mathscr{F}(_R\nabla) = \mathscr{I}^{<\infty}(R)$ then* $\mathrm{ifd}\, R = \mathrm{id}\, _RT' \leqslant s-1$.

Proof. We shall prove (a) only, since the proof of (b) is dual. Let $\mathscr{F}(_R\Delta) = \mathscr{P}^{<\infty}(R)$ then by 2.5 we have that $\mathrm{pfd}\, R = \mathrm{pd}\, \mathscr{F}(_R\Delta) = \mathrm{pd}\, _RT \leqslant s-1$. □

In [12] Platzeck and Reiten gave sufficient conditions, in terms of quivers with relations, for $\mathscr{F}(_R\Delta) = \mathscr{P}^{<\infty}(R)$ when R is standardly stratified. So, by using Theorem 2.5 in [12] we can construct examples of algebras which satisfy the hypothesis of 3.14(a). Therefore, for those algebras we know how to compute their projective finitistic dimension.

The next proposition gives equivalent conditions for the categories $\mathscr{F}(_R\Delta)$ and $\mathscr{P}^{<\infty}(R)$ to be equal.

Proposition 3.18. *Let $({}_R\Delta, \leqslant)$ be standard and T be the characteristic tilting R-module. The following conditions are equivalent*:

(a) $\mathscr{F}({}_R\Delta) = \mathscr{P}^{<\infty}(R)$,
(b) $\mathscr{F}({}_R\Delta) \supseteq (\operatorname{add} T)^{\wedge}$,
(c) $\mathscr{P}^{<\infty}(\mathscr{I}({}_R\Delta)) \subseteq \mathscr{F}({}_R\Delta)$.

Proof. (a) $\Rightarrow$ (b) Follows from 3.1(c).

(b) $\Rightarrow$ (c) Assume that $\mathscr{F}({}_R\Delta) \supseteq (\operatorname{add} T)^{\wedge}$. By 3.1(e) we know that $\mathscr{P}^{<\infty}(T^{\perp}) = (\operatorname{add} T)^{\wedge}$. So the result follows from the fact $\mathscr{I}({}_R\Delta) = T^{\perp}$ (see 3.14).

(c) $\Rightarrow$ (a) Suppose that $\mathscr{P}^{<\infty}(\mathscr{I}({}_R\Delta)) \subseteq \mathscr{F}({}_R\Delta)$ then we have to prove that $\mathscr{P}^{<\infty}(R) \subseteq \mathscr{F}({}_R\Delta)$.

Let $X \in \mathscr{P}^{<\infty}(R)$. By [13, Lemma 4′] we have an exact sequence

$$\varepsilon : 0 \to X \to Y_X \to Q_X \to 0 \quad \text{with } Y_X \in \mathscr{I}({}_R\Delta) \text{ and } Q_X \in \mathscr{F}({}_R\Delta).$$

Since $\operatorname{pd} X < \infty$ and by 2.5 $\operatorname{pd} Q_X < \infty$, we get from the exact sequence ε that $\operatorname{pd} Y_X < \infty$. Hence by assumption we get $Y_X \in \mathscr{F}({}_R\Delta)$. But now, using that $\mathscr{F}({}_R\Delta)$ is closed
under kernels of surjections (see Lemma 1.5 in [8]) we have that $X \in \mathscr{F}({}_R\Delta)$, proving the result. □

We also state the dual version of the previous proposition.

Proposition 3.19. *Let $({}_R\nabla, \leqslant^{op})$ be costandard, ${}_{R^{op}}T$ be the characteristic tilting R^{op}-module associated to $({}_{R^{op}}\Delta, \leqslant)$ and ${}_RT' = D({}_{R^{op}}T)$. The following conditions are equivalent*:

(a) $\mathscr{F}({}_R\nabla) = \mathscr{I}^{<\infty}(R)$,
(b) $\mathscr{F}({}_R\nabla) \supseteq (\operatorname{add} T')^{\vee}$,
(c) $\mathscr{I}^{<\infty}(\mathscr{P}({}_R\nabla)) \subseteq \mathscr{F}({}_R\nabla)$.

Given a stratifying system $(\theta, \preccurlyeq)$, we recall that there is a unique eiss $(\theta, \underline{Y}, \preccurlyeq)$ (resp. epss $(\theta, \underline{Q}, \preccurlyeq)$) associated to it. Moreover, the algebras $A = \operatorname{End}({}_RY)$ and $B = \operatorname{End}({}_RQ)^{op}$ are standardly stratified [11].

Proposition 3.20. *Let $(\theta, \preccurlyeq)$ be a stratifying system and ${}_BT$ the characteristic tilting B-module associated to $({}_B\Delta, \leqslant)$. If* $\operatorname{pd} {}_RQ \leqslant 1$ *then*

(a) $\operatorname{pd} \operatorname{Hom}_R({}_RQ_B, {}_RM) \leqslant \operatorname{pd} {}_RM$ *for all* ${}_RM \in \mathscr{F}(\theta)$,
(b) $\operatorname{pd} \mathscr{F}({}_B\Delta) \leqslant \operatorname{pd} \mathscr{F}(\theta)$, *in particular* $\operatorname{pd} {}_BT \leqslant \operatorname{pd} {}_RY$,
(c) *if $\mathscr{F}({}_B\Delta)$ is closed under submodules then B is quasi-hereditary*, $\operatorname{gl\,dim} B \leqslant 1 + \operatorname{pd} {}_RY$ *and* $\operatorname{id} {}_BT \leqslant 1$.

Proof. (a) We proceed by induction on $n = \operatorname{pd} M$ with $M \in \mathscr{F}(\theta)$. If $n = 0$ then M is projective and so $M \in \operatorname{add} Q$, since by Proposition 2.14 in [11] we know that $\mathscr{P}(\theta) \cap \mathscr{F}(\theta) = \operatorname{add} Q$. Hence $\operatorname{pd} \operatorname{Hom}_R({}_RQ_B, {}_RM) = 0$.

Assume that $n \geqslant 1$. By Proposition 2.10 in [11] there is an exact sequence

$$0 \to N \to Q_0 \to M \to 0 \tag{7}$$

with $N \in \mathscr{F}(\theta)$ and $Q_0 \in \operatorname{add} Q$. Applying the functor $\operatorname{Hom}_R(Q, -)$ to (7) we have the exact sequence

$$0 \to \operatorname{Hom}_R(Q, N) \to \operatorname{Hom}_R(Q, Q_0) \to \operatorname{Hom}_R(Q, M) \to 0. \tag{8}$$

If $n = 1$ then $N \in \operatorname{add} Q$. Indeed, applying $\operatorname{Hom}_R(-, X)$ to (7), with $X \in \mathscr{F}(\theta)$, yields $0 = \operatorname{Ext}^1_R(Q_0, X) \to \operatorname{Ext}^1_R(N, X) \to \operatorname{Ext}^2_R(M, X) = 0$.

Therefore, if $n = 1$ then $\operatorname{Hom}_R(Q, N)$ is B-projective and so by (8) we get $\operatorname{pd} \operatorname{Hom}_R({}_RQ_B, {}_RM) \leqslant 1$. Finally, if $n \geqslant 2$ it follows from (7) and $\operatorname{pd}_R Q \leqslant 1$ that $\operatorname{pd} N \leqslant \operatorname{pd} M - 1$. Hence by induction and (8) we get $\operatorname{pd} \operatorname{Hom}_R({}_RQ_B, {}_RM) \leqslant \operatorname{pd} \operatorname{Hom}_R({}_RQ_{B,N}) + 1 \leqslant \operatorname{pd} N + 1 \leqslant \operatorname{pd} M$.

(b) It follows from (a) using that $\operatorname{Hom}_R({}_RQ_B, -) : \mathscr{F}(\theta) \to \mathscr{F}({}_B\Delta)$ is an equivalence of categories, see [11, Theorem 3.1].

(c) Let ${}_BN$ be a B-module and consider the exact sequence

$$0 \to {}_BK \to {}_BP(N) \to {}_BN \to 0, \tag{9}$$

where ${}_BP(N)$ is the projective cover of ${}_BN$. Since B is standardly stratified and $\mathscr{F}({}_B\Delta)$ is, by assumption, closed under submodules we get that $K \in \mathscr{F}({}_B\Delta)$. Using that $\operatorname{Hom}_R({}_RQ_B, -)$: $\mathscr{F}(\theta) \to \mathscr{F}({}_B\Delta)$ is an equivalence of categories we get that $K = \operatorname{Hom}_R(Q, K')$ for some ${}_RK' \in \mathscr{F}(\theta)$. Hence by item (a) and 2.4 we have that $\operatorname{pd}_B K \leqslant \operatorname{pd}_R K' \leqslant \operatorname{pd}_R Y$. From (9) we obtain that $\operatorname{pd}_B N \leqslant 1 + \operatorname{pd}_B K \leqslant 1 + \operatorname{pd}_R K' \leqslant \operatorname{pd}_R Y + 1$. Hence $\operatorname{gl\,dim} B \leqslant 1 + \operatorname{pd}_R Y$ and $\operatorname{pd}_R Y < \infty$, since $\operatorname{pd}_R Q < \infty$ (see 2.4(c)). Finally, by Lemma 4.1* in [8] we have that $\operatorname{id}_B T \leqslant 1$. $\square$

In the following proposition, we state a necessary condition for the category $\mathscr{F}({}_R\Delta)$ to be closed under submodules. We recall that, in the case of a quasi-hereditary algebra, Dlab and Ringel give in [8, Lemma 4.1*] equivalent conditions for the category $\mathscr{F}({}_R\Delta)$ to be closed under submodules.

Proposition 3.21. *Let $({}_R\Delta, \leqslant)$ be standard and ${}_RT$ be the characteristic tilting R-module associated to it. If the category $\mathscr{F}({}_R\Delta)$ is closed under submodules then R is quasi-hereditary and* $\operatorname{gl\,dim} R \leqslant 1 + \operatorname{pd}_R T$.

Proof. Assume that $\mathscr{F}({}_R\Delta)$ is closed under submodules. So it can be seen that $[{}_R\Delta(i) : S(i)] = 1$ for any i, and hence ${}_R\Delta = {}_R\bar{\Delta}$, that is R is quasi-hereditary. Further, $I = D(R_R) \in \mathscr{F}(\nabla)$ and using inductively Proposition 2 in [13] we get the exact sequence

$$0 \to X \to T_0 \to I \to 0 \quad \text{with } T_0 \in \operatorname{add} T. \tag{10}$$

Since $\mathscr{F}({}_R\Delta)$ is closed under submodules we obtain from (10) that $X \in \mathscr{F}({}_R\Delta)$. Hence $\operatorname{pd} X \leqslant \operatorname{pd} T$, see [2, Proposition 2.2]. On the other hand, using that $\operatorname{gl\,dim} R = \operatorname{pd} I$ and (10) we get that $\operatorname{gl\,dim} R \leqslant \operatorname{pd} T + 1$. $\square$

We recall that a module M is called *torsionless* when it is a submodule of a free module. The following statements generalize a bit the equivalent conditions given in [8, Lemma 4.1*] for quasi-hereditary algebras.

Proposition 3.22. *Let* $({}_R\Delta, \leqslant)$ *be standard and* T *be the characteristic tilting* R*-module. The following conditions are equivalent*:

(a) *the subcategory* $\mathscr{F}({}_R\Delta)$ *is closed under submodules*,
(b) *all torsionless* R*-modules belong to* $\mathscr{F}({}_R\Delta)$,
(c) R *is quasi-hereditary and* $\operatorname{id}\mathscr{F}({}_R\nabla)\leqslant 1$,
(d) R *is quasi-hereditary and* $\operatorname{id} T\leqslant 1$,
(e) $\operatorname{gl\,dim} R\leqslant 1+\operatorname{pd} T$ *and* $\operatorname{id} T\leqslant 1$.

Proof. (a) $\Rightarrow$ (b) By 3.21 we have that R is quasi-hereditary. So the result follows from [8, Lemma 4.1*].

(b) $\Rightarrow$ (a) Since all torsionless modules belong to $\mathscr{F}({}_R\Delta)$ then all torsionless modules have finite projective dimension. Now given any module X, we have that the first syzygy of X is torsionless, so $\Omega(X)$ has finite projective dimension, therefore X itself has finite projective dimension. Then $\mathscr{P}^{<\infty}(R)=\operatorname{mod} R$, and so $\operatorname{gl\,dim} R=\operatorname{pfd} R<\infty$, since R is standardly stratified (see 3.3). Hence R is quasi-hereditary and the implication follows from Lemma 4.1* in [8].

The equivalences of (a), (c), (d) and (e) follows from 3.21 and Lemma 4.1* [8]. $\square$

Remark 3.23. (a) The simple example where $R=k[X]/(X^2)$ shows that the hypothesis of R being quasi-hereditary, in (c) and (d) of 3.22, is necessary.

(b) Let R be a quasi-hereditary algebra with t simple modules, and T be the R-module. If $\operatorname{id} T\leqslant 1$ then $\operatorname{id}{}_R R\leqslant t$. Indeed, by 2.5 we have that $\operatorname{id}{}_R R\leqslant \operatorname{id} T+t-1$.

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