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Sturm–Liouville operators in the half axis with local perturbations

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Abstract

We give conditions which imply equivalence of the Lebesgue measure with respect to a measure μ generated as an average of spectral measures corresponding to Sturm–Liouville operators in the half axis. We apply this to prove that some spectral properties of these operators hold for large sets of boundary conditions if and only if they hold for large sets of positive local perturbations.

Keywords: Sturm–Liouville operators; Mixed spectra; Local perturbations

1. Introduction

A key step in important results concerning localization phenomena has been to establish absolute continuity of measures $\mu(\cdot) := \int \rho_\lambda(\cdot)\, d\lambda$ generated as averages of spectral measures ρ_λ which correspond to self-adjoint operators.

Let us for example consider a family of self-adjoint operators $L_\theta(q)$ in $L_2(0,\infty)$ generated by the differential expression

$$lu = -u'' + q(x)u$$

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and the boundary condition

$$u(0)\cos\theta - u'(0)\sin\theta = 0, \quad \theta \in [0,\pi),$$

and let ρ_θ be the spectral measure corresponding to L_θ. The average $\mu(\cdot) := \int_0^\pi \rho_\theta(\cdot)\,d\theta$ is not only absolutely continuous with respect to Lebesgue measure, but is equal to it! (See e.g. [9, Theorem 1].)

Using the absolute continuity of μ the following can be proven (see e.g. [4, Corollary 3.2]).

Theorem 1. *If for almost every $E \in I$, $I \subset \mathbb{R}$, there exists a nontrivial L^2 solution of $lu = Eu$, then*

$$\sigma_c(L_\theta) \cap I = \emptyset$$

for almost every $\theta \in [0,\pi)$, where σ_c denotes the continuous spectrum.

For a survey of other localization results related to the absolute continuity of averages as μ see [12].

In this paper we give conditions which imply that Lebesgue measure and a properly chosen average measure μ have the same zero sets and apply this to understand some aspects of the spectral theory of Sturm–Liouville operators with local perturbations. In particular, if some spectral property holds for a set of positive Lebesgue measure in the boundary conditions we can show that this property holds for a set of positive measure of local perturbations leaving any boundary condition fixed. This will be a consequence of our main result Theorem 3 below. In Section 2 our main results are proven. The techniques used rely heavily on the Prüfer transform which is a very useful tool in this paper. In Section 3 we show consequences of the previous results, some applications to α-singular and α-continuous spectrum are given and an open question is presented. Some of the methods we used in this paper have their origin in [7].

2. Main results

For each $(\lambda,\theta) \in \mathbb{R} \times [0,\pi)$ let us consider the self-adjoint operator $L_{\lambda\theta}$ in $L_2(0,\infty)$ generated by the differential expression

$$l_\lambda u = -\frac{d^2}{dx^2} + V(x) + \lambda W(x) \tag{1}$$

as follows: $L_{\lambda,\theta}u := l_\lambda u$ for u in the domain of $L_{\lambda\theta}$ given by

$$D(L_{\lambda\theta}) = \bigl\{ f \in L^2(0,\infty):\ f, f' \text{ locally absolutely continuous in } (0,\infty), \\ l_\lambda f \in L_2(0,\infty),\ f(0)\cos\theta - f'(0)\sin\theta = 0,\ \theta \in [0,\pi) \bigr\}.$$

We assume $L_{\lambda\theta}$ is regular at 0 and the limit point case holds at ∞. See for these concepts [2]. The functions V and W are real-valued and locally in L^1. We assume $W(x) > 0$ for a.e. $x \in [0,c]$ and $W(x) = 0$ for every $x \notin [0,c]$.

Denote by ρ_λ^θ the spectral function of the operator $L_{\lambda\theta}$.

Using the same differential expression (1) we define a regular problem in the finite interval $[0,c]$ setting for $\theta \in [0,\pi)$ the boundary conditions

$$\begin{cases} u(0)\cos\theta - u'(0)\sin\theta = 0, \\ u(c) = 0. \end{cases} \tag{2}$$

Definition 1. Fix $E \in \mathbb{R}$ and let us call $\lambda(E)$ an eigenvalue if there is a nontrivial solution u of $l_\lambda u = Eu$ which satisfies the conditions (2).

The following result is due to C. Sturm and J. Liouville [13]. For more recent references see for example [2, Theorem 2.1, Chapter 8, p. 212], [5, Chapter XI, p. 337], [14, Theorem 13.2], [1, Theorems 8.4.5 and 8.4.6].

Theorem 2. *There exists a monotonous decreasing sequence of eigenvalues* $\lambda^{(0)} > \lambda^{(1)} > \lambda^{(2)} > \cdots$ *such that* $\lambda^{(n)} \to -\infty$ *if* $n \to \infty$.

If μ and ν are two measures we use the notation $\mu \preccurlyeq \nu$ when μ is absolutely continuous with respect to ν, that is if $\nu(A) = 0$ implies $\mu(A) = 0$. If $\mu \preccurlyeq \nu$ and $\nu \preccurlyeq \mu$ we say that the two measures are equivalent. We use $|\cdot|$ to denote the Lebesgue measure.

Let A be a Borel set $A \subset (E_1, E_2)$ and define a measure μ_θ as follows:

$$\mu_\theta(A) := \int_{\lambda_1}^{\lambda_2} \rho_\lambda^\theta(A)\, d\lambda.$$

The next result gives conditions which imply equivalence between μ_θ and $|\cdot|$.

Theorem 3.

(I) *If* $\lambda_1 \leqslant \lambda^{(n+1)}(E_1) < \lambda^{(n)}(E_2) \leqslant \lambda_2$ *where* $\lambda^{(n)}$ *are eigenvalues of the regular problem in* $[0, c]$ *mentioned in previous theorem then* $|\cdot| \preccurlyeq \mu_\theta$.

(II) *If* $\lambda_1 \leqslant \lambda_2$ *then* $\mu_\theta \preccurlyeq |\cdot|$.

Before we prove the theorem let us introduce notation and recall some results.

Consider real solutions $u \not\equiv 0$ of $l_\lambda u = Eu$ such that for fixed $a \in \mathbb{R}$,

$$u(a) = \sin\theta,$$
$$u'(a) = \cos\theta.$$

If we write the vector $(u'(x), u(x))$ in polar coordinates we obtain

$$u(x) = r_a(x) \sin\phi_a(x),$$
$$u'(x) = r_a(x) \cos\phi_a(x),$$

where $\phi_a(x, \theta, E, \lambda)$ and $r_a(x, \theta, E, \lambda)$ are called the Prüfer phase and the Prüfer amplitude of u.

We fix a unique value of ϕ_a by requiring $\phi_a(a, \theta, E, \lambda) = \theta$ and continuity in x. These functions r_a and ϕ_a are jointly continuous in x, E and λ (use arguments similar to [14, Theorem 2.1], [2, Theorem, Chapter 2.4], [5, Theorem, Chapter V.3]). This will be very important in what follows.

We shall need the next results

(a) $$\rho_\lambda^\theta\big((E_1, E_2)\big) = \lim_{b\to\infty} \frac{1}{\pi} \int_{E_1}^{E_2} r_0(b, \theta, E, \lambda)^{-2}\, dE$$

if E_1 and E_2 are not discrete points of ρ_λ^θ. See [9, Theorem 2].

(b) (i) $$(\partial_\lambda \phi_a)(x,\lambda) = -\frac{1}{r_a(x,\lambda)^2}\int_a^x W(t) r_a(t,\lambda)^2 \sin^2\phi_a(t,\lambda)\,dt,$$

(ii) $$(\partial_E \phi_a)(x,E) = \frac{1}{r_a(x,\lambda)^2}\int_a^x r_a(t,E)^2 \sin^2\phi_a(t,E)\,dt.$$

See [11, Appendix B].

(c) For any $a, x, \theta, E \in \mathbb{R}$,

$$\frac{1}{\pi}\int_\theta^{\theta+\pi} r_a(x,\beta,E)^{-2}\,d\beta = 1.$$

See [10, Corollary 12] and [11, Appendix B].

(d) For the Prüfer phase ϕ_a of a solution of $l_\lambda u = Eu$ we have

$$\phi_a(p,\lambda)\to\infty \quad \text{as } \lambda\to-\infty$$

for any point $p \in \operatorname{supp} W$ (see [2, formula (2.5) in the proof of Theorem 2.1, p. 212] or [14, Theorem 13.2, part (c)]).

Proof of Theorem 3. We use an argument similar to the one used in [11, Lemma 5.10] where (II) was proven for the case of the whole line. Given any $E_1 < E_2$, the measure ρ_λ^θ is continuous in E_1 and E_2 for almost any λ.

From (a) we know that

$$\mu_\theta\big((E_1,E_2)\big) = \int_{\lambda_1}^{\lambda_2} \rho_\lambda^\theta\big((E_1,E_2)\big)\,d\lambda = \int_{\lambda_1}^{\lambda_2}\left[\lim_{b\to\infty}\frac{1}{\pi}\int_{E_1}^{E_2} r_0(b,\theta,E,\lambda)^{-2}\,dE\right]d\lambda. \tag{3}$$

Let $b > c > 0$ where $c = \sup S$ and $S = \{x\colon\ W(x)\neq 0\} = \operatorname{supp} W$.

Let us estimate

$$\int_{\lambda_1}^{\lambda_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\big(r_0(b,\theta,E,\lambda)\big)^{-2}\,dE\right]d\lambda.$$

Using Fubini we can interchange integrals and from the equality

$$r_0(b,\theta,E,\lambda)^{-2} = r_0(c,\theta,E,\lambda)^{-2} r_c\big(b,\phi_0(c,\theta,E,\lambda),E\big)^{-2}$$

and the fact that r_0 is jointly continuous in E and λ and strictly positive we get

$$\frac{1}{\pi}\sup_{\substack{E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}}\big\{r_0(c,\theta,E,\lambda)^{-2}\big\}\int_{E_1}^{E_2}dE\left[\int_{\lambda_1}^{\lambda_2} r_c\big(b,\phi_0(c,\theta,E,\lambda),E\big)^{-2}\,d\lambda\right]$$
$$\geqslant \int_{\lambda_1}^{\lambda_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\big(r_0(b,\theta,E,\lambda)\big)^{-2}\,dE\right]d\lambda = \frac{1}{\pi}\int_{E_1}^{E_2}\left[\int_{\lambda_1}^{\lambda_2}\big[r_0(b,\theta,E,\lambda)\big]^{-2}\,d\lambda\right]dE$$

$$\geqslant \frac{1}{\pi} \inf_{\substack{E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}} \left\{r_0(c,\theta,E,\lambda)^{-2}\right\} \int_{E_1}^{E_2} dE \left[\int_{\lambda_1}^{\lambda_2} r_c\big(b,\phi_0(c,\theta,E,\lambda),E\big)^{-2}\,d\lambda\right].$$

Define $\beta(\lambda) := \phi_0(c,\theta,E,\lambda)$.

From (b)(i) we know that $\beta'(\lambda) < 0$. Changing variables we obtain

$$\int_{\lambda_1}^{\lambda_2} r_c\big(b,\phi_0(\lambda),E\big)^{-2}\,d\lambda = \int_{\beta(\lambda_1)}^{\beta(\lambda_2)} \frac{r_c(b,\beta,E)^{-2}}{\beta'(\lambda)}\,d\beta = \int_{\beta(\lambda_2)}^{\beta(\lambda_1)} \frac{1}{|\beta'(\lambda)|} r_c(b,\beta,E)^{-2}\,d\beta.$$

Using (b)(i) and the fact that r_0 is jointly continuous in λ and E we obtain

$$\begin{aligned}
\left|\beta'(\lambda)\right| &= \int_0^c W(t) \frac{r_0^2(t,\theta,E,\lambda)\sin^2\phi_0(t,\theta,E,\lambda)\,dt}{r_0^2(c,\theta,E,\lambda)} \\
&\leqslant \int_0^c W(t) \frac{r_0^2(t,\theta,E,\lambda)\,dt}{r_0^2(c,\theta,E,\lambda)} \\
&\leqslant \sup_{\substack{t\in[0,c]\\ E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}} \left\{\frac{r_0^2(t,\theta,E,\lambda)}{r_0^2(c,\theta,E,\lambda)}\right\} \int_0^c W(t)\,dt \\
&\leqslant \frac{\sup_{t,E,\lambda} r_0^2(t,\theta,E,\lambda)}{\inf_{E,\lambda} r_0^2(c,\theta,E,\lambda)} \int_0^c W(t)\,dt
\end{aligned}$$

therefore

$$\frac{1}{|\beta'(\lambda)|} \geqslant \frac{\inf r_0^2(c,\theta,E,\lambda)}{\sup r_0^2(t,\theta,E,\lambda)\int_0^c W(t)\,dt} =: K.$$

On the other hand we have that

$$\frac{r_0^2(t,\theta,E,\lambda)}{r_0^2(c,\theta,E,\lambda)} \geqslant C > 0$$

for $t \in [0,c]$ locally uniformly in E,λ and, since $\sin\phi_0(\cdot,\theta,E,\lambda)$ has only isolated zeros,

$$\int_0^c W(t)\sin^2\phi_0(t,\theta,E,\lambda)\,dt > 0.$$

By continuity in E and λ we arrive at

$$\inf_{\substack{E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}} \int_0^c W(t)\sin^2\phi_0(t,\theta,E,\lambda)\,dt > 0$$

and

$$\frac{1}{|\beta'(\lambda)|} \leqslant C.$$

Altogether we get

$$C\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b,\beta,E)^{-2}\,d\beta \geqslant \int_{\lambda_1}^{\lambda_2} r_c\bigl(b,\phi_0(\lambda),E\bigr)^{-2}\,d\lambda = \int_{\beta(\lambda_2)}^{\beta(\lambda_1)} \frac{r_c(b,\beta,E)^{-2}}{|\beta'(\lambda)|}\,d\beta$$
$$\geqslant K\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b,\beta,E)^{-2}\,d\beta$$

hence

$$\tilde{C}\int_{E_1}^{E_2} dE\left[\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b,\beta,E)^{-2}\,d\beta\right] \geqslant \int_{\lambda_1}^{\lambda_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\bigl(r_0(b,\theta,E,\lambda)\bigr)^{-2}\,dE\right]d\lambda$$
$$= \frac{1}{\pi}\int_{E_1}^{E_2}\left[\int_{\lambda_1}^{\lambda_2} r_0(b,\theta,E,\lambda)^{-2}\,d\lambda\right]dE$$
$$\geqslant \tilde{c}\int_{E_1}^{E_2} dE\left[\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b,\beta,E)^{-2}\,d\beta\right],$$

where

$$\tilde{c} = \frac{1}{\pi}\inf_{\substack{E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}}\bigl\{r_0(b,\theta,E,\lambda)^{-2}\bigr\}K \quad\text{and}\quad \tilde{C} = \frac{1}{\pi}\sup_{\substack{E\in[E_1,E_2]\\ \lambda\in[\lambda_1,\lambda_2]}}\bigl\{r_0(b,\theta,E,\lambda)^{-2}\bigr\}C.$$

In order to handle the limit that appears in (3) we would like to get estimates for

$$\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b,\beta,E)^{-2}\,d\beta$$

which are independent of b. To accomplish this we look for conditions on λ_1, λ_2 which guarantee that $\beta(\lambda_1) - \beta(\lambda_2) \geqslant \pi$ and then apply result (c) mentioned above.

To prove (I) we need an estimate from below. From (b)(i), (ii) and (d) we know that $\beta(\lambda) = \phi_0(c,\theta,E,\lambda)$ is decreasing in λ, increasing in E and $\beta(\lambda)\to\infty$ if $\lambda\to-\infty$ or $E\to\infty$. Moreover, λ is an eigenvalue of the problem with boundary conditions (2) if and only if $\beta(\lambda) = m\pi$. For fixed E choose two consecutive eigenvalues $\lambda^{(n)}, \lambda^{(n+1)}$ of this problem and take

$$\lambda_2 > \lambda^{(n)} > \lambda^{(n+1)} > \lambda_1$$

then we will have $\beta(\lambda_1) - \beta(\lambda_2) \geqslant \pi$. Now since β is increasing in E we can choose $\lambda_2 \geqslant \lambda^{(n)}(E_2) > \lambda^{(n+1)}(E_1) > \lambda_1$ and we get

$$\beta(\lambda_1,E) - \beta(\lambda_2,E) \geqslant \pi \quad \text{for every } E\in[E_1,E_2],$$

then we can conclude

$$\int_{\lambda_1}^{\lambda_2}\frac{1}{\pi}\left[\int_{E_1}^{E_2}\bigl(r_0(b,\theta,E,\lambda)\bigr)^{-2}\,dE\right]d\lambda \geqslant \pi\tilde{c}(E_2-E_1).$$

To get the estimate from above that we need to prove (II) it is not necessary to impose extra conditions on λ_1, λ_2. We split the interval $[\beta(\lambda_2), \beta(\lambda_1)]$ into intervals of length at most π and apply (c),

$$\int_{\beta(\lambda_2)}^{\beta(\lambda_1)} r_c(b, \beta, E)^{-2}\, d\beta \leqslant \pi\left(\frac{\beta(\lambda_1) - \beta(\lambda_2)}{\pi} + 1\right).$$

Remember that $\beta(\lambda) = \phi_0(c, \theta, E, \lambda)$ is bounded locally uniformly in E, then

$$\int_{\lambda_1}^{\lambda_2} \frac{1}{\pi}\left[\int_{E_1}^{E_2} \big(r_0(b, \theta, E, \lambda)\big)^{-2}\, dE\right] d\lambda \leqslant \pi \hat{C}(E_2 - E_1),$$

where

$$\hat{C} = \big(\beta(\lambda_1) - \beta(\lambda_2) + \pi\big)\tilde{C}(E_2 - E_1).$$

Since $\frac{1}{\pi}\int_{E_1}^{E_2}(r_0(b, \theta, E, \lambda))^{-2}\, dE$ is bounded uniformly for every $b > c$ and $\lambda \in [\lambda_1, \lambda_2]$ (use [8, Lemma 1.1, Chapter 2] and joint continuity) we can apply the Lebesgue theorem of dominated convergence to obtain

$$\pi \hat{C}(E_2 - E_1) \geqslant \mu_\theta\big((E_1, E_2)\big) \geqslant \pi \tilde{c}(E_2 - E_1).$$

Using countable additivity the theorem follows for general Borel set A. □

The following result is a consequence of Theorem 3.

Corollary 1. *Let $I := (E_1, E_2) \subset \mathbb{R}$ be open and define $L_{\lambda\theta}$ as above. For any $\lambda \in \mathbb{R}$, the operator $L_{\lambda\theta}$ has singular continuous spectrum in I for a set of positive Lebesgue measure of θ's if and only if for any $\theta \in [0, \pi)$, $L_{\lambda\theta}$ has singular continuous spectrum in I for a set B of λ's of positive Lebesgue measure. Moreover, if $\lambda^{(n)}(E)$ are the eigenvalues of* (1) *with boundary conditions* (2)*, then*

$$\big|B \cap \big[\lambda^{(n+1)}(E_1), \lambda^{(n)}(E_2)\big]\big| > 0 \quad \textit{for every } n \geqslant 0.$$

Proof. ($\Rightarrow$) Let S be the set of points E for which there are subordinate solutions of $l_\lambda u = Eu$ which are not in L_2. It is known that this set is a support of the singular continuous part of $L_{\lambda\theta}$ and it does not depend on λ and θ (see [6]). Since $\rho_\lambda^\theta(S \cap I) > 0$ for a set of positive measure in θ by hypothesis using equality (see e.g. [9])

$$|S \cap I| = \int_0^\pi \rho_\lambda^\theta(S \cap I)\, d\theta$$

we deduce $|S \cap I| > 0$. This implies using Theorem 3(I) that $\mu_\theta(S \cap I) > 0$ for any θ. From here we know $\rho_\lambda^\theta(S \cap I) > 0$ for $\lambda \in B$ and

$$\big|B \cap \big[\lambda^{(n+1)}(E_1), \lambda^{(n)}(E_2)\big]\big| > 0.$$

($\Leftarrow$) Assume $L_{\lambda\theta}$ has singular continuous spectrum in I for a set B of λ's of positive Lebesgue measure. Then $\rho_\lambda^\theta(S \cap I) > 0$ for $\lambda \in B$ and

$$\int_B \rho_\lambda^\theta(S \cap I)\, d\lambda > 0.$$

Therefore there exists an interval J such that

$$\int_J \rho_\lambda^\theta(S \cap I)\,d\lambda \geqslant \int_{B\cap J} \rho_\lambda^\theta(S \cap I)\,d\lambda > 0.$$

Using Theorem 3(II) we obtain $|S \cap I| > 0$ and therefore

$$\int_0^\pi \rho_\lambda^\theta(S \cap I)\,d\theta = |S \cap I| > 0$$

for every fixed λ. Therefore $L_{\lambda\theta}$ has singular continuous spectrum in I for a set of positive Lebesgue measure in θ. $\square$

If instead of taking the support S as above we take the set P of subordinate solutions which are in L_2 we get the same result for the pure point part and taking $P \cup S$ we obtain the result for the singular part of $L_{\lambda\theta}$.

3. Consequences and remarks

For the pure point part of the spectrum the following result can be proven. Define $L_{\lambda\theta}(U)$ as it was done in Section 2 but using instead the differential expression

$$l_\lambda(U) = -\frac{d^2}{dx^2} + V(x) + U(x) + \lambda W(x),$$

where we assume V is bounded from below ($L_{\lambda\theta} = L_{\lambda\theta}(U)$ if $U \equiv 0$). Using Corollary 1.8 of [6] the following version of Corollary 1 above can be proven:

Proposition 1. *For any $\lambda \in \mathbb{R}$ the operator $L_{\lambda\theta}$ has point spectrum in I for a set of positive Lebesgue measure in θ's if and only if for any θ the operator $L_{\lambda\theta}(U)$ has point spectrum in I for a set of λ's of positive Lebesgue measure. We assume the function $U(x)$ is locally in L_1 and*

$$\int \bigl|U(x)\bigr| e^{A|x|}\,dx < \infty$$

for all $A > 0$.

We recall the following related result which was proven in [4].

Corollary 2. *Let $I \subset \mathbb{R}$ be open. If $L_\theta(q)$ has singular continuous spectrum in I for a set of positive measure of θ's, the same is true for $q + v$ where v is a perturbation of compact support (v does not need to be positive).*

The next proposition states that, in a way, the roles of θ and v in Corollary 2 can be exchanged. The proof follows directly from Corollary 1.

Proposition 2. *Let $\theta_0, \theta_1 \in [0, \pi)$. The operator $L_{\lambda\theta_0}$ has singular continuous spectrum in I for a set of positive Lebesgue measure in λ if and only if $L_{\lambda\theta_1}$ has singular continuous spectrum in I for a set of positive measure in λ.*

Similar results can be proven for the α-continuous and α-singular spectrum. Recall that for $\alpha \in (0,1)$ the α-dimensional Hausdorff measure is defined for Borel sets A by

$$h^\alpha(A) \equiv \lim_{\delta\to 0} \inf_{\delta\text{-covers}} \sum_{v=1}^{\infty} |b_v|^\alpha,$$

where a δ-cover is a countable collection of intervals each of length at most δ so $A \subset \bigcup_{v=1}^{\infty} b_v$.

Given $\alpha \in [0,1]$ we define a measure μ to be α-continuous (αc) if $\mu(S) = 0$ for any set S with $h^\alpha(S) = 0$ and α-singular (αs) if it is supported on a set S with $h^\alpha(S) = 0$. For every such α and any measure μ, one can uniquely decompose $\mu = \mu^{\alpha c} + \mu^{\alpha s}$ with $\mu^{\alpha c}$ α-continuous and $\mu^{\alpha s}$, α-singular.

Denote $\rho := \rho_\lambda^\theta$. It is possible to find sets A_α and B_α such that

$$d\rho^{\alpha c} = d\rho(A_\alpha \cap .),$$
$$d\rho^{sc} = d\rho(B_\alpha \cap .)$$

and it happens that A_α and B_α are independent of θ and λ. See [6] and references therein.

Using the same reasoning as above one can prove Proposition 2 and Corollary 1 for α-singular and α-continuous instead of singular continuous spectrum.

Remark. Define B to be the set of λ's such that $L_{\lambda\theta}$ has singular continuous (point, singular, α-continuous, α-singular) spectrum in an interval $I = (E_1, E_2)$. From the results above, it follows in particular that it is not possible to have $|B| > 0$ and $B \subset J$ where J is a finite interval, since $|B \cap [\lambda^{(n+1)}(E_1), \lambda^{(n)}(E_2)]| > 0$ for every $n \geqslant 0$ and $\lambda^{(n)} \to -\infty$ if $n \to \infty$. It remains to answer this for θ. We believe it is possible to have a set T of θ's such that $|T| > 0$ and $T \subset J$ where J is an interval properly contained in $[0,\pi)$. Our belief is based on the following result [3, Theorem 1.2] with regard to abstract rank one perturbations. Let A be a self-adjoint operator with a cyclic vector φ. Denote by A_λ the perturbations of A:

$$A_\lambda = A + \lambda(\cdot,\varphi)\varphi, \quad \lambda \in \mathbb{R}.$$

Theorem 4. *For any measurable set $B \subset \mathbb{R}$ there exists a family of rank one perturbations $\{A_\lambda\}_{\lambda\in\mathbb{R}}$ such that A_λ has dense absolutely continuous and dense singular spectrum for almost every $\lambda \in B$ and dense absolutely continuous (but no singular) spectrum for almost every $\lambda \notin B$.*

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