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Complete hypersurfaces in R^{2n+2} with constant negative $2n$-th curvature

By

Oscar Palmas

Abstract. We use equivariant geometry methods to show the existence of complete hypersurfaces in euclidean spaces with constant negative $2n$-th curvature.

Introduction. A classic theorem by Hilbert states that there are no complete surfaces in R^3 with constant negative gaussian curvature. Attempting to generalize this theorem to higher dimensions, it is natural to analyze first some examples of hypersurfaces in euclidean space with constant negative r-th curvature σ_r, σ_r being the r-th symmetric function of the principal curvatures of the hypersurface. As this universe is still large, we restrict further to those such hypersurfaces which are invariant under a given group of isometries of the euclidean space. Following the classification of the low cohomogeneity isometry groups given by W. Y. Hsiang and H. B. Lawson in [2], the author studied rotational (i.e., $SO(n)$-invariant) hypersurfaces in space forms in [5], and concluded, on one hand, that for each r odd and each $\sigma < 0$ there exists a 1-parameter family of complete rotational hypersurfaces in R^{n+1} with $\sigma_r = \sigma$; and, on the other hand, that there are no complete rotational hypersurfaces with constant negative σ_r for r even. It should be mentioned that these results generalize those obtained by M. L. Leite in [3] and J. Hounie and Leite in [1].

This result would suggest a generalization of Hilbert's theorem for constant negative σ_r and r even, but T. Okayasu [4] built the first example of a complete hypersurface in R^4 with constant negative scalar curvature (so that $\sigma_2 < 0$), using the cohomogeneity two isometry group $O(2) \times O(2)$. Here we extend Okayasu's result to higher dimensions. More precisely, we prove

Theorem. *Given an integer $n > 1$ and σ such that $\sigma \leqq -2n$, there exists a complete hypersurface in R^{2n+2} with constant negative $\sigma_{2n} = \sigma$.*

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1. Notation and results. We will use the standard action of $\mathrm{SO}(n) \times \mathrm{SO}(n)$ over $R^{2n+2} = R^{n+1} \times R^{n+1}$. In this case, the orbit space can be identified with $Q = \{(x, y);\ x \geqq 0, y \geqq 0\}$, in such a way that every interior point of Q corresponds to a principal orbit given as the product of spheres $S^n(x) \times S^n(y)$. We may define a hypersurface M of R^{2n+2} invariant under this action by giving a generating curve $\gamma(s) = (x(s), y(s))$, parametrized by arc length s and contained in Q, so that M is parametrized by

$$X(s, \phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n) = (x(s)\Phi(\phi_1, \dots, \phi_n), y(s)\Psi(\psi_1, \dots, \psi_n))$$

where Φ and Ψ are orthogonal parametrizations of a unit n-dimensional sphere.

Using the parametrization X, it can be shown that the principal curvatures $\kappa_0, \kappa_i, \kappa_j$, $i = 1, \dots, n$, $j = n+1, \dots, 2n$ associated to M are:

$$\kappa_0 = \dot{x}\ddot{y} - \dot{y}\ddot{x}, \quad \kappa_i = \frac{\dot{y}}{x} \quad \kappa_j = -\frac{\dot{x}}{y}$$

where the dot denotes the derivative with respect to s.

Recall that σ_{2n} is the $2n$-th symmetric function of the principal curvatures of M, namely,

$$\sigma_{2n} = \sum_{0 \leqq i_1 < i_2 < \dots < i_{2n} \leqq 2n} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2n}}$$

so that the expression for the curvature σ_{2n} of M in the given parametrization is

$$\sigma_{2n} = \left(-\frac{\dot{x}\dot{y}}{xy}\right)^{n-1}\left(n(\dot{x}\ddot{y} - \dot{y}\ddot{x})\left(\frac{\dot{y}}{x} - \frac{\dot{x}}{y}\right) - \frac{\dot{x}\dot{y}}{xy}\right). \tag{1}$$

Hereafter, we will suppose that σ_{2n} is constant and negative and, for brevity, write σ instead.

Suppose further that y may be written as a function of x. Using y', y'' to denote the derivatives of y with respect to x and using the facts $\dot{x}^2 + \dot{y}^2 = 1$, $\dot{y} = y'\dot{x}$, so that $\dot{x}^2(1 + (y')^2) = 1$ and $\ddot{y} = y''\dot{x}^2 + y'\ddot{x}$, we express (1) as a differential equation on y, namely,

$$\sigma(xy)^n(1 + (y')^2)^{n+1} = (-y')^{n-1}(ny''(yy' - x) - y'(1 + (y')^2)). \tag{2}$$

Denote by $f_\sigma = f_\sigma(x)$, $x \geqq 1$ the solution of (2) satisfying

$$y(1) = 1 \quad \text{and} \quad y'(1) = -1. \tag{3}$$

The study of this solution is the main objective of this note.

Note that the initial value problem (2)–(3) satisfies the usual conditions for existence and uniqueness of their solutions, as long as $y' \neq 0$ and $yy' - x \neq 0$. When $y' = 0$, we obtain directly from (2) that $y = 0$; on the other hand, as the expression $yy' - x$ is negative at $x = 1$, if there is a point for which this expression vanishes, then there must be also another point where $yy' = 0$. So, in both cases we may suppose the existence of a first point $x_\sigma > 1$ such that $f_\sigma(x_\sigma) = 0$ and

$$(4) \qquad 0 < f_\sigma(x) < 1 \quad \text{and} \ -1 < f'_\sigma(x) < 0$$

for every $x \in (1, x_\sigma)$. Note that in this interval we may write

$$(5) \qquad y'' = \frac{1}{n}(1 + (y')^2)(y' + \sigma(xy)^n(1 + (y')^2)^n(-y')^{-(n-1)})(yy' - x)^{-1}$$

and also note that $y'' > 0$ in $(1, x_\sigma)$, since the last two factors are negative.

Using (4) we may estimate (5) as follows:

$$\begin{aligned} y'' &\geqq \frac{1}{n}(y' + \sigma(xy)^n(-y')^{-(n-1)})(yy' - x)^{-1} \\ &\geqq \frac{\sigma}{n}(xy)^n(-y')^{-(n-1)}(yy' - x)^{-1} \geqq \frac{\sigma}{n}y^n(-y')^{-(n-1)}\frac{x}{yy' - x} \\ &\geqq -\frac{\sigma}{2n}y^n(-y')^{-(n-1)} \geqq y^n(-y')^{-(n-1)} \end{aligned}$$

whenever $\sigma \leqq -2n$. We compare f_σ with the solution h to the initial-value problem

$$(6) \qquad y'' = y^n(-y')^{-(n-1)}, \qquad y(1) = 1, \qquad y'(1) = -1,$$

which is given explicitly by $h(x) = e^{-(x-1)}$. We shall prove that $f_\sigma \geqq h$ on $(1, x_\sigma)$. Let A_σ be the set of $x \in (1, x_\sigma)$ for which

$$(7) \qquad f_\sigma(t) > h(t) \quad \text{and} \quad f'_\sigma(t) > h'(t) \qquad \text{for each } t \in (1, x).$$

Note that

$$f''_\sigma(1) = \frac{1 - 2^n\sigma}{n} > 2 = h''(1).$$

By continuity, $f''_\sigma > h''$ holds at least in a small interval $(1, x)$. Integrating over $[1, x]$ we have

$$f'_\sigma(t) - f'_\sigma(1) > h'(t) - h'(1)$$

which implies $f'_\sigma > h'$ on $(1, x]$. Similarly, $f_\sigma > h$ in $(1, x]$, so A_σ is not empty. Let $x_1 = \sup A_\sigma$ and suppose $x_1 < x_\sigma$. Then the *strict* inequalities (7) hold in $(1, x_1)$ and we have

$$f''_\sigma \geqq f^n_\sigma(-f'_\sigma)^{-(n-1)} > h^n(-h')^{-(n-1)} = h''.$$

Integrating and using the initial conditions twice, as before, we have that (7) hold in x_1, so that $x_1 \neq \sup A_\sigma$. This contradiction shows that $x_1 = x_\sigma$.

Suppose now that x_σ is finite. From (7) we have

$$f_\sigma(t) > h(t) \geqq h(x_\sigma).$$

This implies $f_\sigma(x_\sigma) \geqq h(x_\sigma) > 0$, a contradiction with the definition of x_σ. Then x_σ must be infinite and $f_\sigma \neq 0$ in $[1, \infty)$.

Note that f'_σ is also everywhere negative, as $f'_\sigma = 0$ would imply, by (2), that $x = 0$, $f_\sigma = 0$ or $\sigma = 0$ somewhere in $[1, \infty)$.

Proof of the Theorem. We use the function f_σ constructed above. Since $f_\sigma(1) = 1$ and $f'_\sigma(1) = -1$, we can reflect the graph of f_σ with respect to the line $y = x$ to obtain a curve γ contained in the interior of the first quadrant, with infinite length, which gives rise to a complete $SO(n) \times SO(n)$-invariant hypersurface M in R^{2n+2} with $2n$-th curvature equal to σ. □

By taking the standard embedding of R^{2n+2} into R^m for $m \geqq 2n + 2$, we get

Corollary. *Given $n > 1$, $\sigma \leqq -2n$ as above and an integer $m \geqq 2n + 2$, there exists a complete hypersurface in R^m with $\sigma_{2n} = \sigma$.*

Final remarks and questions. The behaviour of our differential equation (2) is quite different from that of Okayasu's ($n = 1$); in this last case, he proved the existence of just one value $\sigma < 0$ for which f_σ (in our notation) defines a complete hypersurface, and conjectured that this value may be the only one with this property. For $n > 1$, we obtain a 1-parameter family of such complete hypersurfaces ($\sigma \leqq -2n$ being the parameter). As the case $n = 1$ shows, our bound $-2n$ may not be optimal, so the existence of complete hypersurfaces with $\sigma_{2n} \in (-2n, 0)$ is still an open question.

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Oscar Palmas
Departamento de Matemáticas
Facultad de Ciencias UNAM
México 04510 DF
México
opv@hp.fciencias.unam.mx