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Journal of Algebra 280 (2004) 472–487
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Stratifying systems via relative simple modules ☆

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Received 5 August 2003

Available online 18 August 2004

Communicated by Kent R. Fuller

Dedicated to Roberto Martínez Villa in his 60th birthday.

Abstract

In this paper we continue the study of stratifying systems, which were introduced by K. Erdmann and C. Sáenz in [Comm. Algebra 31 (7) (2003) 3429–3446]. We show that this new concept provides a categorical generalization of the Δ-modules for a standardly stratified algebra.

Introduction

The algebras to be considered in this paper are basic finite dimensional algebras over an algebraically closed field k. We denote by $\operatorname{mod} R$ the category of finitely generated left R-modules over an algebra R, and $D : \operatorname{mod} R \to \operatorname{mod} R^{\mathrm{op}}$ is the usual duality $\operatorname{Hom}_k(-, k)$.

☆ The first author wants to thank CNPq (Brazil) and CONACyT J200.1221 (Mexico). The second author thanks the Instituto de Matemáticas, UNAM. The third author wants to thank CONACyT G27976-E (Mexico) and Facultad de Ciencias, UNAM.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.06.018

Given $f : M \to N$ and $g : N \to L$ morphisms in $\operatorname{mod} R$ we denote the composition of f and g by gf which is a morphism from M to L.

Given a class $\mathcal{C}$ of R-modules, we denote by $\mathcal{F}(\mathcal{C})$ the full subcategory of $\operatorname{mod} R$ containing the zero module and all modules which are filtered by modules in $\mathcal{C}$. That is, a non-zero R-module M belongs to $\mathcal{F}(\mathcal{C})$, if there is a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in $\mathcal{C}$ for all $i = 1, 2, \ldots, m$. In particular, if $\mathcal{C} = \emptyset$ then $\mathcal{F}(\mathcal{C}) = \{0\}$. It is easy to see that $\mathcal{F}(\mathcal{C})$ is closed under extensions. In general, $\mathcal{F}(\mathcal{C})$ fails to be closed under direct summands, see [7]. Let $\mathcal{X}$ and $\mathcal{Y}$ be full subcategories of $\operatorname{mod} R$. We say that $\operatorname{Ext}^i_R(\mathcal{X}, \mathcal{Y}) = 0$, if $\operatorname{Ext}^i_R(X, Y) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Other categories related with $\mathcal{F}(\mathcal{C})$ are the following:

$$\mathcal{I}(\mathcal{C}) = \bigl\{X \in \operatorname{mod} R\colon \operatorname{Ext}^1_R\bigl(\mathcal{F}(\mathcal{C}), X\bigr) = 0\bigr\},$$
$$\mathcal{P}(\mathcal{C}) = \bigl\{X \in \operatorname{mod} R\colon \operatorname{Ext}^1_R\bigl(X, \mathcal{F}(\mathcal{C})\bigr) = 0\bigr\}.$$

In [3], K. Erdmann and C. Sáenz introduced the concept of stratifying system $(\theta, \underline{Y}, \preceq)$ of size t, where $\theta = \{\theta(i)\}_{i=1}^t$ is a set of non-zero R-modules, and $\underline{Y} = \{Y(i)\}_{i=1}^t$ is a set of indecomposable R-modules in $\mathcal{I}(\theta) \cap \mathcal{F}(\theta)$ satisfying certain conditions related to a total order $\preceq$ given on the set $\Omega_t = \{1, 2, \ldots, t\}$. They showed that the algebra $A = \operatorname{End}({}_R Y)$ is standardly stratified, where ${}_R Y = \coprod_{i=1}^t Y(i)$ and also that, for any R-module M in $\mathcal{F}(\theta)$ the filtration multiplicities $[M : \theta(i)]$ do not depend on a given filtration of M in θ. The R-modules in the set θ will be called relative simple modules of $\mathcal{F}(\theta)$. Observe that the relative simple modules have not proper submodules in $\mathcal{F}(\theta)$. The modules in the set $\underline{Y}$ will be called indecomposable relative injective modules of $\mathcal{F}(\theta)$.

This paper is the first part of a series of two papers. In this one we will study stratifying systems via the set θ of relative simple modules and in the second one via the set of relative projective modules of $\mathcal{F}(\theta)$, see [5].

The paper is organized as follows. In Section 1 we collect some preliminary facts needed later in the paper, which can be found for instance in [1,3,4,7]. Then we give an equivalent definition of a stratifying system $(\theta, \underline{Y}, \preceq)$ of size t, which depends only on the set $\theta = \{\theta(i)\}_{i=1}^t$. We also point out that the category $\mathcal{F}(\theta)$ is closed under direct summands and is a functorially finite subcategory of $\operatorname{mod} R$.

In Section 2 we consider the full subcategories $\mathcal{F}(\theta)$, $\mathcal{I}(\theta)$ and $\mathcal{P}(\theta)$ of $\operatorname{mod} R$ and prove that $\mathcal{F}(\theta) \cap \mathcal{I}(\theta) = \operatorname{add} Y$. Moreover, we prove that the category $\operatorname{add} {}_A T$ is equivalent to the category $\mathcal{F}(\theta) \cap \mathcal{P}(\theta)$, where ${}_A T$ is the characteristic tilting A-module associated to the standardly stratified algebra $A = \operatorname{End}({}_R Y)$. We also show that the concept of a stratifying system generalizes the concept of the standard modules for a standardly stratified algebra and we use our data to show that the determinant of the Cartan matrix of a standardly stratified algebra is not zero.

In Section 3 we study stratifying systems $(\theta, \underline{Y}, \preceq)$ in connection with Y being a generalized tilting R-module.

1. Preliminaries

Let R be an algebra, $\theta = \{\theta(i)\}_{i=1}^{t}$ be a set of R-modules, and $\Omega_t = \{1, 2, \ldots, t\}$. We will consider a total order $\preceq$ on the set Ω_t and we denote by $\leqslant$ (respectively $\leqslant^{\mathrm{op}}$) the natural (opposite natural) total order on Ω_t. It is well known that there is a unique isomorphism $\omega_t : (\Omega_t, \preceq) \to (\Omega_t, \leqslant)$ of ordered sets. Throughout the paper we will also make use of the isomorphism $\sigma_t : (\Omega_t, \leqslant) \to (\Omega_t, \leqslant^{\mathrm{op}})$ of ordered sets given by $\sigma_t(i) = t - i + 1$.

Recall that the category $\mathcal{F}(\theta)$ is the category of R-modules which have a θ-filtration. We start this section by recalling the definition of a stratifying system $(\theta, \underline{Y}, \preceq)$ of size t and then we give a characterization of it, which depends only on the system $(\theta, \preceq)$. We also collect some facts from [1,3,4,7] that will be used later in the paper. At the end of this section we show that for any stratifying system $(\theta, \underline{Y}, \preceq)$ the category $\mathcal{F}(\theta)$ is closed under direct summands and therefore it is a functorially finite subcategory of mod R (see [7]).

Definition 1.1 [3]. Let $\theta = \{\theta(i)\}_{i=1}^{t}$ be a set of non-zero R-modules and $\underline{Y} = \{Y(i)\}_{i=1}^{t}$ be a set of indecomposable R-modules. The system $(\theta, \underline{Y}, \preceq)$ is a stratifying system of size t, if the following three conditions hold:

(1) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(2) For each $i \in \Omega_t$ there is an exact sequence $0 \to \theta(i) \xrightarrow{\alpha_i} Y(i) \to Z(i) \to 0$ such that $Z(i) \in \mathcal{F}(\{\theta(j)\colon\ j \prec i\})$,
(3) $\mathrm{Ext}_R^1(\mathcal{F}(\theta), Y) = 0$, where $Y = \coprod_{i=1}^{t} Y(i)$.

Let $M \in \mathcal{F}(\theta)$. We denote by $\ell_\theta(M)$ the sum $\sum_{i=1}^{t}[M : \theta(i)]$ and call it the θ-length of M. Observe that the set $\underline{Y} = \{Y(1), \ldots, Y(t)\}$ of a given stratifying system $(\theta, \underline{Y}, \preceq)$ consists of pairwise non-isomorphic R-modules. In fact, if $i \prec j$ then $[Y(i) : \theta(j)] = 0$ and $[Y(j) : \theta(j)] = 1$. Therefore, the algebra $A = \mathrm{End}(\coprod_{i=1}^{t} Y(i))$ is basic and the number of simple A-modules (up to isomorphism) is equal to t.

From the following lemma we obtain the uniqueness of the R-morphism $\alpha_i : \theta(i) \to Y(i)$ given in 1.1.

Lemma 1.2. *Let $\alpha_i : \theta(i) \to Y(i)$ be the R-morphism given in* 1.1. *Then α_i is the left minimal $\mathcal{I}(\theta)$-approximation of $\theta(i)$.*

Proof. Since $Y(i)$ is indecomposable it follows that α_i is left minimal. To prove that α_i is left $\mathcal{I}(\theta)$-approximation we consider the exact sequence given in 1.1 and apply to it the functor $\mathrm{Hom}_R(-, X)$ with $X \in \mathcal{I}(\theta)$. So the result follows from the fact that $\mathrm{Ext}_R^1(Z(i), X) = 0$. □

Definition 1.3. Let $(\theta, \underline{Y}, \preceq)$ and $(\theta', \underline{Y'}, \preceq)$ be two stratifying systems of size t, and let $\alpha_i : \theta(i) \to Y(i)$ and $\alpha'_i : \theta'(i) \to Y'(i)$ be the left minimal $\mathcal{I}(\theta)$-approximations of $\theta(i)$ and $\theta'(i)$ respectively.

A morphism $f : (\theta, \underline{Y}, \preceq) \to (\theta', \underline{Y'}, \preceq)$ is a set of morphisms $f = \{f_1(i), f_2(i)\}_{i=1}^{t}$, where $f_1(i) : \theta(i) \to \theta'(i)$ and $f_2(i) : Y(i) \to Y'(i)$ are R-morphisms such that $f_2(i)\alpha_i = \alpha'_i f_1(i)$, for all $i = 1, \ldots, t$.

The morphism $f = \{f_1(i), f_2(i)\}_{i=1}^t : (\theta, \underline{Y}, \preceq) \to (\theta', \underline{Y'}, \preceq)$ is an isomorphism if $f_1(i)$ and $f_2(i)$ are R-isomorphisms for any i.

Proposition 1.4. *Let $(\theta, \underline{Y}, \preceq)$ and $(\theta, \underline{Y'}, \preceq)$ be stratifying systems of size t. Then for each $i = 1, \ldots, t$, there exists a R-isomorphism $f_i : Y(i) \to Y'(i)$, such that $f = \{1_{\theta(i)}, f_i\}_{i=1}^t : (\theta, \underline{Y}, \preceq) \to (\theta, \underline{Y'}, \preceq)$ is an isomorphism of stratifying systems.*

Proof. Let $\alpha_i : \theta(i) \to Y(i)$ and $\alpha_i' : \theta(i) \to Y'(i)$ be the left-minimal $\mathcal{I}(\theta)$-approximations of $\theta(i)$ and $\theta'(i)$ respectively. Hence, there exist R-morphisms $f_i : Y(i) \to Y'(i)$, $g_i : Y'(i) \to Y(i)$ such that $f_i\alpha_i = \alpha_i'$ and $g_i\alpha_i' = \alpha_i$ for all $i = 1, 2, \ldots, t$. Therefore, using that α_i and α_i' are left-minimal we get that $f_i g_i$ and $g_i f_i$ are isomorphisms for all $i = 1, 2, \ldots, t$, proving the result. □

We will make use of the following result, which appears in [3], in order to give a characterization of stratifying systems depending only on the set of relative simple modules. This characterization can be taken as an alternative definition of stratifying systems.

Proposition 1.5 [3]. *Given a set $\theta = \{\theta(i)\}_{i=1}^t$ of non-zero R-modules and a total order $\preceq$ on the set Ω_t, the following conditions* (a) *and* (b) *are equivalent.*

(a) *There exists a set of indecomposable R-modules $\underline{Y} = \{Y(i)\}_{i=1}^t$ such that $(\theta, \underline{Y}, \preceq)$ is a stratifying system of size t.*
(b) *The set θ satisfies the following conditions*:
  (i) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ *for $j \succ i$,*
  (ii) $\mathrm{Ext}_R^1(\theta(j), \theta(i)) = 0$, *for $j \succeq i$,*
  (iii) $\theta(i)$ *is indecomposable, for all $i = 1, 2, \ldots, t$.*

As consequence of 1.5 and 1.4, we state now the promised characterization of stratifying systems.

Characterization 1.6. *A stratifying system $(\theta, \preceq)$ of size t consists of a set $\theta = \{\theta(i)\}_{i=1}^t$ of indecomposable R-modules and a total order $\preceq$ on the set Ω_t satisfying the following conditions*:

(i) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ *for $j \succ i$,*
(ii) $\mathrm{Ext}_R^1(\theta(j), \theta(i)) = 0$, *for $j \succeq i$.*

Definition 1.7. A stratifying system $(\theta, \preceq)$ is *standard* if ${}_RR \in \mathcal{F}(\theta)$. Dually, we say that $(\theta, \preceq)$ is *costandard* if $D(R_R) \in \mathcal{F}(\theta)$.

Next we fix the following notation. Let A be an algebra and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t\}$ be a complete set of primitive orthogonal idempotents. We denote by $P_i = A\varepsilon_i$ the indecomposable projective A-module corresponding to ε_i and by S_i the simple A-module top of P_i for $i = 1, \ldots, t$. The standard A-module ${}_A\Delta(i)$ is by definition the maximal quotient of P_i with composition factors only amongst S_j with $j \leqslant i$. We shall denote by ${}_A\Delta$ the set of all standard modules. By Lemmas 1.2 and 1.3 in [2] we have that ${}_A\Delta$ satisfies 1.6. Hence we get that $({}_A\Delta, \leqslant)$ is always a stratifying system of size t.

It is well known, see [2,4,7], that the category $\mathcal{F}({}_A\Delta)$ is a functorially finite subcategory of $\operatorname{mod} A$, closed under direct summands and kernels of surjections.

Dually, we have the notion of costandard A-modules. Let I_i be the injective envelope of the simple A-module S_i. The costandard A-module ${}_A\nabla(i)$ is the maximal submodule of I_i with composition factors only amongst S_j with $j \leqslant i$. We denote by ${}_A\nabla$ the set of all costandard modules. Using the definition of costandard modules it can be proved that $({}_A\nabla, \leqslant^{\mathrm{op}})$ is always a stratifying system of size t.

Note that the stratifying system $({}_A\Delta, \leqslant)$ is standard if and only if A is a standardly stratified algebra with the same order of the simple modules. Recall that the standardly stratified algebra A is called *quasi-hereditary* if $\dim_k \operatorname{End}({}_A\Delta(i)) = 1$ for any i.

By duality arguments, it can be proved that $({}_A\Delta, \leqslant)$ is standard if and only if $({}_A\nabla^{\mathrm{op}}, \leqslant^{\mathrm{op}})$ is costandard.

We recall that an A-module T is a *generalized tilting* A-module if the following three conditions hold:

(a) T has finite projective dimension,
(b) $\operatorname{Ext}^i_A(T, T) = 0$ for all $i > 0$, and
(c) there exists an exact sequence $0 \to {}_A A \to T_0 \to T_1 \to \cdots \to T_m \to 0$ with $T_j \in \operatorname{add} T$ for all j, where $\operatorname{add} T$ is the full subcategory of $\operatorname{mod} A$ whose objects are direct sums of direct summands of T.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} A$ closed under extensions and $\mathcal{S}_{\mathcal{X}}$ the set of exact sequences in $\operatorname{mod} A$ with terms in $\mathcal{X}$. The pair $(\mathcal{X}, \mathcal{S}_{\mathcal{X}})$ is an *exact subcategory* of $\operatorname{mod} A$. Let $(\mathcal{X}, \mathcal{S}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{S}_{\mathcal{Y}})$ be exact subcategories of $\operatorname{mod} A$ and $\operatorname{mod} A'$ respectively. An additive functor $H : \mathcal{X} \to \mathcal{Y}$ is said to be an *exact functor* if $H(\mathcal{S}_{\mathcal{X}}) \subseteq \mathcal{S}_{\mathcal{Y}}$.

The following result shows that if the stratifying system $({}_A\Delta, \leqslant)$ is standard then there is a distinguished generalized tilting A-module T. Henceforth T will be called *the characteristic tilting* A-module associated to the standard stratifying system $({}_A\Delta, \leqslant)$.

Theorem 1.8 [1,4,7]. *Let A be an algebra and assume that $({}_A\Delta, \leqslant)$ is standard of size t. Then there is a basic generalized tilting A-module T such that $\mathcal{F}({}_A\Delta) \cap \mathcal{I}({}_A\Delta) = \operatorname{add} T$. Moreover, if $A' = \operatorname{End}({}_A T)$ then following statements hold*:

(i) $T = \coprod_{i=1}^t T(i)$*, where $T(i)$ is indecomposable and for each i there is an exact sequence $0 \to {}_A\Delta(i) \to T(i) \to X(i) \to 0$ with $X(i) \in \mathcal{F}(\{{}_A\Delta(j)\colon\ j < i\})$,*
(ii) *let $\{\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_t\}$ be a complete set of primitive orthogonal idempotents of A' such that $A'\varepsilon'_i \simeq \operatorname{Hom}_A(T(\sigma_t(i)), T)$ for any i. Then ${}_{A'}\Delta(i) \simeq \operatorname{Hom}_A({}_A\Delta(\sigma_t(i)), T)$ for each $i = 1, 2, \ldots, t$. Furthermore the stratifying system $({}_{A'}\Delta, \leqslant)$ is standard.*
(iii) *the functor* $\operatorname{Hom}_A(-, T) : \mathcal{F}({}_A\Delta) \to \mathcal{F}({}_{A'}\Delta)$ *is an exact duality.*
(iv) $\mathcal{F}({}_A\Delta) \cap \mathcal{P}({}_A\Delta) = \operatorname{add} {}_A A$.

In the next theorem, we will collect some facts from [3]. Let R be an algebra, $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t and $A = \operatorname{End}({}_R Y)$, where $Y = \coprod_{i=1}^t Y(i)$. Let F be the functor $\mathrm{Hom}_R(-, {}_RY_{A^{\mathrm{op}}}) : \mathrm{mod}\, R \to \mathrm{mod}\, A$, G be the functor $\mathrm{Hom}_A(-, {}_RY_{A^{\mathrm{op}}}) : \mathrm{mod}\, A \to \mathrm{mod}\, R$, and $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t\}$ be a complete set of primitive orthogonal idempotents of A such that $A\varepsilon_i \simeq F(Y(\omega_t^{-1}\sigma_t(i)))$ for all $i = 1, 2, \dots, t$, where $\omega_t : (\Omega_t, \preceq) \to (\Omega_t, \leqslant)$ and $\sigma_t : (\Omega_t, \leqslant) \to (\Omega_t, \leqslant^{\mathrm{op}})$ are the unique isomorphism of ordered sets.

Since $F(\mathcal{F}(\theta)) \subseteq \mathcal{F}({}_A\Delta)$ and $G(\mathcal{F}({}_A\Delta)) \subseteq \mathcal{F}(\theta)$, we will denote also by F and G the functors $F|_{\mathcal{F}(\theta)}$ and $G|_{\mathcal{F}({}_A\Delta)}$ respectively.

Theorem 1.9 [3]. *Let $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t, and let F and G be the functors defined above. Then*:

(i) *the functors $F : \mathcal{F}(\theta) \to \mathcal{F}({}_A\Delta)$ and $G : \mathcal{F}({}_A\Delta) \to \mathcal{F}(\theta)$ are inverse exact dualities,*
(ii) *$F(\theta(\omega_t^{-1}\sigma_t(i))) \simeq {}_A\Delta(i)$, for all $i = 1, 2, \dots, t$,*
(iii) *the stratifying system $({}_A\Delta, \leqslant)$ is standard,*
(iv) *for any $X \in \mathcal{F}(\theta)$, there is an exact sequence $0 \to X \to Y_0 \to Y_1 \to \cdots \to Y_k \to 0$, where $Y_i \in \mathrm{add}\, Y$ and $k < t$,*
(v) *The determinant of the Cartan Matrix C_A of A is given by the formula $\det C_A = \prod_{i=1}^{t} \dim_k \mathrm{End}({}_R\theta(i))$.*

As a consequence of 1.9 we get the following result, which shows that the relative injective modules determine the relative simple modules.

Proposition 1.10. *Let $(\theta', \underline{Y}, \preceq)$ and $(\theta, \underline{Y}, \preceq)$ be stratifying systems of size t. Then, there exists an isomorphism of stratifying systems $f : (\theta', \underline{Y}, \preceq) \to (\theta, \underline{Y}, \preceq)$.*

Proof. By 1.9 we know that the functor $F' = F|_{\mathcal{F}(\theta')} : \mathcal{F}(\theta') \to \mathcal{F}({}_A\Delta)$ induces an isomorphism

$$F(\beta_i) : F\big(\theta(i)\big) \xrightarrow{\sim} F\big(\theta'(i)\big) \quad \text{for any } i.$$

Let $\alpha_i : \theta(i) \to Y(i)$ and $\alpha_i' : \theta'(i) \to Y(i)$ be the R-morphisms given in 1.1. By 1.2 we know that α_i and α_i' are left-minimal. Then the A-morphisms

$$F\big(Y(i)\big) \xrightarrow{F(\beta_i)F(\alpha_i)} F\big(\theta'(i)\big) \quad \text{and} \quad F\big(Y(i)\big) \xrightarrow{F(\alpha_i')} F\big(\theta'(i)\big)$$

are right-minimal for all $i = 1, 2, \dots, t$. Hence there is an A-isomorphism $F(f_i) : F(Y(i)) \to F(Y(i))$ such that $F(\beta_i)F(\alpha_i) = F(\alpha_i')F(f_i)$ for each i. Therefore $f = \{\beta_i, f_i\}_{i=1}^t : (\theta', \underline{Y}, \preceq) \to (\theta, \underline{Y}, \preceq)$ is an isomorphism. □

Observe that Ringel gives in Section 1 of [7] an example, which shows that conditions (ii) and (iii) of 1.5 are not enough for $\mathcal{F}(\theta)$ to be closed under direct summands. The following corollary gives sufficient conditions to obtain this fact.

Corollary 1.11. *For any stratifying system $(\theta, \preceq)$ the category $\mathcal{F}(\theta)$ is a functorially finite subcategory of* $\mathrm{mod}\, R$ *which is closed under direct summands.*

Proof. Using the duality F given in 1.9 and the fact that $\mathcal{F}({}_A\Delta)$ is closed under direct summands, we get that $\mathcal{F}(\theta)$ is also closed under direct summands. The result follows now by Corollary 1 in [7]. $\square$

2. Stratifying systems

Let $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t, $Y = \coprod_{i=1}^{t} Y(i)$, $A = \operatorname{End}({}_R Y)$ and ${}_A T$ be the characteristic tilting A-module associated to the standardly stratified algebra $A = \operatorname{End}({}_R Y)$. In this section we prove that $\mathcal{F}(\theta) \cap \mathcal{I}(\theta) = \operatorname{add} Y$ and also that the category $\mathcal{F}(\theta) \cap \mathcal{P}(\theta)$ corresponds to the category $\operatorname{add} {}_A T$, under the duality $F = \operatorname{Hom}_R(-, {}_R Y_{A^{\mathrm{op}}})$. Finally, we give conditions for a stratifying system of size t to be isomorphic to the standard stratifying system $({}_R\Delta, \leqslant)$ and prove that the determinant of the Cartan matrix of a standardly stratified algebra R is non-zero.

In the following proposition we shall prove that, for a given algebra R, there is always a stratifying system of size t in $\operatorname{mod} R$ for any $t \leqslant n$, where n is the number of simple R-modules (up to isomorphism).

Proposition 2.1. *Let R be an algebra, $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of R, and let ${}_R\Delta_t = \{{}_R\Delta(j)\colon\ j \leqslant t\}$. Then*:

(a) *$({}_R\Delta_t, \leqslant)$ is a stratifying system of size t, for any $t \leqslant n$.*
(b) *Let $({}_R\Delta, \leqslant)$ be standard, and let ${}_R T = \coprod_{i=1}^{n} T(i)$ be the indecomposable decomposition, given in* 1.8, *of the characteristic tilting R-module. Then $({}_R\Delta_t, \underline{T_t}, \leqslant)$ is a stratifying system of size t for any $t \leqslant n$, where $\underline{T_t} = \{T(j)\colon\ j \leqslant t\}$.*

Proof. (a) follows from the fact that $({}_R\Delta, \leqslant)$ is a stratifying system (see 1.2 and 1.3 in [2]). And (b) is a consequence of 1.8. $\square$

We also state the dual result:

Proposition 2.2. *Let R be an algebra, $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of R, and let ${}_R\nabla_t = \{{}_R\nabla(j)\colon\ j \leqslant^{\mathrm{op}} t\}$. Then*:

(a) *$({}_R\nabla_t, \leqslant^{\mathrm{op}})$ is a stratifying system of size $\sigma_n(t)$ for any $t \leqslant n$.*
(b) *Let $({}_R\nabla, \leqslant^{\mathrm{op}})$ be costandard and ${}_R I = \coprod_{i=1}^{n} I_i$, where I_i is the injective envelope of the simple module S_i for each $1 \leqslant i \leqslant n$. Then $({}_R\nabla_t, \underline{I_t}, \leqslant^{\mathrm{op}})$ is a stratifying system of size $\sigma_n(t)$ for any $t \leqslant n$, where $\underline{I_t} = \{I_j\colon\ j \leqslant^{\mathrm{op}} t\}$.*

The following theorem gives necessarily and sufficient conditions on a set of indecomposable modules $\underline{Y}$ for the existence of a stratifying system $(\theta, \underline{Y}, \preceq)$. We recall that $\omega_t : (\Omega_t, \preceq) \to (\Omega_t, \leqslant)$ and $\sigma_t : (\Omega_t, \leqslant) \to (\Omega_t, \leqslant^{\mathrm{op}})$ are the unique isomorphism of ordered sets, where $\Omega_t = \{1, 2, \ldots, t\}$.

Theorem 2.3. *Let $\underline{Y} = \{Y(i)\}_{i=1}^{t}$ be a set of pairwise non-isomorphic indecomposable R-modules, $Y = \coprod_{i=1}^{t} Y(i)$, $A = \operatorname{End}({}_R Y)$, and $\preceq$ be a total order on Ω_t. We fix a* *complete set* $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t\}$ *of primitive orthogonal idempotents of* A *such that* $A\varepsilon_i \simeq \operatorname{Hom}_R(Y(\omega_t^{-1}\sigma_t(i)), {}_RY_{A^{\mathrm{op}}})$ *for each* $i = 1, 2, \dots, t$. *Then the following statements* (1) *and* (2) *are equivalent*:

(1) (a) *the stratifying system* $({}_A\Delta, \leqslant)$ *is standard and* $\operatorname{Ext}^1_A(\mathcal{F}({}_A\Delta), {}_AY) = 0$,
(b) *there is a full subcategory* $\mathcal{A}$ *of* mod R, *closed under extensions, such that* $Y(i) \in \mathcal{A}$ *for each* $i = 1, 2, \dots, t$ *and* $\operatorname{Ext}^1_R(\mathcal{A}, {}_RY) = 0$,
(c) *the functors* $F = \operatorname{Hom}_R(-, {}_RY_{A^{\mathrm{op}}}) : \mathcal{A} \to \mathcal{F}({}_A\Delta)$ *and* $G = \operatorname{Hom}_A(-, {}_RY_{A^{\mathrm{op}}}) : \mathcal{F}({}_A\Delta) \to \mathcal{A}$ *are inverse exact dualities.*

(2) *There exists a family of* R*-modules* $\theta = \{\theta(i)\}_{i=1}^t$ *such that* $(\theta, \underline{Y}, \preceq)$ *is a stratifying system.*

Proof. Assume that (1) holds and let $\mu := \omega_t^{-1}\sigma_t$. We define $\theta(i) := G({}_A\Delta(\mu^{-1}(i)))$ for each i and we shall prove that $(\theta, \underline{Y}, \preceq)$ is a stratifying system.

Let $j \succ i$. Then $\mu^{-1}(j) < \mu^{-1}(i)$ and therefore we get $\operatorname{Hom}_R(\theta(j), \theta(i)) \simeq \operatorname{Hom}_A({}_A\Delta(\mu^{-1}(i)), {}_A\Delta(\mu^{-1}(j))) = 0$. So the first condition of 1.1 holds. For the second condition we fix i and we consider the exact sequence

$$0 \to U\big(\mu^{-1}(i)\big) \to A\varepsilon_{\mu^{-1}(i)} \to {}_A\Delta\big(\mu^{-1}(i)\big) \to 0, \qquad (*)$$

where $U(\mu^{-1}(i))$ is the sum of the images of all morphisms $A\varepsilon_{\mu^{-1}(j)} \to A\varepsilon_{\mu^{-1}(i)}$ with $\mu^{-1}(j) > \mu^{-1}(i)$ (see 1.1 in [2]). Since $({}_A\Delta, \leqslant)$ is standard we get that

$$U\big(\mu^{-1}(i)\big) \in \mathcal{F}\big(\big\{{}_A\Delta\big(\mu^{-1}(j)\big)\colon \mu^{-1}(j) > \mu^{-1}(i)\big\}\big) = \mathcal{F}\big(\big\{{}_A\Delta\big(\mu^{-1}(j)\big)\colon j \prec i\big\}\big).$$

Applying the functor G to $(*)$ we obtain the exact sequence

$$0 \to \theta(i) \to Y(i) \to G\big(U\big(\mu^{-1}(i)\big)\big) \to 0, \qquad (**)$$

and by induction on the ${}_A\Delta$-length of $U(\mu^{-1}(i))$ we get that $G(U(\mu^{-1}(i))) \in \mathcal{F}(\{\theta(j)\colon j \prec i\})$. Hence, the sequence $(**)$ satisfies the condition (2) of 1.1.

Finally, to see that $\operatorname{Ext}^1_R(\mathcal{F}(\theta), Y) = 0$ it is enough to prove that $\mathcal{F}(\theta) \subseteq \mathcal{A}$. This follows from the hypothesis that $\mathcal{A}$ is closed under extensions and $\theta(i) \in \mathcal{A}$ for each $i = 1, 2, \dots, t$.

Now we show that statement (2) implies (1). This implication is a consequence of 1.9, where we take $\mathcal{F}(\theta) = \mathcal{A}$ and the fact that $\operatorname{Ext}^1_A(\mathcal{F}({}_A\Delta), {}_AY) = 0$ (see Theorem 1.6 in [3]). □

We observe that the category $\mathcal{A}$ satisfying the conditions given in 2.3 is uniquely determined by the family $\underline{Y}$. Moreover $\mathcal{A} = \mathcal{F}(\theta)$, where $\theta(i) = G({}_A\Delta(\sigma_t^{-1}\omega_t(i)))$ for any i.

For a standardly stratified algebra A we have that the characteristic tilting A-module T satisfies $\mathcal{F}({}_A\Delta) \cap \mathcal{I}({}_A\Delta) = \operatorname{add} T$. The following result is a generalization of this fact.

Theorem 2.4. *If* $(\theta, \underline{Y}, \preceq)$ *is a stratifying system of size* t *then* $\mathcal{F}(\theta) \cap \mathcal{I}(\theta) = \operatorname{add} Y$, *where* $Y = \coprod_{i=1}^t Y(i)$.

Proof. Let $A = \operatorname{End}({}_RY)$. Since each $Y(i) \in \mathcal{F}(\theta) \cap \mathcal{I}(\theta)$ we have that $\operatorname{add} Y \subseteq \mathcal{F}(\theta) \cap \mathcal{I}(\theta)$. Next we prove that $\mathcal{F}(\theta) \cap \mathcal{I}(\theta) \subseteq \operatorname{add} Y$. Let $N \in \mathcal{F}(\theta) \cap \mathcal{I}(\theta)$ be an indecomposable R-module, and let

$$0 \to M' \to L' \to F(N) \to 0, \qquad (*)$$

be a short exact sequence in $\mathcal{F}({}_A\Delta)$. Using the duality F given in 1.9 we know that there exist M and L in $\mathcal{F}(\theta)$ such that the sequence $(*)$ has the form $0 \to F(M) \to F(L) \to F(N) \to 0$. Then using the functor G giving in 1.9 and the exact sequence $(*)$ we get that the sequence $0 \to N \to L \to M \to 0$ is exact. This sequence splits because $N \in \mathcal{I}(\theta) \cap \mathcal{F}(\theta)$ and $M \in \mathcal{F}(\theta)$. Hence the exact sequence $(*)$ splits. Thus $F(N) \in \mathcal{F}({}_A\Delta) \cap \mathcal{P}({}_A\Delta) = \operatorname{add} {}_AA$ because $({}_A\Delta, \leqslant)$ is a standard stratifying system (see 1.8(iv)). Therefore $F(N) \simeq F(Y(i))$ for some i and so $N \simeq Y(i)$, since F is a duality. $\square$

Theorem 2.5. *Let $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t and $A = \operatorname{End}({}_RY)$, where $Y = \coprod_{i=1}^t Y(i)$. If ${}_AT$ is the characteristic tilting A-module associated to $({}_A\Delta, \leqslant)$ (see* 1.9)*, then $X \in \mathcal{F}(\theta) \cap \mathcal{P}(\theta)$ if and only if $F(X) = \operatorname{Hom}_R(X, {}_RY_{A^{\mathrm{op}}}) \in \operatorname{add} {}_AT$.*

Proof. By 1.9, the functors F and G preserve exact sequences in $\mathcal{F}(\theta)$ and in $\mathcal{F}({}_A\Delta)$ respectively, and both categories are closed by extensions. So there is a natural isomorphism of abelian groups from $\operatorname{Ext}^1_R(M, N)$ to $\operatorname{Ext}^1_A(F(N), F(M))$ given by $[\eta] \mapsto [F(\eta)]$. Therefore, $\operatorname{Ext}^1_R(X, \mathcal{F}(\theta)) = 0$ if and only if $\operatorname{Ext}^1_A(\mathcal{F}({}_A\Delta), F(X)) = 0$ and this last condition is equivalent to $F(X) \in \mathcal{F}({}_A\Delta) \cap \mathcal{I}({}_A\Delta) = \operatorname{add} {}_AT$. $\square$

The next theorem gives necessarily and sufficient conditions for the set θ of a standard stratifying system $(\theta, \underline{Y}, \preceq)$ to coincide with the set of Δ-modules of a standardly stratified algebra R. To do that, we will fix a complete set of primitive orthogonal idempotents of $A = \operatorname{End}({}_RY)$ and $A' = \operatorname{End}({}_AT)$ respectively, as we did in 1.8 and 1.9. We recall that $\omega_t : (\Omega_t, \preceq) \to (\Omega_t, \leqslant)$ and $\sigma_t : (\Omega_t, \leqslant) \to (\Omega_t, \leqslant^{\mathrm{op}})$ are the unique isomorphism of ordered sets, where $\Omega_t = \{1, 2, \ldots, t\}$.

Theorem 2.6. *Let R be an algebra, $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t, $A = \operatorname{End}({}_RY)$ with $Y = \coprod_{i=1}^t Y(i)$, and ${}_AT$ be the characteristic tilting A-module associated to $({}_A\Delta, \leqslant)$. Then the following conditions are equivalent*:

(a) *$\mathcal{F}(\theta)$ is closed under kernels of surjections and $(\theta, \preceq)$ is standard,*
(b) *$\mathcal{F}(\theta) \cap \mathcal{P}(\theta) = \operatorname{add} {}_RR$,*
(c) *the stratifying system $(\theta, \preceq)$ is standard and t is the number of non-isomorphic simple R-modules.*
(d) *$R \simeq \operatorname{End}({}_AY)$ and ${}_AY_{R^{\mathrm{op}}} \simeq {}_AT_{R^{\mathrm{op}}}$,*
(e) *there is a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_t\}$ of R, such that ${}_R\Delta(i) \simeq \theta(\omega_t^{-1}(i))$ for any $i = 1, 2, \ldots, t$, and R is a standardly stratified algebra with the given ordering of the simple R-modules.*

Proof. (a) $\Rightarrow$ (b). Let $X \in \mathcal{F}(\theta) \cap \mathcal{P}(\theta)$ be indecomposable and let $f : P_0(X) \to X$ be the projective cover of X. Since ${}_R R \in \mathcal{F}(\theta)$ we have that $P_0(X) \in \mathcal{F}(\theta)$. Therefore, the exact sequence

$$0 \to \operatorname{Ker} f \to P_0(X) \to X \to 0 \qquad (*)$$

lies in $\mathcal{F}(\theta)$, since $\mathcal{F}(\theta)$ is closed under kernels of surjections. Using the fact that $X \in \mathcal{P}(\theta)$ we obtain that $(*)$ splits. Hence X is a projective R-module, proving that $\mathcal{F}(\theta) \cap \mathcal{P}(\theta) \subseteq \operatorname{add} {}_R R$. The other inclusion follows from the fact that $\mathcal{F}(\theta)$ is closed under extensions and under direct summands (see 1.11).

(b) $\Rightarrow$ (d). Using 2.5 and the assumptions that $\mathcal{F}(\theta) \cap \mathcal{P}(\theta) = \operatorname{add} {}_R R$ and that R is basic, we get that $F(R) = \operatorname{Hom}_R({}_R R_R, {}_R Y_{A^{\mathrm{op}}}) \simeq {}_A T_{R^{\mathrm{op}}}$. On the other hand, ${}_A Y_{R^{\mathrm{op}}} \simeq \operatorname{Hom}_R(R, {}_R Y_{A^{\mathrm{op}}}) = F(R)$. Thus ${}_A Y_{R^{\mathrm{op}}} \simeq {}_A T_{R^{\mathrm{op}}}$. Finally $R \simeq GF(R) = \operatorname{Hom}_A(F(R), {}_A Y) \simeq \operatorname{End}({}_A Y)$.

(d) $\Rightarrow$ (e). Let $A' = \operatorname{End}({}_A T)$ and $\mu := \omega_t^{-1}\sigma_t$. Then by 1.8 and 1.9 we get that ${}_A\Delta(i) \simeq F(\theta(\mu(i)))$ and ${}_{A'}\Delta(i) \simeq \operatorname{Hom}_A({}_A\Delta(\sigma_t(i)), T)$ for any i. Since $({}_A\Delta, \leqslant)$ is standard we get that $({}_{A'}\Delta, \leqslant)$ is standard. On the other hand, $R \simeq \operatorname{End}({}_A Y) \simeq \operatorname{End}({}_A T) = A'$. Hence ${}_R Y_{A^{\mathrm{op}}} \simeq {}_{A'} T_{A^{\mathrm{op}}}$. Then by 1.8 and 1.9 we get

$$\begin{aligned} \theta(i) &\simeq G\big({}_A\Delta\big(\mu^{-1}(i)\big)\big) \simeq \operatorname{Hom}_A\big({}_A\Delta\big(\mu^{-1}(i)\big), {}_{A'}T_{A^{\mathrm{op}}}\big) \simeq {}_{A'}\Delta\big(\sigma_t^{-1}\mu^{-1}(i)\big) \\ &\simeq {}_R\Delta\big(\sigma_t^{-1}\mu^{-1}(i)\big) = {}_R\Delta\big(\omega_t(i)\big) \end{aligned}$$

for any i, proving that (d) implies (e).

(e) $\Rightarrow$ (a). Follows from the well known fact that $\mathcal{F}({}_R\Delta)$ is closed under kernels of surjections.

(c) $\Rightarrow$ (b). We have that the number of simple R-modules (up to isomorphism) is equal to t. On the other hand, the number of indecomposable direct summands of the characteristic tilting A-module ${}_A T$ is equal to t (see 1.8(i) and 1.9(ii)). Then by 2.5 we get that the number of non-isomorphic indecomposable R-modules in $\mathcal{F}(\theta) \cap \mathcal{P}(\theta)$ is equal to t. Therefore, using that ${}_R R \in \mathcal{F}(\theta)$ we obtain that $\mathcal{F}(\theta) \cap \mathcal{P}(\theta) = \operatorname{add} R$.

(b) $\Rightarrow$ (c). This follows from the fact that (b) implies (e). $\square$

Remark 2.7.

(a) Let R be an algebra, s the number of simple R-modules up to isomorphism and $(\theta, \preceq)$ an stratifying system of size t. If $(\theta, \preceq)$ is standard then $s \leqslant t$. Indeed, let $P_1, P_2, \ldots, P_s$ be the non-isomorphic indecomposable projective R-modules. Since $\bigoplus_{i=1}^t P_i \in \mathcal{F}(\theta)$ we get for each i an exact sequence in $\mathcal{F}(\theta)$

$$0 \to \operatorname{Ker} \beta_i \to P_i \xrightarrow{\beta_i} \theta(k_i) \to 0.$$

The morphism β_i is right-minimal, since P_i is indecomposable. Hence the map $\beta : \{P_i\colon 1 \leqslant i \leqslant s\} \to \theta$ defined by $P_i \mapsto \theta(k_i)$ is an injection, proving that $s \leqslant t$.

(b) Consider the algebra $R = kQ/I$, where Q is the following quiver:

$$3 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 2 \xleftarrow{\gamma} 4$$

and I is the ideal generated by $\beta\gamma$. Taking $\theta(1) = S(1) = {}_R\Delta(1) = P(1)$, $\theta(2) = {}_R\Delta(2) = P(2)$, $\theta(3) = {}_R\Delta(3) = P(3)$, $\theta(4) = {}_R\Delta(4) = P(4) = I(2)$ and $\theta(5) = S(4)$, we get that the stratifying system $(\theta, \leqslant)$ is standard of size 5, whereas the canonical stratifying system $({}_R\Delta, \leqslant)$ is standard of size 4.

The following result was also obtained by C. Xi, see Theorem 4.5 in [4]. To state it we shall consider, as in 1.8, the indecomposable decomposition ${}_RT = \coprod_{i=1}^{t} {}_RT(i)$ of the characteristic tilting R-module ${}_RT$ associated to the standard stratifying system $({}_R\Delta, \leqslant)$ of size t. We fix a complete set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t\}$ of primitive orthogonal idempotents of $A = \operatorname{End}({}_RT)$ such that $A\varepsilon_i \simeq \operatorname{Hom}_R({}_RT(\sigma_t(i)), {}_RT_{A^{\mathrm{op}}})$ for any i.

Corollary 2.8. *Let R be a standardly stratified algebra and ${}_RT$ be the characteristic tilting R-module associated to $({}_R\Delta, \leqslant)$. Let $A = \operatorname{End}({}_RT)$ and ${}_AT'$ be the characteristic tilting A-module associated to $({}_A\Delta, \leqslant)$. Then $R \simeq \operatorname{End}({}_AT')$ and ${}_AT'_{R^{\mathrm{op}}} \simeq {}_AT_{R^{\mathrm{op}}}$.*

Proof. Since the stratifying system $({}_R\Delta, \underline{{}_RT}, \leqslant)$ is standard (see 2.1) we obtain by 1.8 that the stratifying system $({}_A\Delta, \underline{{}_AT'}, \leqslant)$ is standard. Then the result follows now from the items (d) and (e) of the previous theorem. □

Let R be an standardly stratified algebra. The following corollary allows us to compute the determinant of the Cartan matrix C_R and shows that it is non-zero. We also get that $\det C_R = 1$ is equivalent to the fact that R is quasi-hereditary.

We recall that $C_R(ij) := \dim_k \operatorname{Hom}_R(Re_i, Re_j)$, where $\{e_1, e_2, \ldots, e_t\}$ is a complete set of primitive orthogonal idempotents of R.

Corollary 2.9. *If the stratifying system $({}_R\Delta, \leqslant)$ is standard of size t and C_R is the Cartan matrix of R, then $\det C_R = \prod_{i=1}^{t} \dim_k \operatorname{End}({}_R\Delta(i))$.*

Proof. The result follows from 1.9(v) and the previous corollary. □

Corollary 2.10. *Let $({}_R\Delta, \leqslant)$ be standard. Then R is quasi-hereditary if and only if $\det C_R = 1$.*

Proof. The result follows from 2.9. □

3. Finite projective dimension

Let R be an algebra. For any R-module M we denote by $\operatorname{pd} M$ the projective dimension of M. We introduce the following well known full subcategories of $\operatorname{mod} R$:

$$\mathcal{P}^{\leqslant t}(R) = \{M \in \operatorname{mod} R\colon \operatorname{pd} M \leqslant t\} \quad \text{and} \quad \mathcal{P}^{<\infty}(R) = \bigcup_{t \geqslant 0} \mathcal{P}^{\leqslant t}(R).$$

Let $\mathcal{C}$ be a class of modules in $\operatorname{mod} R$. We denote by $\mathcal{C}^\vee$ the full subcategory of $\operatorname{mod} R$ whose objects are the R-modules having a finite $\mathcal{C}$-coresolution. That is, $X \in \mathcal{C}^\vee$ if there is a long exact sequence $0 \to X \to X_0 \to X_1 \to \cdots \to X_m \to 0$ with $X_i \in \mathcal{C}$ for all $i = 0, 1, \ldots, m$. We denote by $\mathcal{P}_R/\mathcal{F}({}_R\Delta)$ the full subcategory of $\operatorname{mod} R$ having as objects the R-modules M such that there is an exact sequence $0 \to X \to P \to M \to 0$, where $X \in \mathcal{F}({}_R\Delta)$ and P is a projective R-module. In this section we study standard stratifying systems $(\theta, \underline{Y}, \preceq)$ in connection with Y being a generalized tilting R-module. We give an example where $\mathcal{F}(\theta) \neq (\operatorname{add} Y)^\vee$ even when by 1.9(iv) we know that $\mathcal{F}(\theta) \subseteq (\operatorname{add} Y)^\vee$. Moreover, if $A = \operatorname{End}({}_RY)$ and $\mathcal{F}({}_A\Delta) = \mathcal{P}^{<\infty}(A)$ we prove that $\mathcal{F}(\theta) = (\operatorname{add} Y)^\vee$, and from this fact we obtain the following result: if T is an indecomposable R-module such that $\operatorname{Ext}^1_R(T, T) = 0$ then $\operatorname{add} T = (\operatorname{add} T)^\vee$. We recall that $\omega_t : (\Omega_t, \preceq) \to (\Omega_t, \leqslant)$ and $\sigma_t : (\Omega_t, \leqslant) \to (\Omega_t, \leqslant^{\mathrm{op}})$ are the unique isomorphism of ordered sets, where $\Omega_t = \{1, 2, \ldots, t\}$.

In the next theorem R is an algebra, $(\theta, \underline{Y}, \preceq)$ is an stratifying system of size t and $Y = \coprod_{i=1}^t Y(i)$.

Theorem 3.1. *If ${}_RY$ is a generalized tilting R-module and $(\theta, \preceq)$ is standard then*:

(a) *there is a set $\{e_1, e_2, \ldots, e_t\}$ of primitive orthogonal idempotents of R such that ${}_R\Delta(i) \simeq \theta(\omega_t^{-1}(i))$ for any $i = 1, 2, \ldots, t$,*
(b) *the stratifying system $({}_R\Delta, \leqslant)$ is standard and t is the number of non-isomorphic simple R-modules.*

Proof. Since ${}_RY$ is a generalized basic tilting R-module we have that the number of non-isomorphic simple R-modules is equal to t. Then the result follows from items (c) and (e) of 2.6. □

Corollary 3.2. *Let R be an algebra. Then the following conditions are equivalent.*

(a) *The stratifying system $({}_R\Delta, \leqslant)$ is standard.*
(b) *There exists a standard stratifying system $(\theta, \underline{Y}, \preceq)$ of size t such that ${}_RY$ is a generalized tilting R-module, where ${}_RY = \coprod_{i=1}^t Y(i)$.*

Proof. (a) ⇒ (b). Let ${}_RT = \coprod_{i=1}^t T(i)$ be the characteristic tilting R-module associated to $({}_R\Delta, \leqslant)$, see 1.8. Then by 2.1 we get that $({}_R\Delta(i), T(i), \leqslant)_{i=1}^t$ is a stratifying system.

(b) ⇒ (a). This implication follows from the previous theorem. □

We recall that a full subcategory $\mathcal{C}$ of $\operatorname{mod} R$ is *coresolving*, if $\mathcal{C}$ is closed under extensions, cokernels of injections and $D(R_R) \in \mathcal{C}$.

The following proposition gives sufficient conditions for an R-module Y to be generalized tilting.

Proposition 3.3. *Let $(\theta, \underline{Y}, \preceq)$ be a stratifying system of size t and $Y = \coprod_{i=1}^t Y(i)$. Then*:

(a) $\operatorname{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$ *if and only if $\mathcal{I}(\theta)$ is coresolving,*
(b) *if $\mathcal{I}(\theta)$ is coresolving then* $\operatorname{Ext}^i_R(Y, Y) = 0$ *for any $i > 0$,*

(c) *if* $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$, $\mathrm{pd}\, Y < \infty$ *and* $(\theta, \preceq)$ *is standard, then* Y *is a generalized tilting* R*-module.*

Proof. (a) Assume that $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$. Let $0 \to M \to E \to N \to 0$ be an exact sequence with $M, E \in \mathcal{I}(\theta)$. So we get the exact sequence

$$\mathrm{Ext}^1_R(X, E) \to \mathrm{Ext}^1_R(X, N) \to \mathrm{Ext}^2_R(X, M) \quad \text{for any } X \in \mathcal{F}(\theta).$$

Since $\mathrm{Ext}^1_R(\mathcal{F}(\theta), E) = 0$ and $\mathrm{Ext}^2_R(\mathcal{F}(\theta), M) = 0$ we obtain that $N \in \mathcal{I}(\theta)$, proving that $\mathcal{I}(\theta)$ is closed under cokernels of injections.

Suppose that $\mathcal{I}(\theta)$ is coresolving. Let $M \in \mathcal{F}(\theta)$ and $N \in \mathcal{I}(\theta)$. Since $\mathcal{I}(\theta)$ is coresolving there exists an exact sequence $0 \to N \to I_0(N) \to \Omega^{-1}(N) \to 0$ with $\Omega^{-1}(N) \in \mathcal{I}(\theta)$. Therefore $\mathrm{Ext}^2_R(M, N) \simeq \mathrm{Ext}^1_R(M, \Omega^{-1}(N)) = 0$.

(b) Since $Y \in \mathcal{F}(\theta) \cap \mathcal{I}(\theta)$ and $\mathcal{I}(\theta)$ is coresolving we get for any $i > 0$ the exact sequence $0 \to Y \to I_0(Y) \to I_1(Y) \to \cdots \to I_{i-1}(Y) \to \Omega^{-i}(Y) \to 0$, where $I_m(Y)$ is an injective R-module for $m = 0, 1, \ldots, i-1$ and $\Omega^{-i}(Y) \in \mathcal{I}(\theta)$. Therefore $\mathrm{Ext}^{i+1}_R(Y, Y) \simeq \mathrm{Ext}^1_R(Y, \Omega^{-i}(Y)) = 0$.

(c) This item follows from (a), (b) and the fact that ${}_RR \in \mathcal{F}(\theta) \subseteq (\mathrm{add}\, Y)^\vee$ (see 1.9(iv)). □

Remark 3.4. Observe that $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$ if and only if $\mathrm{Ext}^2_R(\theta, \mathcal{I}(\theta)) = 0$.

We recall that the global dimension of R is defined by $\mathrm{gldim}\, R = \sup\{\mathrm{pd}\, X\colon X \in \mathrm{mod}\, R\}$, and the finitistic dimension of R is $\mathrm{fin.dim}\, R = \sup\{\mathrm{pd}\, X\colon X \in \mathcal{P}^{<\infty}(R)\}$.

Theorem 3.5. *The algebra* R *is quasi-hereditary if and only if* $\mathrm{gldim}\, R < \infty$ *and there is a standard stratifying system* $(\theta, \preceq)$ *such that* $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$.

Proof. Recall that an algebra R is quasi-hereditary if and only if R is a standardly stratified algebra and has finite global dimension (see [1,4]).

Assume that R is quasi-hereditary. Then the stratifying system $({}_R\Delta, \leqslant)$ is standard and $\mathrm{gldim}\, R < \infty$. Hence by Theorem 1.6 from [1] we obtain that $\mathcal{I}({}_R\Delta)$ is coresolving. Therefore by 3.3(a) we get that $\mathrm{Ext}^2_R(\mathcal{F}({}_R\Delta), \mathcal{I}({}_R\Delta)) = 0$.

For the reverse implication assume that $(\theta, \underline{Y}, \preceq)$ is standard, $\mathrm{gldim}\, R < \infty$ and $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$. Hence 3.3(c) holds and so ${}_RY$ is a generalized tilting R-module. The result follows now from 3.2. □

Remark 3.6. Note that the condition $\mathrm{Ext}^2_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$ generalizes condition (iv) in [2, Theorem 1].

The following result appears in [1,6]. The two examples, that follow the statement, show that none of the conclusions hold for a general stratifying system.

Proposition 3.7 [1,6]. *Let* R *be a standardly stratified algebra and* ${}_RT$ *be the characteristic tilting* R*-module associated to* $({}_R\Delta, \leqslant)$*. Then*:

(a) $\mathcal{F}({}_R\Delta) \subseteq \mathcal{P}^{<\infty}(R)$,
(b) $\mathcal{F}({}_R\Delta) = (\operatorname{add} {}_RT)^{\vee}$.

Example 3.8. Let $R \simeq kQ$, where Q is the quiver $\bullet^1 \to \bullet^2$. Consider the stratifying system of size 2 given by: $\theta(1) = Y(1) = I_2$, and $\theta(2) = Y(2) = I_1$. We have the exact sequence

$$0 \to S_2 \to Y(1) \to Y(2) \to 0$$

and the simple $S_2 \notin \mathcal{F}(\theta)$. This example shows that in general $\mathcal{F}(\theta) \neq (\operatorname{add} Y)^{\vee}$ and that $\mathcal{F}(\theta)$ is not necessarily closed under kernels of surjections.

The following example shows that in general $\mathcal{F}({}_R\Delta)$ fails to be contained in $\mathcal{P}^{<\infty}(R)$, if $({}_R\Delta, \leqslant)$ is not standard.

Example 3.9. Let $R \simeq kQ/I$, where Q is the quiver $\circlearrowleft \bullet^1 \rightleftarrows \bullet^2 \circlearrowright$ and $r^2 = 0$. It can be seen that ${}_A\Delta(1) \notin \mathcal{P}^{<\infty}(R)$.

In the following three results we shall assume that $(\theta, \underline{Y}, \preceq)$ is a stratifying system of size t and $A = \operatorname{End}({}_RY)$, where $Y = \coprod_{i=1}^{t} Y(i)$.

Lemma 3.10. *If $\mathcal{F}({}_A\Delta) = \mathcal{P}^{<\infty}(A)$ then the following statements hold*:

(a) *if $0 \to M \to Y_0 \xrightarrow{f} Y_1 \to 0$ is an exact sequence with $Y_0, Y_1 \in \operatorname{add} Y$, then $M \in \mathcal{F}(\theta)$,*
(b) *if $0 \to M \to Y_0 \xrightarrow{f} K \to 0$ is an exact sequence with $Y_0 \in \operatorname{add} Y$, $K \in \mathcal{F}(\theta)$, then $M \in \mathcal{F}(\theta)$.*

Proof. We will make use of the functors F and G defined in 1.9.

(a) Let $0 \to M \to Y_0 \xrightarrow{f} Y_1 \to 0$ be an exact sequence with $Y_0, Y_1 \in \operatorname{add} Y$. Since $\operatorname{Ext}^1_R(Y_1, Y) = 0$ we get the exact sequence

$$0 \to F(Y_1) \xrightarrow{F(f)} F(Y_0) \to F(M) \to 0. \qquad (*)$$

Therefore $N = F(M) \in \mathcal{P}^{<\infty}(A) = \mathcal{F}({}_A\Delta)$. Applying the functor G to $(*)$ we obtain the exact sequence

$$0 \to G(N) \to Y_0 \xrightarrow{f} Y_1 \to 0.$$

Hence $M \simeq G(N) \in \mathcal{F}(\theta)$.

(b) Let $0 \to M \to Y_0 \xrightarrow{f} K \to 0$ be an exact sequence with $Y_0 \in \operatorname{add} Y$, $K \in \mathcal{F}(\theta)$. Using that $\operatorname{Ext}^1_R(K, Y) = 0$ we get the exact sequence

$$0 \to F(K) \xrightarrow{F(f)} F(Y_0) \to F(M) \to 0. \qquad (**)$$

Since $F(K) \in \mathcal{F}({}_A\Delta) = \mathcal{P}^{<\infty}(A)$ we have that $N = F(M) \in \mathcal{P}^{<\infty}(A) = \mathcal{F}({}_A\Delta)$. Applying the functor G to the exact sequence $(**)$ we obtain an exact sequence

$$0 \to G(N) \to Y_0 \xrightarrow{f} K \to 0.$$

Hence $M \simeq G(N) \in \mathcal{F}(\theta)$. □

Corollary 3.11. *If* $\mathcal{F}({}_A\Delta) = \mathcal{P}^{<\infty}(A)$ *then* $\mathcal{F}(\theta) = (\operatorname{add}{}_R Y)^\vee$.

Proof. The inclusion $\mathcal{F}(\theta) \subseteq (\operatorname{add}{}_R Y)^\vee$ follows from 1.9(iv). The other inclusion is a consequence of the previous lemma. □

Proposition 3.12. *If* $\mathcal{F}(\theta) = (\operatorname{add}{}_R Y)^\vee$, $\operatorname{Ext}^1_A(\mathcal{P}^{\leqslant 1}(A), {}_A Y) = 0$ *and* $\operatorname{Ext}^1_A(\mathcal{P}_A/\mathcal{F}({}_A\Delta), {}_A Y) = 0$ *then* $\mathcal{F}({}_A\Delta) = \mathcal{P}^{<\infty}(A)$.

Proof. The proof is similar to the one given in the previous corollary and will be omitted. □

The following well known result will be needed to prove the curious Corollary 3.14.

Lemma 3.13. *If R is a local algebra then* $\operatorname{fin.dim} R = 0$.

Corollary 3.14. *If T is an indecomposable R-module such that* $\operatorname{Ext}^1_R(T, T) = 0$ *then* $\operatorname{add} T = (\operatorname{add} T)^\vee$.

Proof. Let T be an indecomposable R-module such that $\operatorname{Ext}^1_R(T, T) = 0$. Consider the system $\theta = \{T\}$, $\underline{Y} = \{T\}$. Then $(\theta, \underline{Y}, \preceq)$ is a stratifying system of size 1. Therefore $\mathcal{F}(\theta) = \operatorname{add} T$ and $A = \operatorname{End}({}_R T)$ is a local algebra. We will prove that $\mathcal{P}^{<\infty}(A) = \mathcal{F}({}_A\Delta)$ and then by 3.11 we will get the result. Since A is a local algebra we have that ${}_A\Delta = \{{}_A\Delta(1) = {}_A A\}$ and $\mathcal{F}({}_A\Delta) = \operatorname{add}{}_A A$. On the other hand, by 3.13 we have $\mathcal{P}^{<\infty}(A) = \operatorname{add}{}_A A$. Then $\mathcal{P}^{<\infty}(A) = \mathcal{F}({}_A\Delta)$, proving the result. □

Acknowledgment

We thank María Inés Platzeck for helpful suggestions.

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