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LINDELÖF Σ-PROPERTY IN $C_p(X)$ AND $p(C_p(X)) = \omega$ DO NOT IMPLY COUNTABLE NETWORK WEIGHT IN X

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Abstract. We prove that there are Tychonoff spaces X for which $p\big(C_p(X)\big) = \omega$ and $C_p(X)$ is a Lindelöf Σ-space while the network weight of X is uncountable. This answers Problem 75 from [4]. An example of a space Y is given such that $p(Y) = \omega$ and $C_p(Y)$ is a Lindelöf Σ-space, while the network weight of Y is uncountable. This gives a negative answer to Problem 73 from [4]. For a space X with one non-isolated point a necessary and sufficient condition in terms of the topology on X is given for $C_p(X)$ to have countable point-finite cellularity.

0. Introduction

The *point-finite cellularity* $p(X)$ of a space X is the supremum of cardinalities of point-finite families of open sets in X [16]. In this paper we study the point-finite cellularity of the spaces $C_p(X)$ of continuous functions endowed with the topology of pointwise convergence.

It is easy to see that always $c(X) \leqq p(X)$, and in fact, if X is a Baire space, then $p(X)$ and $c(X)$ coincide. On the other hand, if τ is a calibre of X, then $p(X) \leqq \tau$.

It is known [16, Theorem 1] that for a space X, always $p(X) = a\big(C_p(X)\big)$ where $a(Z)$ is the supremum of cardinalities of compact subspaces of Z with one nonisolated point; it is also true that $a(X) \leqq p\big(C_p(X)\big)$ [16], but not always $a(X) = p\big(C_p(X)\big)$ (in particular, $p\big(C_p(X)\big)$ is countable if $X = \beta\omega$ [16]). However, if X is a Gul'ko compact space (that is, such that $C_p(X)$ is a Lindelöf Σ-space), then X is metrizable (see [5, Proposition 2.10] and [9, Theorem 2]). Note that the Lindelöf Σ-property of $C_p(X)$ implies that

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X "almost" has this property, (in fact, that the Hewitt extension υX is a Lindelöf Σ-space [12]; see also Section IV.9 in [3]), and that every compact subspace of X is Gul'ko. Arhangel'skiĭ points out in [4, Theorem 11.13], that if ω_1 is a calibre of X and $C_p(X)$ is a Lindelöf Σ-space, then under Martin's Axiom and the negation of the Continuum Hypothesis, the space X has a countable network. Since "ω_1 is a calibre" is a stronger condition than the countability of the point-finite cellularity, this motivates Problem 73 in [4]: *Does MA+¬CH imply the existence of a countable network in a space X with $C_p(X)$ a Lindelöf Σ-space and $p\big(C_p(X)\big) = \omega$?*

Another question is mentioned on page 50 in [4]: *Find necessary and sufficient conditions for $p\big(C_p(X)\big) = \omega$ in terms of the topology of a space X.* One criterion was obtained by Arhangel'skiĭ and Tkachuk for compact spaces [5, Proposition 2.7]. In this paper we give a criterion for the spaces with at most one non-isolated point.

We further present a simple machinery for constructing from a subset B of a real line an uncountable Lindelöf Σ-space $X(B)$ with one non-isolated point so that $C_p\big(X(B)\big)$ is a Lindelöf Σ-space; the above criterion allows to establish that for some particular B we have $p\Big(C_p\big(X(B)\big)\Big) = \omega$. This provides negative answers (in ZFC) to Problems 73 and 75 in [4]. We further show that it is possible to choose a Bernstein set B so that $p\Big(C_p\big(X(B)\big)\Big) \neq \omega$; for this space $a\Big(L_p\big(X(B)\big)\Big) \neq a\Big(C_pC_p\big(X(B)\big)\Big)$. This leads to an example related to joint factorization of sets of continuous functions on $C_p(X)$.

Terminology and notation not explained below are as in [7]. All topological spaces are assumed to be Tychonoff. If X is a space, then $\mathcal{T}(X)$ is its topology, and $\mathcal{T}(x, X) = \{\, U \in \mathcal{T}(X) \,:\, x \in U \,\}$. The space $\mathbf{R}$ is the real line with its natural topology. If X is a space, then $C_p(X)$ is the set of all real-valued continuous functions on X endowed with the topology of pointwise convergence. The last statement means that the sets of the form

$$[x_1, \dots, x_n; O_1, \dots, O_n] = \{\, f \in C_p(X) \,:\, f(x_i) \in O_i \quad i = 1, \dots, n \,\},$$

where $n \in \omega$, $x_1, \dots, x_n \in X$, and $O_1, \dots, O_n$ are open subsets of $\mathbf{R}$ (see [3]), constitute a base in $C_p(X)$. The *point-finite cellularity $p(Z)$ of a space Z* is the supremum of the cardinalities of point-finite families of non-empty open subsets of Z. A space X is called an *Eberlein–Grothendieck* space (shortly, an *EG-space*) if X can be embedded into $C_p(Y)$ for some compact space Y. A space Z is called *ω-monolithic* if the closure of every countable subset of Z has a countable network [3] (see also Section II.6 in [3]). A subspace A of a topological space X is *C-embedded* (*C^*-embedded*) in X if every (bounded) continuous function $g : A \to \mathbf{R}$ has a continuous extension over X.

A space X is called *a Lindelöf Σ-space* if it is a continuous image of a space Y that can be perfectly mapped onto a separable metrizable space.

The symbol $\mathfrak{c}$ denotes the cardinality of the continuum.

1. Countable point-finite cellularity in $C_p(X)$

We start with establishing a criterion that characterizes the inequality $C_p(X) \leqq \tau$ in terms of the topology of a space X when X has a unique non-isolated point.

1.1. THEOREM. *Let X be a space with one non-isolated point: $X = \{a\} \cup Y$, where all points of Y are isolated and $a \notin Y$. For every infinite cardinal τ the following conditions are equivalent:*

(1) *$p\big(C_p(X)\big) \leqq \tau$;*

(2) *if $\{A_\alpha : \alpha < \tau^+\}$ is a disjoint family of finite subsets of Y, then there is an infinite $S \subset \tau^+$ such that $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$;*

(3) *if $\{A_\alpha : \alpha < \tau^+\}$ is a family of finite subsets of Y, then there is an infinite $S \subset \tau^+$ such that $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$.*

PROOF. (1) $\Longrightarrow$ (2). Let $\{A_\alpha : \alpha < \tau^+\}$ be a disjoint family of finite subsets of Y. By an obvious counting argument, we may assume that all A_α have the same cardinality $n \in \omega$; let $A_\alpha = \{x_1^\alpha, \dots, x_\alpha^n\}$. Put

$$V_\alpha = \big[a, x_1^\alpha, \dots, x_n^\alpha; (-1,1), (2,3), \dots, (n+1, n+2)\big]$$
$$= \big\{ f \in C_p(X) : f(a) \in (-1,1), \quad f(x_i^\alpha) \in (i+1, i+2), \quad i = 1, \dots, n \big\}.$$

The sets V_α are non-empty and open in $C_p(X)$; by the condition (1), the family $\{V_\alpha : \alpha < \tau^+\}$ cannot be point-finite, so there is an infinite $S \subset \omega_1$ and a $g \in C_p(X)$ such that $g \in \bigcap\{V_\alpha : \alpha \in S\}$.

The set $W = g^{-1}\big((-1,1)\big)$ is open and contains a. If $\alpha \in S$ then $A_\alpha \cap W = \emptyset$, because otherwise we would have $g(x) \in (-1,1)$ for some $x \in A_\alpha$, while from $g \in V_\alpha$ follows $g(A_\alpha) \subset (1, n-k+2)$. Thus, $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$.

(2) $\Longrightarrow$ (3). Let $\{A_\alpha : \alpha < \tau^+\}$ be a family of finite subsets of Y. By the Δ-lemma there is a finite set $F \subset Y$ and a $T \subset \tau^+$ such that $|T| = \tau^+$ and $A_\alpha \cap A_\beta = F$ whenever $\alpha, \beta \in T$ and $\alpha \neq \beta$. For every $\alpha \in T$ put $A'_\alpha = A_\alpha \setminus F$. Then $\{A'_\alpha : \alpha \in T\}$ is a disjoint uncountable family of finite sets in Y. By the condition (2), there is an infinite $S \subset T$ such that a is not a limit point of $\bigcup\{A'_\alpha : \alpha \in S\}$. Since $\bigcup\{A_\alpha : \alpha \in S\} = F \cup \bigcup\{A'_\alpha : \alpha \in S\}$ and F is finite, a is not a limit point for $\bigcup\{A_\alpha : \alpha \in S\}$.

(3) $\Longrightarrow$ (1). Suppose that $\gamma = \{U_\alpha : \alpha < \tau^+\}$ is a family of non-empty open subsets of $C_p(X)$. Applying the Δ-lemma once more, considering a subfamily of γ and smaller open sets if necessary, we may assume that there exist natural numbers m, n and open non-empty disjoint subsets $W_0, \dots, W_n$, $V_1, \dots V_m$ of $\mathbf{R}$ such that

(i) $U_\alpha = [a, x_1^\alpha, \dots, x_n^\alpha, y_1, \dots y_m; W_0, W_1, \dots, W_n, V_1, \dots, V_m]$ for every $\alpha < \tau^+$;

(ii) the sets $A_\alpha = \{x_1^\alpha, \dots, x_n^\alpha\}$ are disjoint and disjoint from the set $F = \{y_1, \dots, y_m\}$.

By the condition (3), there is an infinite $S \subset \tau^+$ such that $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$. The normality of X and the fact that the set $\{a\} \cup F \cup \bigcup\{A_\alpha : \alpha \in S\}$ is closed and discrete in X imply the existence of a function $g \in C_p(X)$ such that $g(a) \in W_0, g(x_i^\alpha) \in W_i$, $i = 1, \ldots, n$ and $g(y_i) \in V_i$, $i = 1, \ldots, m$ for every $\alpha \in S$. Then $g \in \bigcap\{U_\alpha : \alpha \in S\}$, which shows that the family γ is not point-finite. $\square$

1.2. COROLLARY. *Let X be a space with a unique non-isolated point. If X has no non-trivial convergent sequences, then the point-finite cellularity of $C_p(X)$ is countable.*

PROOF. Let $X = \{a\} \cup Y$ where all points of Y are isolated in X. Suppose that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is a family of disjoint finite subsets of Y. Choosing if necessary an appropriate subfamily of $\mathcal{A}$, we may assume that $|A_\alpha| = n \in \omega$ for every $\alpha < \omega_1$. For every $\alpha < \omega_1$, let $A_\alpha = \{x_1^\alpha, \ldots, x_n^\alpha\}$. Consider the set $B_1 = \{x_1^\alpha : \alpha < \omega_1\}$. Since the set B_1 is infinite and $\{a\} \cup B_1$ contains no convergent sequences, there is an infinite $S_1 \subset \omega_1$ such that a is not a limit point for $\{x_1^\alpha : \alpha \in S_1\}$.

Assume that $k < n$, and we have already constructed infinite subsets $S_1 \supset \ldots \supset S_k$ of ω_1 so that for every $i \leqq k$, a is not a limit point for the set $\{x_i^\alpha : \alpha \in S_i\}$. The set $B_{k+1} = \{x_{k+1}^\alpha : \alpha \in S_k\}$ is infinite and $\{a\} \cup B_{k+1}$ contains no convergent sequences, so there exists an infinite $S_{k+1} \subset S_k$ such that a is not a limit point for B_{k+1}.

At the nth step of this process we obtain an infinite set $S = S_n \subset \omega_1$ such that for every $i \leqq n$, the point a is not in in the closure of the set $D_i = \{x_i^\alpha : \alpha \in S\}$. Clearly, a is not a limit point for $D_1 \cup \ldots \cup D_n$. By Theorem 1.1, we have $p\big(C_p(X)\big) = \omega$. $\square$

1.3. REMARK. It is easy to see from the proof of Theorem 1.1 that (1) $\Longrightarrow$ (2) is true for any space X and any $a \in X$. It is generally not true that (2) $\Longrightarrow$ (1). Indeed, the condition (2) holds in any first countable space X: given a family $\mathcal{A}$ of finite subsets not containing a, for every $A \in \mathcal{A}$ we can find a basic neighborhood of a disjoint from A. If $\mathcal{A}$ is uncountable, some basic neighborhood of a avoids uncountably many elements of $\mathcal{A}$.

There exist first countable compact spaces X such that the point-finite cellularity of $C_p(X)$ is uncountable. For example, if X is the Alexandroff double circumference, then $p\big(C_p(X)\big) > \omega$, because $C_p(X)$ is homeomorphic to $C_p(Y)$ where Y is the discrete union of the circumference and the Alexandroff one-point compactification of the discrete space of cardinality $\mathfrak{c}$ [15, Corollary 1]. Therefore $p\big(C_p(X)\big) = p\big(C_p(Y)\big) = a(Y) = \mathfrak{c}$.

Let us introduce for any infinite cardinal τ a property of a space X which implies that the point-finite cellularity of $C_p(X)$ is $\leqq \tau$.

1.4. DEFINITION. We say that a family $\mathcal{A}$ of finite subsets of X is *concentrated* if there is no infinite $\mathcal{A}' \subset \mathcal{A}$ such that the set $\bigcup \mathcal{A}'$ is discrete (in itself) and C^*-embedded in X.

Thus, Theorem 1.1 says that if X has one non-isolated point, then $p\bigl(C_p(X)\bigr) \leqq \tau$ if and only if every concentrated family of finite sets in X has cardinality $\leqq \tau$.

1.5. PROPOSITION. *Let τ be an infinite cardinal. If every concentrated family of finite sets in X has cardinality $\leqq \tau$, then $p\bigl(C_p(X)\bigr) \leqq \tau$.*

PROOF. Suppose that $\mathcal{A} = \{U_\alpha : \alpha < \tau^+\}$ is a family of non-empty open subsets of $C_p(X)$. Considering a subfamily of $\mathcal{A}$ and smaller open sets if necessary, we may assume that the family $\mathcal{A}$ has the following properties:

(i) there is a natural number n and open non-empty disjoint intervals $W_1, \ldots, W_n$ of $\mathbf{R}$ such that $U_\alpha = [x_1^\alpha, \ldots, x_n^\alpha; W_1, \ldots, W_n]$ for every $\alpha < \tau^+$;

(ii) there is an $F \subset X$ such that $\alpha < \beta < \tau^+$ implies $A_\alpha \cap A_\beta = A$ where $A_\alpha = \{x_1^\alpha, \ldots, x_n^\alpha\}$ for each $\alpha < \tau$;

Choose an infinite $S \subset \tau^+$ so that the set $A_S = \bigcup\{A_\alpha : \alpha \in S\}$ is discrete and C^*-embedded in X. For every $i \leqq n$ pick $t_i \in W_i$ and define the function $h : A_S \to \mathbf{R}$ by $h(x_i^\alpha) = t_i$ for all $\alpha \in S$. It is easy to see that h is a well-defined bounded function on A_S (continuous, because A_S is discrete). Let $h_1 \in C_p(X)$ be a continuous extension of h. Then $h_1 \in \bigcap\{U_\alpha : \alpha \in S\}$, whence $\mathcal{A}$ is not point-finite. $\square$

It was proved in [5] that if X is a dyadic or Gul'ko compact space and $p\bigl(C_p(X)\bigr) = \omega$, then X is metrizable. The next example shows that it is consistent with ZFC that this is not true if X is a perfectly normal compact space. It is worth mentioning that this appears to be the first example of a compact non-metrizable space X with $p\bigl(C_p(X)\bigr) = \omega$ that is not extremally disconnected [5].

1.6. EXAMPLE. The Jensen's principle ($\diamondsuit$) implies the existence of a perfectly normal non-metrizable compact space X such that $p\bigl(C_p(X)\bigr) = \omega$.

Namely, assuming $\diamondsuit$, Ivanov constructed an example of a perfectly normal nonmetrizable compact space X such that X^n is hereditarily separable for all $n \in \omega$ [10, Theorem 2]. By a theorem of Zenor [17], the space $C_p(X)$ is hereditarily Lindelöf. In particular, $C_p(X)$ has no uncountable discrete subspaces. By Proposition 1 in [16], the point-finite cellularity of $C_p(X)$ is countable. $\square$

1.7. REMARK. The compact space X in 1.6 has uncountable concentrated families of one-point sets. Indeed, a subset of a first countable compact space is C^*-embedded if and only if it is closed. Since there are no infinite closed discrete sets in X, any uncountable family of one-point sets is concentrated.

2. The spaces $X(B)$

If M is a space and $A \subset M$, then M_A is the set M with the same topology at the points of A and with all points of $M \setminus A$ isolated (see e.g., [7, 5.1.22]). A set $B \subset I = [0,1]$ is called a *Bernstein set* if every compact subset of B is countable, or, equivalently, if $I_{I \setminus B}$ is a Lindelöf space. In [14] it is proved that there is a set $A \subset I$ such that $(I_A)^\omega$ is Lindelöf (such sets are called *holding* in [14]) and $B = I \setminus A$ has cardinality $\mathfrak{c}$, and that for every $n \leqq \omega$ there is a set $A_n \subset I$ such that $(I_{A_n})^m$ is Lindelöf for all $m < n$, but $(I_{A_n})^n$ has a closed discrete subset of cardinality $\mathfrak{c}$.

Let B be a subspace of I. Let $AD(B)$ be the Alexandroff duplicate of B (see [8]; we use the notation $AD(B)$ rather than $A(B)$ to avoid collision with the notation for the one-point compactification of a discrete set). Thus, $AD(B) = B \times \{0,1\}$, all points of $B \times \{1\}$ are isolated in $AD(B)$, and for every $b \in B$, the sets of the form $W(U) = \big(U \times \{0\}\big) \cup \big(U \times \{1\}\big) \setminus \big\{(b,1)\big\}$ where U is an open neighborhood of b in B, form a base of open neighborhoods of $AD(B)$ at $(b,0)$. We denote $B_0 = B \times \{0\}$ and $B_1 = B \times \{1\}$. Obviously, B is homeomorphic to the subspace B_0 of $AD(B)$, and B_1 is a dense, open discrete set in $AD(B)$. It is easy to see that $AD(B)$ is first-countable (recall that B is a subset of $I = [0,1]$).

Let $r : AD(B) \to B$ be the projection, $r(b,0) = r(b,1) = b$ for every $b \in B$. It is easy to check that r is a perfect map. Define $r_0 : AD(B) \to B_0$ by $r_0(x) = \big(r(x), 0\big)$. Obviously, r_0 is a retraction of $AD(B)$ onto B_0, and the mapping r_0 is homeomorphic to r.

2.1. Lemma. *A subset $P \subset B_1$ is closed in $AD(B)$ if and only if $r(P)$ is closed and discrete in B.*

Proof. If P is closed in $AD(B)$, then $r(P)$ is closed and discrete in B as the image of a closed discrete set under a perfect map. Conversely, if $r(P)$ is closed and discrete, then $P \cup r_0(P) = r^{-1}\big(r(P)\big)$ is closed in $AD(B)$. Therefore, all limit points of P are in $r_0(P)$. Take any $y \in r(P)$. There is an open neighborhood U of y in B such that $U \cap r(P) = \{y\}$. Then $W(U)$ is an open neighborhood of $y \times \{0\}$ in $AD(B)$ disjoint from P. This proves that P is closed and discrete. $\square$

2.2. Lemma. *$AD(B)$ is a Lindelöf Σ-space.*

Indeed, r is a perfect mapping of $AD(B)$ onto the separable metrizable space B.

Let $X(B)$ be the space obtained by identifying to a point the set B_0 in $AD(B)$ (formally, $X(B)$ is the quotient space of $AD(B)$ corresponding to the decomposition $\big\{B_0, \{x\} : x \in B_1\big\}$). Let $p : AD(B) \to X(B)$ be the natural projection; obviously, p is closed, and all points of $X(B)$ except the point $x_* = B_0$ are isolated.

2.3. Proposition. *$X(B)$ is a Lindelöf Σ-space.*

This follows from Lemma 2.2 and the fact that $X(B)$ is a continuous image of $AD(B)$.

2.4. PROPOSITION. *$C_p(X(B))$ is a Lindelöf Σ-space.*

PROOF. The fact that B is an EG-space (it is homeomorphic to a subspace of $C_p(K)$ where K is a singleton) and Proposition 1.11 in [13] imply that $AD(B)$ is an EG-space. By Corollary 2.11 in [13], the space $C_p(AD(B))$ is a Lindelöf Σ-space.

Since $X(B)$ is a quotient of the space $AD(B)$, the space $C_p(X(B))$ is homeomorphic to a closed subspace of $C_p(AD(B))$ (see e.g. Corollary 0.4.8 in [3]). The statement of the proposition now follows from the fact that the class of Lindelöf Σ-spaces is hereditary with respect to closed subspaces. $\square$

Obviously, $nw(X(B)) = |X(B)| = |B|$. The following statement gives a direct description of the neighborhoods of x_* in $X(B)$.

2.5. PROPOSITION. *Suppose that $x_* \in V \subset X(B)$, and $F = X(B) \setminus V$. Then V is a neighborhood of x_* in $X(B)$ if and only if the set $r(p^{-1}(F))$ is closed and discrete in B.*

PROOF. If V is a neighborhood of x_*, then $F = X(B) \setminus V$ is closed; since p is continuous, $p^{-1}(F)$ is closed in $AD(B)$. Since $x_* \notin F$, we have $p^{-1}(F) \subset B_1$. By Lemma 2.1, the set $r(p^{-1}(F))$ is closed and discrete in B.

Conversely, if $r(p^{-1}(F))$ is closed and discrete in B, then by Lemma 2.1, the set $p^{-1}(F)$ is a closed subset of $AD(B)$, and F is closed in $X(B)$ because p is quotient. $\square$

Thus, $X(B)$ may be viewed as the union of B and a singleton $\{x_*\}$ endowed with the topology in which all points of B are isolated and a set containing x_* is open if and only if its complement is closed and discrete in the original topology of B.

2.6. PROPOSITION. *If every compact subspace of B is countable, then every compact subspace of $X(B)$ is countable.*

PROOF. It is sufficient to verify that every uncountable set C in $X(B)$ such that $x_* \notin C$ contains an infinite subset closed in $X(B)$. By Proposition 2.5, this is equivalent to a statement that every uncountable subset of B contains an infinite closed discrete set, which is obviously true given that every compact set in B is at most countable. $\square$

Let $A = I \setminus B$.

PROPOSITION 2.7. *If for every $n \in \omega$ the space $(I_A)^n$ is Lindelöf, then $p(C_p(X(B))) = \omega$.*

PROOF. Suppose that for every $n \in \omega$ the space $(I_A)^n$ is Lindelöf. To prove $p(C_p(X(B))) = \omega$, by Theorem 1.1, it is enough to prove that for every disjoint family $\{A_\alpha : \alpha < \omega_1\}$ of finite sets in $X(B) \setminus \{x_*\}$ there is an infinite $S \subset \omega_1$ such that x_* is not a limit point of $\bigcup\{A_\alpha : \alpha \in S\}$.

Without loss of generality, there is an $n \in \omega$ such that $|A_\alpha| = n$ for all $\alpha < \omega_1$. Let $A_\alpha = \{x_1^\alpha, \dots, x_n^\alpha\}$ for each $\alpha < \omega_1$. The map $p' = p|\big(AD(B) \setminus B_0\big)$ is a bijection, so $(p')^{-1}$ is a function. For all $\alpha < \omega_1$ and $i = 1, \dots, n$ put $b_i^\alpha = r\big((p')^{-1}(x_i^\alpha)\big)$ and $z_\alpha = (b_1^\alpha, \dots, b_n^\alpha) \in I^n$.

Consider the subset $H = \{z_\alpha : \alpha < \omega_1\} \subset B^n$ of the space $(I_A)^n$. Obviously, all points of H are isolated in $(I_A)^n$. Since H is uncountable, it has a limit point $z = (z_1, \dots, z_n)$ in the Lindelöf space $(I_A)^n$. The projection to each factor of $(I_A)^n$ restricted to H is injective, because the sets A_α are disjoint, and the image of H under each projection is contained in the set B of isolated points of I_A; it follows that none of the coordinates of the point z can lie in B. Thus, $z \in A^n$. Pick a sequence $\big\{ h_k = (h_k^1, \dots, h_k^n) : k \in \omega \big\}$ of points of H that converges to z (note that the topologies at $z \in A^n$ are the same in the spaces I^n and $(I_A)^n$, so the character of $(I_A)^n$ at z is countable). For each $i = 1, \dots, n$ the sequence $H_i = \{h_k^i : k \in \omega\}$ converges to z_i, hence the set H_i is closed and discrete in B. For each $k \in \omega$ there is an $\alpha_k < \omega_1$ such that $h_k = z_{\alpha_k}$; let $S = \{\alpha_k : k \in \omega\}$. Then we have

$$H_i = r\Big(p^{-1}\big(\{x_i^\alpha : \alpha \in S\}\big)\Big),$$

and by Proposition 2.5, the set $T_i = \{x_i^\alpha : \alpha \in S\}$ is closed and discrete in $X(B)$. Hence, x_* is not a limit point for each T_i, $i = 1, \dots, n$, and therefore is not a limit point for the set $\bigcup\{A_\alpha : \alpha \in S\} = \bigcup\{T_i : i = 1, \dots, n\}$. $\square$

The following corollaries answer Problems 73 and 75 from [4].

2.8. COROLLARY. *There exists a Lindelöf Σ-space X with a unique non-isolated point such that $C_p(X)$ is a Lindelöf Σ-space, $p\big(C_p(X)\big) = \omega$ and $nw(X) = \mathfrak{c}$.*

PROOF. As shown in [14], there exists a subset B of I such that $|B| = \mathfrak{c}$ and all finite powers of $I_{I \setminus B}$ are Lindelöf. Let $X = X(B)$. The required properties of X follow from Propositions 2.3, 2.4 and 2.7. $\square$

2.9. COROLLARY. *There exists a space Y such that $C_p(Y)$ is a Lindelöf Σ-space, $p(Y) = \omega$, and $nw(Y) = \mathfrak{c}$.*

PROOF. Let X be the space as in Corollary 2.8. Put $Y = C_p(X)$. We already know that $p(Y) = \omega$. Since X and $Y = C_p(X)$ are Lindelöf Σ-spaces, we can apply Theorem 2.12 in [12] to conclude that $C_p(Y) = C_p\big(C_p(X)\big)$ is a Lindelöf Σ-space. Finally, $nw(Y) = nw(X) = \mathfrak{c}$ (see [3], Theorem I.1.1). $\square$

The network weight of the space X constructed in Corollary 2.8 is equal to $\mathfrak{c}$. In fact, this is the maximum network weight such a space can have.

2.10. PROPOSITION. *Suppose that X is a Lindelöf Σ-space such that $C_p(X)$ is a Lindelöf Σ-space and $p\big(C_p(X)\big) = \omega$. Then $|X| \leqq \mathfrak{c}$.*

PROOF. Any Lindelöf Σ-space is a union of at most $\mathfrak{c}$ compact subspaces. This follows easily from the fact that Z is a Lindelöf Σ-space if and only if there exists a family $\mathcal{F} = \{F_n : n \in \omega\}$ of closed subsets of Z such that for some cover $\mathcal{C}$ of Z with compact elements, the family $\mathcal{F}$ is a network at each $C \in \mathcal{C}$, i.e. if $U \in \mathcal{T}(Z)$ and $C \subset U$, then $P \subset F_n \subset U$ for some $n \in \omega$ [11]. Let F be a compact subspace of the space X. The restriction mapping gives is a continuous mapping of $C_p(X)$ onto $C_p(F)$. Hence, $C_p(F)$ is a Lindelöf Σ-space and $p\big(C_p(F)\big) = \omega$. Therefore, F is a Gul'ko compact space, so it follows from $p\big(C_p(F)\big) = \omega$ that F is metrizable (see [5, Proposition 2.10] and [9, Theorem 2]). Thus, X is a union of $\leqq \mathfrak{c}$ of metrizable compact subspaces, whence $|X| \leqq \mathfrak{c}$. $\square$

3. A set B with $p\Big(C_p\big(X(B)\big)\Big) = \mathfrak{c}$

It is not clear whether the inverse of Proposition 2.7 is true, that is, whether $p\Big(C_p\big(X(B)\big)\Big) = \omega$ implies that all finite powers of $I_{I\setminus B}$ are Lindelöf. It is not immediately clear in fact that any condition on B stronger than that B is a Bernstein set is needed here at all. In this section we give an example of a Bernstein set $B \subset I$ such that $p\Big(C_p\big(X(B)\big)\Big) > \omega$.

We say that a subset D of a Polish space M is *n-big* for some $n \in \omega$ if the space $(M_D)^n$ is Lindelöf. Thus, B is a Bernstein set if and only if $M \setminus B$ is 1-big. Obviously, if $D_1 \subset D_2$, then the identity mapping $M_{D_1} \to M_{D_2}$ is continuous; it follows that a superset of an n-big set is n-big.

We denote by $\boldsymbol{C}$ the Cantor cube 2^ω (with the usual product topology).

The next statement is a reformulation of Theorem 1.12 in [14]:

THEOREM 3.1. *Suppose M is a Polish space and $n \in \omega$, $n \geqq 1$. Then a set $D \subset M$ is n-big if and only if for every family of n continuous one-to-one functions $f_1, \dots, f_n : \boldsymbol{C} \to M$, we have*

$$\bigcap \big\{ f_i^{-1}(D) : i = 1, \dots, n \big\} \neq \emptyset. \quad \square$$

LEMMA 3.2. *Let M be an uncountable Polish space, $n \in \omega$ and $n > 1$. If D is an n-big set in M, then there is a $D_1 \subset D$ such that D_1 is $(n-1)$-big and not n-big.*

PROOF. Note that for every family of n continuous one-to-one functions $f_1, \dots, f_n : \boldsymbol{C} \to M$, the intersection $\bigcap \big\{ f_i^{-1}(D) : i = 1, \dots, n \big\}$ has cardinality $\mathfrak{c}$. Indeed, $\boldsymbol{C}$ contains $\mathfrak{c}$ disjoint homeomorphic copies of $\boldsymbol{C}$, and by Theorem 3.1, the intersection of the preimages of D under $f_1, \dots, f_n$ must meet each of them (otherwise the criterion will fail for the restrictions of f_i to one of these copies of $\boldsymbol{C}$).

Fix a family $\Phi = \{g_1, \dots, g_n\}$ of continuous one-to-one functions from $\boldsymbol{C}$ to M so that $g_i(\boldsymbol{C}) \cap g_j(\boldsymbol{C}) = \emptyset$ if $i \neq j$ (this is possible, because the uncountable Polish space M contains $\mathfrak{c}$ disjoint copies of $\boldsymbol{C}$). Let $\{\Psi_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all families of $n-1$ continuous one-to-one functions from $\boldsymbol{C}$ to M.

For every $P \subset \boldsymbol{C}$, $Q \subset M$ and a family Ψ of continuous functions from $\boldsymbol{C}$ to M denote

$$\Psi(P) = \bigcup \left\{ f(P) : f \in \Psi \right\} \quad \text{and} \quad \Psi^{-1}(Q) = \bigcup \left\{ f^{-1}(Q) : f \in \Psi \right\}.$$

Construct by induction on $\alpha < \mathfrak{c}$ sets $A_\alpha \subset M$ and points $x_\alpha \in \boldsymbol{C}$ so that

(1) $A_\alpha = \Psi_\alpha\bigl(\{x_\alpha\}\bigr)$, and

(2) $x_\alpha \in \Bigl(\bigcap \left\{ f^{-1}(D) : f \in \Psi_\alpha \right\}\Bigr) \setminus \Bigl(\bigcup \left\{ \Psi_\beta^{-1}\Psi_\beta\Phi^{-1}(A_\beta) : \beta < \alpha \right\}\Bigr)$

for all $\alpha < \mathfrak{c}$.

Thus, at the step α we pick a point x_α in $P_\alpha = \bigcap \left\{ f^{-1}(D) : f \in \Psi_\alpha \right\}$ that does not belong to $\bigcup \left\{ \Psi_\beta^{-1}\Psi_\beta\Phi^{-1}(A_\gamma) : \gamma < \alpha \right\}$; the choice of x_α is possible, because $|P_\alpha| = \mathfrak{c}$, and the cardinality of the union does not exceed $\omega \cdot |\alpha| < \mathfrak{c}$. Since $x_\alpha \in P_\alpha$ the set A_α is contained in D.

Let $D_1 = \bigcup\{A_\alpha : \alpha < \mathfrak{c}\}$. Then $D_1 \subset D$. If Ψ is a family of $n-1$ continuous one-to-one functions from $\boldsymbol{C}$ to M, then $\Psi = \Psi_\alpha$ for some $\alpha < \mathfrak{c}$. Since $A_\alpha \subset D_1$, we have $x_\alpha \in \bigcap \left\{ f^{-1}(D_1) : f \in \Psi \right\}$, so $\bigcap \left\{ f^{-1}(D_1) : f \in \Psi \right\} \neq \emptyset$, and by Theorem 3.1, D_1 is $(n-1)$-big.

Let us now check that D_1 is not n-big. By Theorem 3.1, it suffices to check that $\bigcap \left\{ g_i^{-1}(D_1) : i = 1, \dots, n \right\} = \emptyset$. Suppose for contradiction that $x \in \bigcap \left\{ g_i^{-1}(D_1) : i = 1, \dots, n \right\}$. Then $g_i(x) \in D_1$ and there are $\alpha_i < \mathfrak{c}$, $i = 1, \dots, n$, such that $g_i(x) \in A_{\alpha_i} = \Psi_{\alpha_i}(x_{\alpha_i})$. Since the images of $\boldsymbol{C}$ under g_i, $i = 1, \dots, n$ are disjoint, we have $g_i(x) \neq g_j(x)$ whenever $i \neq j$. Furthermore, $\bigl|\Psi_\beta(x)\bigr| \leqq n-1$ for all $\beta < \mathfrak{c}$ and $x \in \boldsymbol{C}$. It follows that the set $\{\alpha_1, \dots, \alpha_n\}$ contains at least two distinct ordinals. Let $j, k < n$ be such that $\alpha_j < \alpha_k$. But then $x_{\alpha_k} \in \Psi_{\alpha_k}^{-1}\bigl(\Phi(x)\bigr)$, and $x \in \Phi^{-1}(A_{\alpha_j})$, so $x_{\alpha_k} \in \Psi_{\alpha_k}\Phi\Phi^{-1}(A_{\alpha_j})$, in contradiction with the property (2) of the construction. $\square$

EXAMPLE 3.3. There exists a Bernstein set B in I such that $p\Bigl(C_p\bigl(X(B)\bigr)\Bigr) = \mathfrak{c}$.

Namely, by Theorem 1.13 in [14], there are disjoint 2-big sets B_1 and B_2 in I. Let A be a 1-big subset of B_1 that is not 2-big, and let $B = I \setminus A$. Since $B_2 \subset B$, the set B is 2-big. The set B is Bernstein, because $A = I \setminus B$ is 1-big.

By Theorem 1.2, to show that the point-finite cellularity of $C_p\bigl(X(B)\bigr)$ is equal to $\mathfrak{c}$, it suffices to find a family $\mathcal{A}$ of cardinality $\mathfrak{c}$ of two-point subsets in $X(B) \setminus \{x_*\}$ so that for every infinite subfamily $\mathcal{S}$ of $\mathcal{A}$, the point x_* is in the closure of $\bigcup \mathcal{S}$.

By Theorem 3.2, there are continuous one-to-one functions $g_1, g_2 : \boldsymbol{C} \to I$ such that $g_1^{-1}(A) \cap g_2^{-1}(A) = \emptyset$. On the other hand, since B is 2-big, the intersection of $g_1^{-1}(B) \cap g_2^{-1}(B)$ with every set in $\boldsymbol{C}$ homeomorphic to $\boldsymbol{C}$ is not empty, so the set $P = g_1^{-1}(B) \cap g_2^{-1}(B)$ has the cardinality $\mathfrak{c}$.

For each $c \in P$ put $B_c = \{g_1(c), g_2(c)\}$ and $A_c = p\bigl(B_c \times \{1\}\bigr)$ where $p : AD(B) \to X(B)$ is the natural projection.

Let us verify that the family $\mathcal{A} = \{A_c : c \in P\}$ is as required. Let $H = \bigl\{ \bigl(g_1(c), g_2(c)\bigr) : c \in P \bigr\} \subset I \times I$. The set

$$K = (g_1 \times g_2)(\boldsymbol{C} \times \boldsymbol{C}) = \bigl\{ \bigl(g_1(c), g_2(d)\bigr) : c, d \in \boldsymbol{C} \bigr\}$$

is compact and contains H. On the other hand, the intersection of K with $A \times A$ is empty, because $g_1^{-1}(A) \cap g_2^{-1}(A) = \emptyset$. Hence, the closure of H in $I \times I$ is disjoint with A.

Suppose $\mathcal{B}$ is an infinite subfamily of in $\mathcal{A}$. Then there is an infinite set $S \subset P$ such that $\mathcal{B} = \{A_c : c \in S\}$. The set $H_S = \bigl\{ \bigl(g_1(c), g_2(c)\bigr) : c \in S \bigr\}$ is infinite, and hence has a limit point (a_0, b_0) in the compact space $I \times I$. Since $H_S \subset H$ and the closure of H does not meet $A \times A$, either a_0 or b_0 is in B. This implies that at least one of the sets $\bigl\{g_1(c) : c \in S\bigr\}$, $\bigl\{g_2(c) : c \in S\bigr\}$ has a limit point in B, and hence the union $\bigcup\{B_c : c \in S\}$ is not closed discrete in B. By Proposition 2.5, the point x_* is in the closure of the set $\bigcup\{A_c : c \in S\}$. $\square$

Recall that $L_p(X)$ is the subspace of $C_p(X)$ formed by all linear continuous functions on $C_p(X)$. For every $x \in X$ define $\widehat{x} : C_p(X) \to \mathbf{R}$ by the rule: $\widehat{x}(f) = f(x)$ for all $f \in C_p(X)$. The set $\widehat{X} = \{\widehat{x} : x \in X\}$ is a Hamel base of the space $L_p(X)$ (see [3], Section 0.5).

For every $M \subset L_p(X)$ define the set $\operatorname{supp} M$ as the minimum subset L of X such that M lies in the linear hull of L. A subset A of a topological space X is called *bounded* in X if every real-valued continuous function on X is bounded on A.

The next assertion is proved in [2].

THEOREM 3.4. *If K is a compact set in $L_p(X)$, then the set* $\operatorname{supp} K$ *is bounded in X.* $\square$

EXAMPLE 3.5. There exists a space X with a unique non-isolated point such that

(1) X and $C_p(X)$ are Lindelöf Σ-spaces,

(2) Every compact set in $L_p(X)$ is metrizable, in particular, $a\bigl(L_p(X)\bigr) = \omega$;

(3) $p\bigl(C_p(X)\bigr) = \mathfrak{c}$, and hence, $a\Bigl(C_p\bigl(C_p(X)\bigr)\Bigr) = \mathfrak{c}$.

Namely, let B be as in Theorem 3.3 and $X = X(B)$. Then X and $C_p(X)$ are Lindelöf Σ-spaces by Propositions 2.3 and 2.4. By the choice of X, we have $p\big(C_p(X)\big) = \mathfrak{c}$, and by Theorem 1 in [16], we conclude that $a\Big(C_p\big(C_p(X)\big)\Big) = \mathfrak{c}$.

Let K be a compact set in $L_p(X)$. Then $\operatorname{supp} K$ is bounded in X, and since X has only one non-isolated point, the closure of $\operatorname{supp} K$ in X is compact. It follows from Proposition 2.6 that $\operatorname{supp} K$ is countable. Hence, K lies in a linear subspace of $L_p(X)$ generated by a countable set, so the network weight of K is countable. Since K is compact, it is metrizable. $\square$

Let K be a subset of $C_p\big(C_p(X)\big)$ and $L \subset X$. We say that K *admits a (continuous) factorization through* L if for every $\phi \in K$ there is a (continuous) function $\psi : C_p\big(X|L\big) \to \mathbf{R}$ such that $\phi = \psi \circ r_L$; here $r_L : C_p(X) \to C_p(L)$ is the restriction mapping, and $C_p\big(X|L\big) = r_L\big(C_p(X)\big)$. It is easy to see that if $L \subset L_1$ and K admits a (continuous) factorization through L, then it also admits a (continuous) factorization through L_1. Note that if L is closed, then every factorization through L is continuous, because the restriction mapping r_L is open (see Proposition 0.4.1 in [3]).

Theorem 3.4 implies that every compact subset of $L_p(X)$ admits a continuous factorization through a bounded set in X, and by the factorization theorem in [1], every singleton in $C_p\big(C_p(X)\big)$ admits a continuous factorization through a countable set. This led the first author to conjecture some time ago that every compact set in $C_p\big(C_p(X)\big)$ admits a factorization through a σ-bounded set in X (that is, a countable union of bounded sets in X). It turns out that this conjecture fails for X as in Example 3.5.

EXAMPLE 3.6. There exist a space X with one non-isolated point and a compact set $K \subset C_p\big(C_p(X)\big)$ such that

(1) X and $C_p(X)$ are Lindelöf Σ-spaces,

(2) K is homeomorphic to the one-point compactification $A(\mathfrak{c})$ of the discrete space of cardinality $\mathfrak{c}$,

(3) K does not admit factorization through a σ-bounded subset of X.

Namely, let X be as in Example 3.5; since $a\big(C_p(C_p(X)\big) = \mathfrak{c}$, there is a compact set K in $C_p(X)$ homeomorphic to $A(\mathfrak{c})$. Suppose L_0 is a σ-bounded subset of X. Then L_0 is countable. Indeed, it is easy to see that in a space with one non-isolated point the closure of every bounded set is compact; by Proposition 2.6, every compact set in X is countable. Let $L = L_0 \cup \{x_*\}$; then L is closed in X and $L_0 \subset L$. Since L is countable, the weight of the space $C_p\big(X|L\big) \subset C_p(L)$ is countable. Let $r_L^{\#} : C_p\Big(C_p\big(X|L\big)\Big) \to C_p\big(C_p(X)\big)$ be the dual mapping of r_L (defined by $r_L^{\#}(\psi) = \psi \circ r_L$ for every $\psi \in C_p\Big(C_p\big(X|L\big)\Big)$; see Section 0.4 in [3]). Since $r_L^{\#}$ is continuous and $nw\Big(C_p\big(C_p(X|L)\big)\Big) = nw\Big(C_p\big(X|L\big)\Big) = \omega$, we have

$nw\Big(r_L^{\#}\Big(C_p\big(C_p\big(X|L\big)\big)\Big)\Big) \leqq \omega$, and since $nw(K) = \mathfrak{c}$, there is a function $\phi_0 \in K$ such that $\phi_0 \notin r_L^{\#}\big(C_p\big(C_p(X|L)\big)\big)$. This means that there is no continuous function $\psi : C_p\big(X|L\big) \to \mathbf{R}$ such that $\phi_0 = \psi \circ r_L$, and since r_L is open, there is no function $\psi : C_p\big(X|L\big) \to \mathbf{R}$ such that $\phi_0 = \psi \circ r_L$ (if there were one, it would be continuous). Thus, K does not admit a factorization through L, and hence through L_0. $\square$

4. Unsolved problems

The most intriguing one is to characterize for an arbitrary Tychonoff space X the countability of the point-finite cellularity of $C_p(X)$ in terms of the topology of the space X. Since there is no general criterion yet, many questions on particular cases remain open.

5.1. Question. Is it consistent with ZFC that every perfectly normal compact space X with $p\big(C_p(X)\big) = \omega$ is metrizable?

5.2. Question. Let X be a Souslin continuum, i.e. a non-separable linearly ordered perfectly normal compact space. Is it true that $p\big(C_p(X)\big) = \omega$?

5.3. Question. Is there a ZFC example of a non-metrizable first countable compact space X such that $p\big(C_p(X)\big) = \omega$?

5.4. Question. Let X be a Corson compact space such that $p\big(C_p(X)\big) = \omega$. Must X be metrizable?

5.5. Question. Suppose that $p\big(C_p(X)\big) = \omega$ and $C_p(X)$ is a Lindelöf Σ-space. Is it true that $|X| \leqq \mathfrak{c}$?

5.6. Question. Suppose that $p(X) = \omega$ and $C_p(X)$ is a Lindelöf Σ-space. Is it true that $nw(X) \leqq \mathfrak{c}$?

5.7. Question. Suppose that $p\big(C_p(X)\big) = \omega$. Is it true that $p\big(C_p(\upsilon X)\big) = \omega$?

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