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On the structure of strong 3-quasi-transitive digraphs

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ABSTRACT

In this paper, $D = (V(D), A(D))$ denotes a loopless *directed graph* (digraph) with at most one arc from u to v for every pair of vertices u and v of $V(D)$. Given a digraph D, we say that D is *3-quasi-transitive* if, whenever $u \to v \to w \to z$ in D, then u and z are adjacent or $u = z$. In Bang-Jensen (2004) [3], Bang-Jensen introduced 3-quasi-transitive digraphs and claimed that the only strong 3-quasi-transitive digraphs are the strong semicomplete digraphs and strong semicomplete bipartite digraphs. In this paper, we exhibit a family of strong 3-quasi-transitive digraphs distinct from strong semicomplete digraphs and strong semicomplete bipartite digraphs and provide a complete characterization of strong 3-quasi-transitive digraphs.

Classes of graphs characterized by forbidding families of induced graphs play an important role in Graph Theory. Given a family of graphs $\mathcal{F}$, we say that a graph G is $\mathcal{F}$*-free* if G has no induced subgraph in $\mathcal{F}$. Perfect graphs are probably the best known class of such graphs.

The class of connected graphs with no induced 2-paths are the complete graphs. The class of connected graphs with no induced 3-paths are the so-called cographs or complement-reducible graphs which were characterized in [7].

In this paper we study a directed version of $\mathcal{F}$-free graphs. In order to do this, given $\mathcal{F}$, a family of oriented graphs, we say that D is an *orientedly* $\mathcal{F}$*-free* digraph if there is no digraph in $\mathcal{F}$ isomorphic to any induced subdigraph of any orientation of D (an *orientation* of a digraph D is a spanning subdigraph of D in which we choose only one arc between any two adjacent vertices of D; for example, any orientation of any semicomplete digraph is a tournament). If $\mathcal{F} = \{F\}$, we say orientedly F-free instead of orientedly $\mathcal{F}$-free.

There are three different possible orientations of the 2-path, see Fig. 1. In $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ any arc between the two vertices with a dotted edge between them is forbidden. Orientedly $\mathcal{J}_1$-free digraphs (respectively orientedly $\mathcal{J}_2$-free digraphs) were introduced by Bang-Jensen in [1] as a generalization of semicomplete digraphs. They are known as *locally in-semicomplete* (resp. *locally out-semicomplete*) digraphs. Orientedly $\{\mathcal{J}_1, \mathcal{J}_2\}$-free digraphs were introduced in the same paper and were characterized by Bang-Jensen, Guo, Gutin, and Volkmann in [4]. Orientedly $\mathcal{J}_3$-free digraphs were introduced by Ghouila-Houri in [9]. Observe that orientedly $\mathcal{J}_3$-free digraphs are the same family which were characterized by Bang-Jensen and Huang in [6].

There are four different possible orientations of the 3-path, see Fig. 2. In $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, any arc between the two vertices with a dotted edge between them is forbidden. Orientedly $\{\mathcal{H}_1, \mathcal{H}_2\}$-free digraphs were introduced by Bang-Jensen as a common generalization of both semicomplete digraphs and semicomplete bipartite digraphs in [2]. They are the *arc-locally semicomplete* digraphs which were characterized by Galeana-Sánchez and Goldfeder in [8]. Orientedly $\mathcal{H}_1$-free digraphs (resp. orientedly $\mathcal{H}_2$-free digraphs) were introduced by Wang and Wang as *arc-locally in-semicomplete*

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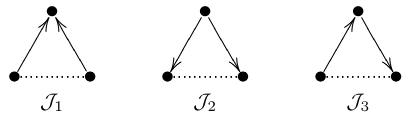

Fig. 1. Different possible orientations of the 2-path.

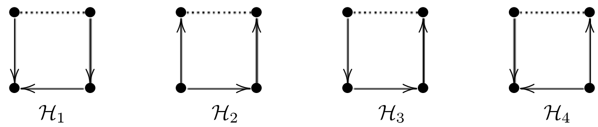

Fig. 2. Different possible orientations of the 3-path.

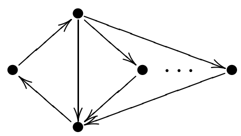

Fig. 3. A family of strong 3-quasi-transitive digraphs.

digraphs (resp. *arc-locally out-semicomplete* digraphs) in [10]. This paper also characterized strong arc-locally in- and out-semicomplete digraphs. Orientedly $\mathcal{H}_3$-free digraphs are the so-called 3-*quasi-transitive* digraphs, introduced in [3] by Bang-Jensen. This paper claims that the only strong 3-quasi-transitive digraphs are the strong semicomplete digraphs and the strong semicomplete bipartite digraphs. The orientedly $\mathcal{H}_4$-free digraphs remain unknown.

In [3], Bang-Jensen conjectured that an orientedly $\mathcal{H}_i$-free digraph, for $i = 1, 2, 3, 4$, is Hamiltonian if and only if it is strong and has a cycle factor, where a *cycle factor* in a digraph D is a collection of vertex disjoint cycles which cover $V(D)$. Wang and Wang proved the conjecture true for $i = 1, 2$ in [10].

Consider the family of digraphs which appear in Fig. 3. These digraphs are strong 3-quasi-transitive digraphs which are not semicomplete digraphs, nor semicomplete bipartite digraphs. This family of digraphs is missing from Bang-Jensen's characterization. In this paper we show that any strong 3-quasi-transitive digraph is either a semicomplete digraph, a semicomplete bipartite digraph, or belongs to the family of Fig. 3. This proves Bang-Jensen's conjecture for $i = 3$.

1. Preliminaries

For general concepts we refer the reader to [5]. In this paper, $D = (V(D), A(D))$ denotes a loopless *directed graph* (digraph) with at most one arc from u to v for every pair of vertices u and v of $V(D)$. For each vertex u in D, $N^+(u)$ (respectively $N^-(u)$) denotes the *out-neighborhood* (resp. *in-neighborhood*) of u. For any $x, y \in V(D)$, we will write $\overrightarrow{xy}$ or $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $\overrightarrow{xy}$ or $\overrightarrow{yx}$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. All our paths and cycles are directed. An n-path (respectively n-cycle) is a path (resp. cycle) of length n. An *independent set* is a set of pairwise nonadjacent vertices of D.

If S is a set of vertices of D, we denote the *subdigraph induced* by S in D by $D[S]$.

Let D and E be digraphs. E is a *subdigraph* of D (denoted by $E \le D$) if $V(E) \subseteq V(D)$ and $A(E) \subseteq A(D)$.

2. On the structure of strong 3-quasi-transitive digraphs

Lemma 2.1. *Let D be a 3-quasi-transitive digraph. For a pair x, y of $V(D)$, if there exists an (x, y)-path of odd length, then x and y are adjacent.*

Proof. Let $P = x_0x_1 \cdots x_{2k+1}$ be an (x, y)-path, where $x_0 = x$ and $x_{2k+1} = y$. The case $k = 0$ is trivial. Since D is 3-quasi-transitive, x_i and x_{i+3} are adjacent, for $i = 0, 1, \ldots, 2k-2$. So the case $k = 1$ is verified. The proof for the case $k \ge 2$ is by induction on k. By induction, x_0 and x_{2k-1} are adjacent. If $x_0 \to x_{2k-1}$, then $x_0 \to x_{2k-1} \to x_{2k} \to x_{2k+1}$ implies that $\overline{x_0x_{2k+1}}$. Next we assume that $x_{2k-1} \to x_0$. Note that $\overline{x_{2k-2}x_{2k+1}}$. If $x_{2k+1} \to x_{2k-2}$, then $x_{2k+1} \to x_{2k-2} \to x_{2k-1} \to x_0$ implies that $\overline{x_{2k+1}x_0}$. Now we assume that $x_{2k-2} \to x_{2k+1}$. Note that $x_0x_1 \cdots x_{2k-2}x_{2k+1}$ is a path of length $2(k-1)+1$. Hence, by the induction hypothesis, $\overline{x_0x_{2k+1}}$. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let D be a strong 3-quasi-transitive digraph with at least three vertices. If D is not a bipartite digraph, then it contains a 3-cycle T. Moreover, every vertex of $V(D)$ is adjacent to at least two vertices of $V(T)$. Finally, suppose that $|V(D)| \ge 4$ and let $T = x_1x_2x_3x_1$. For every $s \in V(D) \setminus V(T)$, if $s \to x_i$, then s and x_{i+2} are adjacent; if $x_i \to s$, then x_{i-2} and s are adjacent, where subscripts are taken modulo 3.*

Proof. Since D is a strong non-bipartite digraph, D contains an odd cycle. Let $C = x_1x_2\cdots x_{2k+1}x_1$ be an odd cycle in D. We will proceed by induction on k. If $k = 1$, then we are done. Next, we assume $k \geq 2$. Note that $x_2x_3\cdots x_{2k+1}$ is an (x_2, x_{2k+1})-path of odd length. By Lemma 2.1, $\overline{x_2x_{2k+1}}$. If $x_2 \to x_{2k+1}$, then $x_2x_{2k+1}x_1x_2$ is a 3-cycle. Now assume $x_{2k+1} \to x_2$. Since $x_{2k+1} \to x_1 \to x_2 \to x_3$, it follows from Lemma 2.1 that $\overline{x_{2k+1}x_3}$. If $x_3 \to x_{2k+1}$, then $x_{2k+1}x_2x_3x_{2k+1}$ is a 3-cycle. Now assume $x_{2k+1} \to x_3$. Note that $x_3x_4\cdots x_{2k+1}x_3$ is an odd cycle of length $2(k-1)+1$. By the induction hypothesis, the assertion is true.

Let $T = x_1x_2x_3x_1$ and $s \in V(D)$ be any. If $s \in V(T)$, then the assertion is true. Next assume $s \notin V(T)$. First, we show that s and one vertex of $V(T)$ are adjacent. Since D is strong, there exists a path from s to T. Let $P = sy_1\cdots y_k$ be a shortest path from s of T, where $y_k \in V(T)$. If k is odd, then, by Lemma 2.1, we have $\overline{sy_k}$. Suppose that k is even. Since T is strong, there exists $u \in V(T)\setminus\{y_k\}$ such that $y_k \to u$. Observe that Pu is an (s, u)-path of odd length. If follows from Lemma 2.1 that $\overline{su}$. Hence, s and one vertex of $V(T)$ are adjacent, say x_1. If $s \to x_1$, then we have $s \to x_1 \to x_2 \to x_3$, which implies $\overline{sx_3}$. If $x_1 \to s$, then we have $x_2 \to x_3 \to x_1 \to s$, which implies that $\overline{sx_2}$. By the arbitrariness of s, the assertion is true.

Finally, if $s \to x_i$, then, since $s \to x_i \to x_{i+1} \to x_{i+2}$, we have $\overline{sx_{i+2}}$. If $x_i \to s$, then, since $x_{i-2} \to x_{i-1} \to x_i \to s$, we have $\overline{sx_{i-2}}$. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let D be a strong 3-quasi-transitive digraph of order $n \geq 4$. If D contains a 3-cycle T as a subdigraph and there exists a vertex $u \in V(D)\setminus V(T)$ such that u and every vertex of T are adjacent, then D is a semicomplete digraph.*

Proof. Let $T = x_1x_2x_3x_1$. From now on, all subscripts appearing in this proof are taken modulo 3.

Claim 1. *For any $w \in V(D)\setminus V(T)$, if w and every vertex of $V(T)$ are adjacent, then w and every vertex of $V(D)\setminus(V(T)\cup\{w\})$ are adjacent.*

Since w and every vertex of $V(T)$ are adjacent and $|V(T)| = 3$, there exist two arcs in the same direction between w and T. Since the converse of a 3-quasi-transitive digraph is also a 3-quasi-transitive digraph. We may, without loss of generality, assume that $w \to x_1$, $w \to x_2$. If $V(D)\setminus(V(T)\cup\{w\}) = \emptyset$, then there is nothing more to prove. Now assume $V(D)\setminus(V(T)\cup\{w\}) \neq \emptyset$. Let $v \in V(D)\setminus(V(T)\cup\{w\})$ be arbitrary. First, we claim that there exists a path of odd length between w and v. Since D is strong, there exists a path from T to v. Let $P = yy_1\cdots y_s$ be a shortest path from T to v.

Suppose that s is even. If $y = x_1$ or $y = x_2$ then wP is a (w, v)-path of odd length. If $y = x_3$, then wx_1x_2P is a (w, v)-path of odd length.

Suppose that s is odd. If $y = x_3$, then wx_2P is a (w, v)-path of odd length. If $y = x_1$, then x_2x_3P is an (x_2, v)-path of odd length, which implies that $\overline{vx_2}$ form Lemma 2.1. If $x_2 \to v$, then wx_1x_2v is a (w, v)-path of odd length. Now assume $v \to x_2$. By the hypothesis, $\overline{wx_3}$. If $\overrightarrow{wx_3}$, then wx_3P is a (w, v)-path of odd length. If $\overrightarrow{x_3w}$, then vx_2x_3w is a (v, w)-path of odd length. If $y = x_2$, then wx_1P is a (w, v)-path of odd length. Hence there exists a path of odd length between w and v. Combining this with Lemma 2.1, we have $\overline{wv}$. By the arbitrariness of v, the claim is true.

Claim 2. *Every vertex of $V(D)\setminus V(T)$ is adjacent to every vertex of $V(T)$.*

If $V(D)\setminus(V(T)\cup\{u\}) = \emptyset$, then there is nothing to prove. Now assume that $V(D)\setminus(V(T)\cup\{u\}) \neq \emptyset$. Let $v \in V(D)\setminus(V(T)\cup\{u\})$ be arbitrary. By the hypothesis and Claim 1, $\overline{uv}$. Without loss of generality, assume that $v \to u$. By Lemma 2.2, v and at least two vertices of $V(T)$ are adjacent, say x_1 and x_2. Next we will show that v and x_3 are also adjacent. If $v \to x_1$ or $x_2 \to v$, then, by Lemma 2.2 we have that $\overline{vx_3}$. So assume that $x_1 \to v$ and $v \to x_2$. If there exists a path of odd length between v and x_3, then, by Lemma 2.1, $\overline{vx_3}$. Now we will show that there exists such a path. We consider two cases:

Case 1. There exists an arc from u of T.

If $u \to x_2$, then vux_2x_3 is a (v, x_3)-path of odd length. Now assume that $x_2 \mapsto u$. If $u \to x_3$, then vx_2ux_3 is a (v, x_3)-path of odd length. So assume $x_3 \mapsto u$ and it must be $u \to x_1$. Observe that x_3ux_1v is an (x_3, v)-path of odd length.

Case 2. There exists no arc from u to T.

Hence $V(T) \mapsto u$. Let $W = \{x \in V(D) : V(T) \mapsto x\}$. Clearly, $u \in W$. If there exists $u' \in W$ such that $u' \to v$, then $x_3x_1u'v$ is an (x_3, v)-path of odd length. So assume $(W, v) = \emptyset$. Combining this with Claim 1, we have that $v \mapsto W$. Since D is strong, $N^+(W) \neq \emptyset$. Let $w \in N^+(W)$ be arbitrary. Then there exists $u'' \in W$ such that $u'' \to w$. By $V(T) \mapsto u''$ and $u'' \to w$, it is not difficult to obtain that w and every vertex of $V(T)$ are adjacent. Then, by Claim 1, w and v are adjacent. If $w \to v$, then $x_3u''wv$ is an (x_3, v)-path of odd length. So assume that $v \mapsto w$. Since $w \notin W$, there exists an arc from w to T. By a similar argument to Case 1, we can find a path of odd length between x_3 and v.

The proof of Lemma 2.3 is complete. □

Consider the digraph F_n with vertex set $\{x_1, x_2, \ldots, x_n\}$ and arc set $\{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_1x_{i+3}, x_{i+3}x_2 : i = 1, 2, \ldots, n-3\}$, where $n \geq 4$.

Lemma 2.4. *Let D be a 3-quasi-transitive digraph of order $n \geq 4$. If D contains a 3-cycle T as a subdigraph and there exists no vertex $V(D)\setminus V(T)$ adjacent to every vertex of T, then D is isomorph to F_n.*

Proof. Let $T = x_1x_2x_3x_1$. Since $n \geq 4$, we have $U = V(D) \setminus V(T) \neq \emptyset$. For any $x \in U$, by Lemma 2.2, x and at least two vertices $V(T)$ are adjacent, say x_1, x_2. If either $x \to x_1$ or $x_2 \to x$, then, by Lemma 2.2, $\overline{xx_3}$, a contradiction. So assume that $x_1 \to x$ and $x \to x_2$. First we show that, for any $x' \in U \setminus \{x\}$, we have that $x_1 \to x'$ and $x' \to x_2$. By Lemma 2.2 and the hypothesis, x' and exactly two vertices of $V(T)$ are adjacent. Suppose that the arcs between x' and T are in the same direction and, without loss of generality, assume that the arcs between x' and $\{x_1, x_2\}$ are in the same direction. If $x' \to \{x_1, x_2\}$, then $x' \to x_1 \to x_2 \to x_3$ implies that $\overline{x'x_3}$, a contradiction. If $\{x_1, x_2\} \to x'$, then $x_3 \to x_1 \to x_2 \to x'$ implies that $\overline{x'x_3}$, a contradiction. Next assume that the arcs between x' and T are in different direction. If $x' \to x_1$, then, by $x' \to x_1 \to x_2 \to x_3$ and $x' \to x_1 \to x \to x_2$, we have that $\overline{x'x_3}$ and $\overline{x'x_2}$, respectively, a contradiction. If $x' \to x_3$, then, by Lemma 2.2, $\overline{x'x_2}$ and so $x_2 \to x'$. Then $x \to x_2 \to x' \to x_3$ implies that $\overline{xx_3}$, a contradiction. So $x' \to x_2$. By Lemma 2.2, $\overline{x'x_1}$ and so $x_1 \to x'$. Next we claim that U is an independent set. If $|U| = 1$, the claim is trivial. Now assume $|U| \geq 2$. Suppose, on the contrary, there exists a pair u, v of U such that $\overline{uv}$. Without loss of generality, assume that $u \to v$. Then $u \to v \to x_2 \to x_3$ implies $\overline{ux_3}$, a contradiction. So the claim is true and the proof of Lemma 2.4 is complete. □

Theorem 2.5. *Let D be a strong 3-quasi-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph, or isomorphic to F_n.*

Proof. If D is a bipartite digraph with bipartition (X, Y). For any pair $x \in X$, $y \in Y$, since D is strong, there exists an (x, y)-path P. Since x, y belong to different parts, the length of P is odd. By Lemma 2.1, x and y are adjacent. So D is a semicomplete bipartite digraph. Now suppose that D is a non-bipartite digraph. By Lemma 2.2, D contains a 3-cycle T as a subdigraph. If $n = 3$, then D is a semicomplete digraph. So assume $n \geq 4$. If there exists a vertex which is adjacent to every vertex of T, then, by Lemma 2.3, D is a semicomplete digraph. If not, then, by Lemma 2.4, D is isomorphic to F_n. The proof of Theorem 2.5 is complete. □

Theorem 2.6 (*[5]*)**.** *A semicomplete digraph is Hamiltonian if and only if it is strong.*

Theorem 2.7 (*[5]*)**.** *A semicomplete bipartite digraph is Hamiltonian if and only if it is strong and contains a cycle factor.*

Corollary 2.8. *A 3-quasi-transitive digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

Proof. Let D be a 3-quasi-transitive digraph. If D is Hamiltonian, then it is strong and has a cycle factor. So assume that D is strong and has a cycle factor. If D is semicomplete or semicomplete bipartite, the result follows from Theorem 2.6 or Theorem 2.7, respectively. Otherwise, D is isomorphic to F_n. Since D has a cycle factor, every vertex x_i with $i \geq 4$ must be contained in some cycle. But any cycle which contains vertex x_i, must contain vertex x_2. Therefore, $n = 4$. Clearly, F_4 is Hamiltonian. □

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