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Note

On monochromatic paths and monochromatic 4-cycles in edge coloured bipartite tournaments

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Abstract

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

(i) For every pair of different vertices $u, v \in N$, there is no monochromatic directed path between them.
(ii) For every vertex $x \in (V(D) - N)$, there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

In this paper it is proved that if D is an m-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic, then D has a kernel by monochromatic paths.

MSC: 05C20

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1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. An arc $(u_1,u_2)\in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2,u_1)\notin A(D)$ (resp. $(u_2,u_1)\in A(D)$). The asymmetrical part of D (resp. symmetrical part of D) which is denoted $\mathrm{Asym}(D)$ (resp. $\mathrm{Sym}(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D; D is called an asymmetrical digraph if $\mathrm{Asym}(D)=D$. We recall that a subdigraph D_1 of D is a spanning subdigraph if $V(D_1)=V(D)$. If S is a nonempty set of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph having vertex set S, and whose arcs are all those arcs of D joining vertices of S. An arc (u_1,u_2) of D will be called an S_1S_2-arc whenever $u_1\in S_1$ and $u_2\in S_2$.

A set $I\subseteq V(D)$ is independent if $A(D[I])=\emptyset$. A kernel N of D is an independent set of vertices such that for each $z\in V(D)-N$ there exists a zN-arc in D. A digraph D is called a kernel-perfect digraph or *KP*-digraph when every induced subdigraph of D has a kernel. A digraph D is called a bipartite tournament if its vertices can be partitioned into two sets V_1 and V_2 such that:

(i) Every arc of D has an endpoint in V_1 and the other endpoint in V_2.
(ii) For all $x_1\in V_1$ and for all $x_2\in V_2$, we have $|\{(x_1,x_2),(x_2,x_1)\}\cap A(D)|=1$. We will write $D=(V_1,V_2)$ to indicate the partition.

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If $T=(z_0,z_1,\ldots,z_n)$ is a directed path, we denote by $\ell(T)=n$ its length and if $z_i,\ z_j\in V(T)$ with $i\leqslant j$, we denote (z_i,T,z_j) the z_iz_j-directed path contained in T. For a directed cycle γ, $\ell(\gamma)$ will denote its length; a directed cycle is quasi-monochromatic if with at most one exception, all of its arcs are coloured alike.

If D is an m-coloured digraph then the closure of D, denoted $\mathscr{C}(D)$ is the m-coloured multidigraph defined as follows:

$$V(\mathscr{C}(D))-V(D),$$

$$A(\mathscr{C}(D))=A(D)\cup\{(u,v) \text{ with colour } i \mid \text{there exists a } uv\text{-monochromatic directed path coloured } i \text{ contained in } D\}.$$

Notice that for any digraph D, $\mathscr{C}(\mathscr{C}(D))\cong\mathscr{C}(D)$ and D has a kernel by monochromatic paths if and only if $\mathscr{C}(D)$ has a kernel.

In [7] Sands et al. have proved that any 2-coloured digraph has a kernel by monochromatic paths. In particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must $\mathscr{C}(T)$ have a kernel? In [6] Shen Minggang proved that if in the problem we ask that every transitive tournament of order 3 be quasi-monochromatic, the answer will be yes. In [4] it was proved that if T is an m-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then $\mathscr{C}(T)$ is kernel-perfect and hence T has a kernel by monochromatic paths. Results similar to those in [6] and [4] were proved for the digraph obtained from a tournament by the deletion of a single arc, in [5] and [3], respectively. The known sufficient conditions for the existence of a kernel by monochromatic paths in m-coloured ($m\geqslant 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of small subdigraphs as directed cycles of length at most 4 or transitive tournaments of order 3.

In this paper it is proved that if D is an m-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic then D has a kernel by monochromatic paths and the result is best possible.

We will need the following result.

Theorem 1.1 (Duchet [2]). *If D is a digraph such that every directed cycle has at least one symmetrical arc, then D is a kernel-perfect digraph.*

2. The main result

First we prove the following lemmas which will be useful in the proof of the main result:

Lemma 2.1. *Let $D=(V_1,V_2)$ be a bipartite tournament and $C=(u_0,u_1,\ldots,u_n)$ a directed walk in D. For $\{i,j\}\subseteq\{0,1,\ldots,n\}$ $(u_i,u_j)\in A(D)$ or $(u_j,u_i)\in A(D)$ if and only if $j-i\equiv 1\,(\mathrm{mod}\,2)$.*

Proof. Without loss of generality we may assume $u_0\in V_1$, then we clearly have $u_i\in V_1$ iff $i\equiv 0\,(\mathrm{mod}\,2)$ and $u_i\in V_2$ iff $i\equiv 1\,(\mathrm{mod}\,2)$. $\square$

Lemma 2.2. *For a bipartite tournament $D=(V_1,V_2)$, every closed directed walk of length at most 6 in D is a directed cycle of D.*

Proof. Let C be a closed directed walk with $\ell(C)\leqslant 6$. We will prove that C is a directed cycle. Since D is bipartite $\ell(C)$ is even (as every closed odd directed walk contains an odd directed cycle); $\ell(C)=2$ is impossible as a bipartite tournament is an asymmetrical digraph. Suppose $\ell(C)=4$, and let $C=(u_0,u_1,u_2,u_3,u_0)$ we may assume w.l.o.g. $u_i\in V_1$ for $i\in\{0,2\}$ and $u_j\in V_2$ for $j\in\{1,3\}$ which implies $u_i\neq u_j$ for $i\in\{0,2\}$, $j\in\{1,3\}$. Since $(u_1,u_2)\in A(D)$ and $(u_2,u_3)\in A(D)$ we have $u_1\neq u_3$ (as D is an asymmetrical digraph) and analogously $u_0\neq u_2$; so C is a directed cycle. Finally suppose $\ell(C)=6$ and let $C=(u_0,u_1,u_2,u_3,u_4,u_5,u_0)$, clearly we may assume w.l.o.g. that $u_i\in V_1$ for $i\in\{0,2,4\}$ and $u_j\in V_2$ for $j\in\{1,3,5\}$ which implies $u_i\neq u_j$ for $i\in\{0,2,4\}$ and $j\in\{1,3,5\}$.

Moreover, since $\{(u_i,u_{i+1}),(u_{i+1},u_{i+2})\}\subseteq A(D)$ for $i\in\{0,1,\ldots,5\}$ (notation (mod 6)) and D is asymmetrical, we have $u_i\neq u_{i+2}$ for $i\in\{0,1,\ldots,5\}$. $\square$

Lemma 2.3. *Let D be an m-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic and $u,\ v\in V(D)$. If there exists a uv-monochromatic directed path and there is no vu-monochromatic directed path* (*in* D), *then at least one of the two following conditions holds*:

(i) $(u,v)\in A(D)$;
(ii) *there exists* (*in* D) *a* uv-*directed path of length* 2.

Proof. Let D, u, $v\in V(D)$ be as in the hypothesis. We proceed by induction on the length of a uv-monochromatic directed path. Clearly Lemma 2.3 holds when there exists a uv-monochromatic directed path of length at most 2. Suppose that Lemma 2.3 holds when there exists a uv-monochromatic directed path of length ℓ with $2\leqslant \ell\leqslant n$. Now assume that there exists a uv-monochromatic directed path say $T=(u=u_0,u_1,\ldots,u_{n+1}=v)$ with $\ell(T)=n+1$; we may assume w.l.o.g. T is coloured 1.

Claim 1. *If* $(u_i,v)\in A(D)$ *for some* $i\in\{0,1,\ldots,n-2\}$ *then* $(u,v)\in A(D)$ *or there exists a* uv-*directed path of length* 2.

Assume $(u_i,v)\in A(D)$ for some $i\in\{0,1,\ldots,n-2\}$ and let $i_0=\min\{i\in\{0,1,\ldots,n-2\}\mid (u_i,v)\in A(D)\}$. If $i_0=0$ then $(u,v)\in A(D)$ and if $i_0=1$ then (u,u_1,v) is a uv-directed path of length 2, so we can assume $i_0\in\{2,\ldots,n-2\}$.

Since $i_0\equiv i_0-2\,(\mathrm{mod}\,2)$ and $i_0\not\equiv n+1\,(\mathrm{mod}\,2)$ (as $(u_{i_0},v)\in A(D)$) we have $i_0-2\not\equiv n+1\,(\mathrm{mod}\,2)$ and it follows from Lemma 2.1 that $(u_{i_0-2},v)\in A(D)$ or $(v,u_{i_0-2})\in A(D)$. Now the choice of i_0 implies $(v,u_{i_0-2})\in A(D)$ and hence $C_4=(u_{i_0-2},u_{i_0-1},u_{i_0},v,u_{i_0-2})$ is a directed cycle of length 4 which by hypothesis is monochromatic, moreover, since (u_{i_0-1},u_{i_0}) is coloured 1 (as it is an arc of T), it follows that C_4 is coloured 1. Then we obtain that $T'=(u,T,u_{i_0})\cup(u_{i_0},v)$ is a uv-monochromatic directed path with $\ell(T')<n+1$; and the inductive hypothesis implies that $(u,v)\in A(D)$ or there exists a uv-directed path of length 2.

Now, it follows from Lemma 2.1 that for each $i\in\{0,1,\ldots,n-2\}$ $(u_i,u_{i+3})\in A(D)$ or $(u_{i+3},u_i)\in A(D)$ (as $i\not\equiv i+3\,(\mathrm{mod}\,2)$).

We will analyze two possible cases:

Case a: There exists $i\in\{0,1,\ldots,n-2\}$ such that $(u_i,u_{i+3})\in A(D)$. Let $j_0=\max\{j\in\{i+3,\ldots,n+1\}\mid (u_i,u_j)\in A(D)\}$ (notice that Lemma 2.1 implies $i\not\equiv j_0\,(\mathrm{mod}\,2)$).

Case a.1: $j_0=n+1$.

Is this case the result follows from Claim 1.

Case a.2: $j_0=n$ and $i=0$.

We have $(u_0=u_i,u_{j_0}=u_n,u_{n+1})$ is a uv-directed path of length 2.

Case a.3: $j_0=n$ and $i\geqslant 1$.

Since $i\not\equiv j_0\,(\mathrm{mod}\,2)$, we have $i-1\not\equiv j_0+1=n+1\,(\mathrm{mod}\,2)$ and it follows from Lemma 2.1 that $(u_{i-1},u_{n+1}=v)\in A(D)$ or $(v,u_{i-1})\in A(D)$. When $(u_{i-1},v)\in A(D)$, the affirmation of Lemma 2.3 follows from Claim 1. When $(v,u_{i-1})\in A(D)$ we obtain $C_4=(u_{i-1},u_i,u_{j_0}=u_n,v,u_{i-1})$ a directed cycle of length 4 which by hypothesis is monochromatic; in fact C_4 is coloured 1 (as $(u_{i-1},u_i)\in A(T)\cap A(C_4)$); and then $T'=(u,T,u_i)\cup(u_i,u_{j_0}=u_n,u_{n+1}=v)$ is a uv-monochromatic directed path with $\ell(T')\leqslant n$. Now it follows from the inductive hypothesis that $(u,v)\in A(D)$ or there exists a uv-directed path of length 2.

Case a.4: $j_0\leqslant n-1$.

$i\not\equiv j_0+2\,(\mathrm{mod}\,2)$ (as $i\not\equiv j_0\,(\mathrm{mod}\,2)$), so it follows from Lemma 2.1 that $(u_i,u_{j_0+2})\in A(D)$ or $(u_{j_0+2},u_i)\in A(D)$; now the choice of j_0 implies $(u_{j_0+2},u_i)\in A(D)$. Thus $C_4=(u_i,u_{j_0},u_{j_0+1},u_{j_0+2},u_i)$ is a directed cycle of length 4 which by hypothesis is monochromatic and coloured 1 (as $(u_{j_0},u_{j_0+1})\in A(T)\cap A(C_4)$); in particular (u_i,u_{j_0}) is coloured 1 and then $T'=(u,T,u_i)\cup(u_i,u_{j_0})\cup(u_{j_0},T,v)$ is a uv-monochromatic directed path with $\ell(T')\leqslant n$ and the inductive hypothesis implies $(u,v)\in A(D)$ or there exists a uv-directed path of length 2.

Case b: For each $i\in\{0,1,\ldots,n-2\}$, $(u_{i+3},u_i)\in A(D)$.

$C_4^i=(u_i,u_{i+1},u_{i+2},u_{i+3},u_i)$ is a directed cycle of length 4 and by hypothesis it is monochromatic, moreover C_4^i is coloured 1 because $(u_i,u_{i+1})\in(A(T)\cap A(C_4^i))$, hence for each $i\in\{0,1,\ldots,n-2\}$, (u_{i+3},u_i) is coloured 1. Let $k\in\{1,2,3\}$ such that $k\equiv n+1\,(\mathrm{mod}\,3)$, then $(v=u_{n+1},u_{n-2},u_{n-5},\ldots,u_k)\cup(u_k,T,u_3)\cup(u_3,u_0)$ is a vu-monochromatic directed path, contradicting the hypothesis, thus this case is impossible. □

Theorem 2.1. *Let* D *be an* m-*coloured bipartite tournament. If every directed cycle of length* 4 *in* D *is monochromatic, then* $\mathscr{C}(D)$ *is kernel-perfect.*

Proof. During the proof we will use the fact that each closed directed walk of length at most 6 is a directed cycle (Lemma 2.2) without any more explanation.

In view of Theorem 1.1 it suffices to prove (and we will prove) that each directed cycle of $\mathscr{C}(D)$ has a symmetrical arc.

We proceed by contradiction; suppose that there exists a directed cycle of $\mathscr{C}(D)$, $C=(u_0,u_1,\ldots,u_n,u_0)$ with $C\subseteq \operatorname{Asym}\mathscr{C}(D)$.

Claim 2. *For each* $i\in\{0,1,\ldots,n\}$, $(u_i,u_{i+1})\in A(D)$ *or there exists a* u_iu_{i+1}*-directed path of length* 2 (*notation* mod $n+1$).

Let $i\in\{0,1,\ldots,n\}$. Since $(u_i,u_{i+1})\in A(\mathscr{C}(D))$ we have that there exists a u_iu_{i+1}-monochromatic directed path in D, and the fact that C has no symmetrical arcs implies there is no $u_{i+1}u_i$-monochromatic directed path in D, so Claim 2 follows from Lemma 2.3.

Now we consider two possible cases:

Case a: $n=2$.

Since D has no odd directed cycles, we have that for some $i\in\{0,1,2\}$, $(u_i,u_{i+1})\notin A(D)$ (notation (mod 3)). W.l.o.g we may assume $(u_0,u_1)\notin A(D)$, then it follows from Claim 2 that there exists a u_0u_1-directed path of length 2 in D, say (u_0,v_0,u_1).

Case a.1: $\{(u_1,u_2),\ (u_2,u_0)\}\subseteq A(D)$.

In this case (u_0,v_0,u_1,u_2,u_0) is a directed cycle of length 4 in D, which by hypothesis is monochromatic; and then (u_1,u_2,u_0) is a u_1u_0-monochromatic directed path in D; thus (u_0,u_1) is a symmetrical arc of C in $\mathscr{C}(D)$, contradicting our assumption.

Case a.2: $\{(u_1,u_2),(u_2,u_0)\}\nsubseteq A(D)$.

Claim 3. *We may assume* $\{(u_1,u_2),(u_2,u_0)\}\cap A(D)=\emptyset$.

If $(u_1,u_2)\notin A(D)$ then $(u_2,u_0)\notin A(D)$; since $(u_1,u_2)\notin A(D)$ it follows from Claim 2 that there exists a u_1u_2-directed path of length 2, say (u_1,v_1,u_2), so when $(u_2,u_0)\in A(D)$ we obtain $(u_0,v_0,u_1,v_1,u_2,u_0)$ a directed cycle of length five contained in D which is impossible. Analogously it can be proved that: If $(u_2,u_0)\notin A(D)$ then $(u_1,u_2)\notin A(D)$.

Now it follows from Claim 2 that there exists a u_1u_2-directed path of length 2 in D, say (u_1,v_1,u_2), and a u_2u_0-directed path of length 2 in D, say (u_2,v_2,u_0). Thus $(u_0,v_0,u_1,v_1,u_2,v_2,u_0)$ is a directed cycle of length 6 in D; and it follows from Lemma 2.1 that $(u_0,v_1)\in A(D)$ or $(v_1,u_0)\in A(D)$. When $(u_0,v_1)\in A(D)$ we obtain (u_0,v_1,u_2,v_2,u_0) is a directed cycle of length 4 in D and by hypothesis it is monochromatic, in particular (u_0,v_1,u_2) is a u_0u_2-monochromatic directed path in D which implies (u_2,u_0) is a symmetrical arc of C in $\mathscr{C}(D)$ contradicting our assumption. When $(v_1,u_0)\in A(D)$ we have (u_0,v_0,u_1,v_1,u_0) is a directed cycle of length 4 in D and by hypothesis is monochromatic, thus (u_1,v_1,u_0) is a u_1u_0-monochromatic directed path in D and then (u_0,u_1) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction.

Case b: $n\geqslant 3$.

In what follows the notation is taken modulo $n+1$.

In view of Claim 2, for each $i\in\{0,1,\ldots,n\}$ we can take a u_iu_{i+1}-directed path as follows:

$$T_i=\begin{cases}(u_i,u_{i+1})\text{ when }(u_i,u_{i+1})\in A(D)\text{ and}\\ \text{a }u_iu_{i+1}\text{-directed path of length 2 when }(u_i,u_{i+1})\notin A(D).\end{cases}$$

Let $C'=\bigcup_{i=1}^{n}T_i$. Then C' is a closed directed walk in D, so we may let $C'=(z_0,z_1,\ldots,z_k,z_0)$ and define the function $\varphi:\{0,1,\ldots,k\}\to V(C)$ as follows: For each $i\in\{0,1,\ldots,n\}$ if $T_i=(u_i=z_{i_0},z_{i_0+1}=u_{i+1})$ then $\varphi(i_0)=z_{i_0}=u_i$; and if $T_i=(u_i=z_{i_0},z_{i_0+1},z_{i_0+2}=u_{i+1})$ then $\varphi(i_0)=\varphi(i_0+1)=z_{i_0}$.

We will say that an index $j\in\{0,1,\ldots,k\}$ is a principal index when $\varphi(j)=z_j$; and we will denote by I_p the set of principal indexes. Notice that in C' the indexes are all different and also notice that a vertex u_j may correspond to a principal index ℓ and also to a non principal index p.

Suppose w.l.o.g. that $u_0=z_0$. Since D is a bipartite tournament, we have $k\equiv 1\,(\mathrm{mod}\,2)$ and by Lemma 2.1, for each $i\in\{1,\ldots,\frac{k-3}{2}\}$ $(z_0,z_{2i+i})\in A(D)$ or $(z_{2i+1},z_0)\in A(D)$. We consider the following cases:

Case b.1: $(z_3,z_0)\in A(D)$.

In this case we have (z_0,z_1,z_2,z_3,z_0) is a directed cycle of length 4 and by hypothesis is monochromatic. The definition of C' implies $z_1=u_1$ or $z_2=u_1$. If $z_1=u_1$ then $(u_1=z_1,z_2,z_3,z_0=u_0)$ is a u_1u_0-monochromatic directed path in D which implies that (u_0,u_1) is a symmetrical arc of C in $\mathscr{C}(D)$, contradicting our assumption on C. So $z_1\neq u_1$, consequently $z_2=u_1$ and then $(u_1=z_2,z_3,z_0=u_0)$ is a u_1u_0-monochromatic directed path in D, thus (u_0,u_1) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction.

Case b.2: $(z_0,z_{k-2})\in A(D)$.

The assumption in subcase b.2 implies $(z_0,z_{k-2},z_{k-1},z_k,z_0)$ is a directed cycle of length 4 which by hypothesis is monochromatic. The construction of C' implies that $z_k=u_n$ or $z_{k-1}=u_n$. When $z_k=u_n$ we have that $(u_0=z_0,z_{k-2},z_{k-1},z_k=u_n)$

is a u_0u_n-monochromatic directed path in D which implies that (u_n, u_0) is a symmetrical arc of C in $\mathscr{C}(D)$, contradicting our assumption. Hence $z_k \neq u_n$ and then $z_{k-1} = u_n$; now $(u_0 = z_0, z_{k-2}, z_{k-1} = u_n)$ is a u_0u_n-monochromatic directed path in D which implies that (u_n, u_0) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction.

*Case b.*3: $(z_0, z_3) \in A(D)$ and $(z_{k-2}, z_0) \in A(D)$.

Since $\{(z_0, z_1), (z_0, z_3), (z_{k-2}, z_0)\} \subseteq A(D)$ we have $k-2 \geqslant 5$ and there exists $j \in \{1, \ldots, \frac{k-5}{2}\}$ such that $(z_0, z_{2j+1}) \in A(D)$ and $(z_{2j+3}, z_0) \in A(D)$. Let $i_0 = \min\{ j \in \{1, \ldots, \frac{k-5}{2}\} \mid \{(z_0, z_{2j+1}), (z_{2j+3}, z_0)\} \subseteq A(D)\}$. Hence $\tilde{C} = (z_0, z_{2i_0+1}, z_{2i_0+2}, z_{2i_0+3}, z_0)$ is a directed cycle of length 4 in D which by hypothesis is monochromatic. Now we consider two possible cases.

*Case b.*3.1: $2i_0 + 1 \in I_p$.

In this case $z_{2i_0+1} = u_j$ for some $j \in \{2, \ldots, n-2\}$ (as $3 \leqslant 2i_0 + 1 \leqslant k-4$). By the construction of C' we have $z_{2i_0+2} = u_{j+1}$ or $z_{2i_0+3} = u_{j+1}$. If $z_{2i_0+2} = u_{j+1}$ then $(u_{j+1} = z_{2i_0+2}, z_{2i_0+3}, z_0, z_{2i_0+1} = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path in D which implies that (u_j, u_{j+1}) is a symmetrical arc of C in $\mathscr{C}(D)$ contradicting our assumption. Hence $z_{2i_0+2} \neq u_{j+1}$ and consequently $z_{2i_0+3} = u_{j+1}$ thus $(u_{j+1} = z_{2i_0+3}, z_0, z_{2i_0+1} = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path in D and then (u_j, u_{j+1}) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction.

*Case b.*3.2: $2i_0 + 1 \notin I_p$.

Now, by construction of C' we have that $\{2i_0, 2i_0 + 2\} \subseteq I_p$, i.e. $z_{2i_0} = u_{j-1}$ and $z_{2i_0+2} = u_j$ for some $j \in \{2, \ldots, n-1\}$. Lemma 2.1 implies $(z_{2i_0}, z_{2i_0+3}) \in A(D)$ or $(z_{2i_0+3}, z_{2i_0}) \in A(D)$. When $(z_{2i_0+3}, z_{2i_0}) \in A(D)$ we obtain that $(z_{2i_0}, z_{2i_0+1}, z_{2i_0+2}, z_{2i_0+3}, z_{2i_0})$ is a directed cycle of length 4 and by hypothesis is monochromatic; thus $(u_j = z_{2i_0+2}, z_{2i_0+3}, z_{2i_0} = u_{j-1})$ is a u_ju_{j-1}-monochromatic directed path and (u_{j-1}, u_j) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction. So we have $(z_{2i_0}, z_{2i_0+3}) \in A(D)$; observe that the choice of i_0 implies $(z_0, z_{2i_0-1}) \in A(D)$ (when $(z_{2i_0-1}, z_0) \in A(D)$, the fact $(z_0, z_1) \in A(D)$ implies that there exists $j \leqslant i_0 - 2$ such that $(z_0, z_{2j+1}) \in A(D)$ and $(z_{2j+3}, z_0) \in A(D)$ contradicting the choice of i_0), thus $C'' = (z_0, z_{2i_0-1}, z_{2i_0}, z_{2i_0+3}, z_0)$ is a directed cycle of length 4 which by hypothesis must be monochromatic; since $(z_{2i_0+3}, z_0) \in A(\tilde{C}) \cap A(C'')$ we have that $\tilde{C}$ and C'' are of the same colour; so $(u_j = z_{2i_0+2}, z_{2i_0+3}, z_0, z_{2i_0-1}, z_{2i_0} = u_{j-1})$ is a monochromatic directed path in D and (u_{j-1}, u_j) is a symmetrical arc of C in $\mathscr{C}(D)$, a contradiction. $\square$

The following result is a direct consequence of Theorem 2.1:

Theorem 2.2. *Let D be an m-coloured bipartite tournament. If every directed cycle of length 4 in D is monochromatic, then D has a kernel by monochromatic paths.*

Remark 2.1. The hypothesis that every directed cycle of length 4 is monochromatic in Theorem 2.2 is tight.

Let D be the 3-coloured bipartite tournament defined as follows:

$V(D) = \{u, v, w, x, y, z\}$ and $A(D) = \{(u, x), (x, v), (v, y), (y, w), (w, z), (z, u), (x, w), (y, u), (z, v)\}$; the arcs $(x, w), (w, z)$ and (z, u) are coloured 1; the arcs $(y, u), (u, x)$ and (x, v) are coloured 2; and the arcs $(z, v), (v, y)$ and (y, w) are coloured 3. The only directed cycles of length 4 of D are $(u, x, w, z, u), (v, y, u, x, v)$ and (w, z, v, y, w) which are quasi-monochromatic and the digraph $\mathscr{C}(D)$ is a complete digraph which has no kernel; hence D has no kernel by monochromatic paths. Moreover, we can construct an infinite family of digraphs all of whose directed cycles of length 4 are quasi-monochromatic and which have no kernel by monochromatic paths as follows: Let D_n be the digraph obtained from D by adding vertices $z_1, z_2, \ldots, z_n$ and arcs coloured 3 from each one of these vertices to u, v and w, respectively.

Remark 2.2. The assumption that every directed cycle of length 4 in a bipartite tournament D is monochromatic, does not imply that every directed cycle of length 6 in D is monochromatic.

Remark 2.3. For each m there exists an m-coloured Hamiltonian bipartite tournament such that every directed cycle of length 4 is monochromatic.

Proof. Let D be the m-coloured digraph defined as follows:

$$V(D) = X \cup Y \cup Z \cup W \quad \text{where;} \quad X = \{x_1, x_2, \ldots, x_m\}, \quad Y = \{y_1, y_2, \ldots, y_m\}$$

$$Z = \{z_1, z_2, \ldots, z_m\}, \quad W = \{w_1, w_2, \ldots, w_m\}.$$

$$A(D) = X_Y \cup Y_z \cup Z_W \cup W_X \cup Z_Y \cup W_Z \cup X_W \quad \text{where:}$$

$$X_Y = \{(x_i, y_j) \mid i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, \ldots, m\}\},$$

$$Y_Z = \{(y_i, z_i) \mid i \in \{1, 2, \ldots, m\}\}, \quad Z_W = \{(z_i, w_i) \mid i \in \{1, 2, \ldots, \ldots, m\}\},$$

$$W_X = \{(w_i, x_{i+1}) \mid i \in \{1, 2, \ldots, m-1\}\} \cup \{(w_m, x_1)\},$$

$$Z_Y = \{(z_i, y_j) \mid i \in \{1,2,\ldots,m\},\ j \in \{1,2,\ldots,\ldots,m\},\ i \neq j\},$$

$$W_Z = \{(w_i, z_j) \mid i \in \{1,2,\ldots,m\},\ j \in \{1,2,\ldots,\ldots,m\},\ i \neq j\},$$

$$X_W = \{(x_i, w_j) \mid i \in \{1,2,\ldots,m\},\ j \in \{1,2,\ldots,\ldots,m\},\ i \neq j+1\},$$

(notation mod m).

For each $i \in \{1,2,\ldots,m\}$ the arc (x_i, y_i) is colored i and any other arc is coloured 1.

Clearly D is an m-coloured bipartite tournament. □

Claim 3. *D is Hamiltonian. It follows from the definition of D that for each* $i \in \{1,2,\ldots,m\}$ *we have the directed path* $T_i = (x_i, y_i, z_i, w_i, x_{i+1})$ *and clearly* $V(T_i) \cap V(T_j) = \emptyset$ *for* $j \neq i+1$, *and* $V(T_i) \cap V(T_{i+1}) = \{x_{i+1}\}$. *So* $C = \bigcup_{i=1}^{m} T_i$ *is a Hamiltonian directed cycle of D.*

Claim 4. *Every directed cycle of length* 4 *of D is monochromatic. Proceeding by contradiction, suppose that* $C_4 = (u_1, u_2, u_3, u_4, u_1)$ *is a non monochromatic directed cycle of D, so* C_4 *must contain at least one arc coloured i for some* $i \in \{2,\ldots,m\}$, *so we may assume that* $u_1 = x_2$ *and* $u_2 = y_2$; *it follows from the definition of D that* $u_3 = z_2$ *and* ($u_4 = w_2$ *or* $u_4 = y_i$ *for some* $i \neq 2$). *When* $u_4 = w_2$, *we obtain that* $(x_2, w_2) \in A(D)$ *and hence* $(w_2, x_2) \notin A(D)$, *a contradiction. When* $u_4 = y_i$ *for some* $i \neq 2$ *we obtain that* $(x_2, y_i) \in A(D)$ *contradicting that* $(u_4 = y_i, u_1 = x_2) \in A(D)$.

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