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Contractive Completions of Hankel Partial Contractions

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A Hankel partial contraction is a Hankel matrix such that not all of its entries are determined, but in which every well-defined submatrix is a contraction. We address the problem of whether a Hankel partial contraction in which the upper left triangle is known can be completed to a contraction. It is known that the 2×2 and 3×3 cases can be solved, and that 4×4 Hankel partial contractions cannot always be completed. We introduce a technique that allows us to exhibit concrete examples of such 4×4 matrices, and to analyze in detail the dependence of the solution set on the given data. At the same time, we obtain necessary and sufficient conditions on the given cross-diagonals in order for the matrix to be completed. We also study the problem of extending a contractive Hankel block of size n to one of size $n + 1$.

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1. INTRODUCTION

Let

$$H = H(a_1, a_2, \dots, a_n; x_1, \dots, x_{n-1}) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix}$$

be a Hankel matrix, where $a_1, \dots, a_n$ are *given* real numbers and $x_1, \dots, x_{n-1}$ are real numbers *to be determined*. We say that H is a *partial contraction* if all completely determined submatrices of H are contractions (in the sense that their *operator* norms are at most 1). In this article, we first study the following problem.

Problem 1.1. Given $a_1, \dots, a_n$, find $x_1, \dots, x_{n-1}$ such that $H(a_1, \dots, a_n; x_1, \dots, x_{n-1})$ is contractive.

We say that Problem 1.1 is well-posed if $H(a_1, a_2, \dots, a_n; x_1, \dots, x_{n-1})$ is partially contractive, and that it is soluble if $H(a_1, a_2, \dots, a_n; x_1, \dots, x_{n-1})$ is contractive for some $x_1, \dots, x_{n-1}$.

For 2×2 *operator* matrices (with *no* required Hankel condition), a solution to the completion problem

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

has been given by G. Arsene and A. Gheondea [ArG], by C. Davis *et al.* [DKW] (see also [Dav; Cra]), by C. Foiaş and A. Frazho [FoF] (using Redheffer products), by S. Parrott [Par], and by Y. L. Shmul'yan and R. N. Yanovskaya [SY], and it is implicit in the work of W. Arveson [Arv] (see also [Pow]).

There is a formulation of Problem 1.1 for Toeplitz matrices, which has been studied by C. R. Johnson and L. Rodman [JR] and by H. J. Woerdeman [Woe]. Reference [JR, Theorem 1] implies that the 3×3 case is always soluble, and that there exist real numbers a, b, c, d such that $H(a, b, c, d; x, y, z)$ is partially contractive but not contractive for all choices of x, y, z. Reference [Woe, Theorem 7.4] implies that for Hankel matrices of the form $H(0, \dots, 0, a_n; x_1, \dots, x_{n-1})$, Problem 1.1 is soluble if $|a_n| < 1$. Woerdeman also exhibits a concrete set of 5×5 data for which Problem 1.1 is well-posed but not soluble, namely $a_1 = a_3 = a_5 = 0$, $a_2 = a_4 = 7/10$. As we shall see below, our techniques allow us to describe general collections of 4×4 matrices with such a property, while contributing to a more quantitative understanding of the geometric conditions on $a_1, \ldots, a_n$ required for the existence of a solution. In particular, we easily recover Woerdeman's example. (A related result, where control of the norm of a Hankel extension is desired, appears in [HW]; see also [Ove].)

The main goal of this paper is to introduce a new method, and to illustrate its usefulness by applying it to the 3×3 and 4×4 cases, for which we can obtain detailed information. To analyze Problem 1.1, the 2×2 operator matrix case has qualitative significance, in that one can guarantee that certain partial contractions admit contractive completions. However, a detailed (quantitative) description of the solutions, in addition to the analysis of the *Hankel* condition, requires that we be much more explicit. For this reason, we are somewhat forced to reconsider the case of a $(k+1) \times (l+1)$ matrix over $\mathbf{R}$ of the form

$$T := \begin{pmatrix} Q & \mathbf{r} \\ \mathbf{s}^* & x \end{pmatrix}, \tag{1.1}$$

where Q is a $k \times l$ matrix, $\mathbf{r}$ and $\mathbf{s}$ are column vectors of length k and l, and x is a real number to be determined. To determine for which values of x T is a contraction, we let $P := I - TT^*$ $(= D_{T^*}^2)$ and study the condition $P \geq 0$. Along the lines of the situation for 2×2 operator matrices, we must obviously require that both $(Q \;\; \mathbf{r})$ and $\binom{Q}{\mathbf{s}^*}$ are contractive, i.e.,

$$A(\mathbf{r}) := I - (Q \quad \mathbf{r})\begin{pmatrix} Q^* \\ \mathbf{r}^* \end{pmatrix} = I - QQ^* - \mathbf{r}\mathbf{r}^* \geq 0$$

and

$$B(\mathbf{s}) := I - \begin{pmatrix} Q \\ \mathbf{s}^* \end{pmatrix}(Q^* \quad \mathbf{s}) = \begin{pmatrix} I - QQ^* & -Q\mathbf{s} \\ -\mathbf{s}^*Q^* & 1 - \mathbf{s}^*\mathbf{s} \end{pmatrix} \geq 0. \tag{1.2}$$

LEMMA 1.2. *Assume that* $A(\mathbf{r})$ *is positive and invertible. Then* $P \geq 0$ *if and only if* $\det P \geq 0$.

THEOREM 1.3. *Let* T *and* P *be as above, and assume that* $A(\mathbf{r})$ *is positive and invertible. Then*

$$\det P = \alpha x^2 + \beta x + \gamma,$$

where

$$\alpha = -\det(I - QQ^*)$$

$$\beta = -\det A(\mathbf{r})\big(\mathbf{r}^* A(\mathbf{r})^{-1} Q\mathbf{s} + \mathbf{s}^* Q^* A(\mathbf{r})^{-1}\mathbf{r}\big)$$

$$\gamma = \det A(\mathbf{r})\big(1 - \mathbf{s}^*\mathbf{s} - \mathbf{s}^* Q^* A(\mathbf{r})^{-1} Q\mathbf{s}\big).$$

COROLLARY 1.4. *With T, P, and $A(\mathbf{r})$ as in Theorem* 1.3, *the discriminant of* $\det P$ *is*

$$\beta^2 - 4\alpha\gamma = 4 \det A(\mathbf{r}) \det B(\mathbf{s}).$$

COROLLARY 1.5. *With T, P, and $A(\mathbf{r})$ as in Theorem* 1.3, *the graph of* $\det P$ *is a downwards parabola which meets the x-axis at points x_l and x_r, with $x_l \le x_r$. Then any value of x in the closed interval $[x_l, x_r]$ gives rise to a contractive T.*

We shall give the proofs of all these preliminary results in Section 2, where we shall also indicate how to deal with the situation in which $A(\mathbf{r})$ is not invertible. As a consequence, simple proofs of the well-known cases $k = 1$ and $k = 2$ can be given, and the (previously unknown) case $k = 3$ can be discussed in detail. If we recall that

$$H = H(a, b, c, d; x, y, z) = \begin{pmatrix} a & b & c & d \\ b & c & d & x \\ c & d & x & y \\ d & x & y & z \end{pmatrix},$$

a straightforward application of Corollary 1.5 and of its generalization to the case of noninvertible $A(\mathbf{r})$ (Theorem 5.5) shows that in order to find x, y, z making H contractive, (i) it suffices to find x, and (ii) the search for x leads to the study of *two* parabolas, according to the following situations:

$$\begin{pmatrix} Q & \mathbf{r} \\ \mathbf{s}^* & x \end{pmatrix} = \left(\begin{array}{ccc|c} a & b & c & d \\ \cdots & \cdots & \cdots & \cdots \\ b & c & d & x \end{array}\right)$$

and

$$\begin{pmatrix} Q & \mathbf{r} \\ \mathbf{s}^* & x \end{pmatrix} = \left(\begin{array}{cc|c} a & b & c \\ b & c & d \\ \cdots & \cdots & \cdots \\ c & d & x \end{array}\right).$$

As a result, the 4×4 case consists of analyzing the relative position of the two associated parabolas, and H is contractive if and only if there is a common positivity interval. We can then provide a direct proof of the relevant portion of [JR, Theorem 1].

THEOREM 1.6. *There exist real numbers a, b, c, d such that* (*i*) *H is partially contractive*; (*ii*) *H is not contractive for all choices of x, y, z.*

In the positive direction, we establish a number of criteria for the existence of contractive completions, under suitable restrictions on a, b, c, d. Of particular interest are the cases with two of a, b, c, d equal to zero, for which we determine explicitly the ranges of values admitting contractive completions. As a consequence, we see that even simple cases like $a = 0$, $b = -\frac{3}{4}$, $c = \frac{3}{8}$, $d = 0$, which produce partial contractions, do not admit contractive completions. These results are presented in Section 3.

We now turn our attention to a related problem, treated in Section 4.

Problem 1.7. Given real numbers $a_1, \dots, a_n$, find a real number x such that the Hankel *triangle*

$$H_\Delta(a_1, \dots, a_n, x) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n & x \\ a_2 & a_3 & \cdots & a_n & x & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{n-1} & a_n & \cdots & & & \\ a_n & x & \cdots & & & \\ x & & & & & \end{pmatrix}$$

is partially contractive.

As with Problem 1.1, we say that Problem 1.7 is well-posed if $H_\Delta(a_1, \dots, a_n)$ is partially contractive, and that it is soluble if $H_\Delta(a_1, \dots, a_n, x)$ is partially contractive for some x. This problem is again related to the work in [HW, JR, Woe].

The techniques employed in our work on Problem 1.1 allow us to study Problem 1.7 as well. In Section 4, after giving a detailed discussion of the case $n = 2$, we proceed with the analysis of the case $n = 3$, with particular emphasis in the situations arising when one of the given numbers equals zero. We also establish a link between Problem 1.1 and 1.7, which allows us to claim that $H_\Delta(a_2, \dots, a_n, x)$ is partially contractive whenever $H(a_1, a_2, \dots, a_n; x, x_2, \dots, x_{n-1})$ is partially contractive for some $a_1, x_2, \dots, x_{n-1}$.

For both Problem 1.1 and Problem 1.7, we also study the *extremal* cases, corresponding to the equation $\det(I - QQ^* - \mathbf{r}\mathbf{r}^*) = 0$. As we explain in Section 5, the extremal cases can be derived from the non-extremal ones by using a limit argument; we present in Section 3, nevertheless, a concrete analysis of some extremal cases. As a matter of fact, we can subsume the study of the extremal cases in the study of Problem 1.1 and Problem 1.7 for $A(\mathbf{r})$ invertible, via an explicit algorithm (Theorem 5.8). (A detailed routine in Mathematica to determine whether Problem 1.1 for

$H(a,b,c;x,y)$, $H(a,b,c,d;x,y,z)$, and $H(a,b,c,d,e;x,y,z,w)$ is well-posed and, if so, soluble, is available from the authors by electronic mail.)

The present work was motivated by a question of C. Foiaş on Hankel completions. We wish to thank Professor Foiaş for helpful discussions related to the topics presented here, and for his encouragement. We are also indebted to T. Constantinescu, J. W. Helton, L. Rodman, and H. Woerdeman for correspondence and insightful comments on the material herein, and to the referee for many valuable suggestions which improved the presentation. Much of the research was done while the second and third named authors spent a sabbatical leave at The University of Iowa. Most of the calculations in this paper, and some of the ideas, were first obtained through computer experiments using the software tool Mathematica [Wol].

2. SOME BASIC RESULTS

We begin by recalling that an $n \times n$-matrix T is a contraction if and only if the matrix

$$P = P(T) := I - TT^*$$

is positive, where I is the identity matrix and T^* is the adjoint of T. In order to test if P is positive, we use the following version of Cholesky's Algorithm.

LEMMA 2.1. *Assume that*

$$P = \begin{pmatrix} P_0 & \mathbf{t} \\ \mathbf{t}^* & u \end{pmatrix}, \tag{2.1}$$

where P_0 is an $(n-1) \times (n-1)$ matrix, $\mathbf{t}$ is a column vector, and u is a real number.

(i) *If P_0 is invertible, then* $\det P = \det P_0(u - \mathbf{t}^* P_0^{-1} \mathbf{t})$.

(ii-a) *If P_0 is invertible and positive, then* $P \geq 0 \Leftrightarrow (u - \mathbf{t}^* P_0^{-1} \mathbf{t}) \geq 0 \Leftrightarrow \det P \geq 0$.

(ii-b) *If $u > 0$ then* $P \geq 0 \Leftrightarrow P_0 - \mathbf{t} u^{-1} \mathbf{t}^* \geq 0$.

(iii) *If $P \geq 0$ and $p_{ii} = 0$ for some i, $1 \leq i \leq n$, then $p_{ij} = p_{ji} = 0$ for all $j = 1, \ldots, n$.*

As a consequence, we can now give the proofs of Lemma 1.2, Theorem 1.3, Corollary 1.4, and Corollary 1.5.

Proof of Lemma 1.2. Observe that

$$P = I - TT^* = \begin{pmatrix} I - QQ^* - \mathbf{r}\mathbf{r}^* & -Q\mathbf{s} - x\mathbf{r} \\ -\mathbf{s}^*Q^* - x\mathbf{r}^* & 1 - \mathbf{s}^*\mathbf{s} - x^2 \end{pmatrix} = \begin{pmatrix} A(\mathbf{r}) & * \\ * & * \end{pmatrix}.$$

Since $A(\mathbf{r})$ is positive and invertible by hypothesis, it follows that $P \geq 0$ if and only if $\det P \geq 0$. ∎

LEMMA 2.2. *Let* $D := \left(\begin{smallmatrix} A & \mathbf{b} \\ \mathbf{c} & 1 \end{smallmatrix}\right)$ *and let* $\mathbf{d}$ *be of equal size as* $\mathbf{b}$. *Then*

$$\det D = \det\begin{pmatrix} A - \mathbf{b}\mathbf{d}^* & \mathbf{b} \\ \mathbf{c} - \mathbf{d}^* & 1 \end{pmatrix}.$$

Proof. Observe that $\mathbf{b}\mathbf{d}^*$ is a matrix whose ith-column is $d_i \mathbf{b}$. Now use the multilinearity of det. ∎

Proof of Theorem 1.3. Since

$$P = \begin{pmatrix} A(\mathbf{r}) & -Q\mathbf{s} - x\mathbf{r} \\ -\mathbf{s}^*Q^* - x\mathbf{r}^* & 1 - \mathbf{s}^*\mathbf{s} - x^2 \end{pmatrix}, \tag{2.2}$$

Lemma 2.1 states that

$$\det P = \det A(\mathbf{r})\left(1 - \mathbf{s}^*\mathbf{s} - x^2 - (-\mathbf{s}^*Q^* - x\mathbf{r}^*)A(\mathbf{r})^{-1}(-Q\mathbf{s} - x\mathbf{r})\right), \tag{2.3}$$

from which it follows that the coefficient of x^2 in $\det P$ is

$$\alpha = -\det A(\mathbf{r})\left(1 + \mathbf{r}^*A(\mathbf{r})^{-1}\mathbf{r}\right).$$

Another application of Lemma 2.1 reveals that

$$\begin{aligned} -\det A(\mathbf{r})\left(1 + \mathbf{r}^*A(\mathbf{r})^{-1}\mathbf{r}\right) &= \det A(\mathbf{r})\left(-1 - \mathbf{r}^*A(\mathbf{r})^{-1}\mathbf{r}\right) \\ &= \det\begin{pmatrix} A(\mathbf{r}) & \mathbf{r} \\ \mathbf{r}^* & -1 \end{pmatrix} = -\det\begin{pmatrix} I - QQ^* - \mathbf{r}\mathbf{r}^* & -\mathbf{r} \\ \mathbf{r}^* & 1 \end{pmatrix}. \end{aligned}$$

An application of Lemma 2.2 now shows that this in turn equals $-\det(I - QQ^*)$, if one takes $A = I - QQ^* - \mathbf{r}\mathbf{r}^*$, $\mathbf{b} = -\mathbf{r}$, $\mathbf{c} = \mathbf{r}^*$, and $\mathbf{d} = \mathbf{r}$. Thus,

$$\alpha = -\det(I - QQ^*),$$

as desired.

From (2.3) we also see at once that

$$\beta = -\det A(\mathbf{r})\left(\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s} + \mathbf{s}^*Q^*A(\mathbf{r})^{-1}\mathbf{r}\right)$$

and

$$\gamma = \det A(\mathbf{r})\left(1 - \mathbf{s}^*\mathbf{s} - \mathbf{s}^*Q^*A(\mathbf{r})^{-1}Q\mathbf{s}\right),$$

which completes the proof. ∎

Proof of Corollary 1.4. We wish to prove that

$$\beta^2 - 4\alpha\gamma = 4\det A(\mathbf{r})\det B(\mathbf{s}). \tag{2.4}$$

Observe first that since $A(\mathbf{r})$ is positive and invertible, the same is true of $A(\mathbf{0})$ (because $A(\mathbf{0}) = A(\mathbf{r}) + \mathbf{r}\mathbf{r}^*$). Also, $\mathbf{s}^*Q^*A(\mathbf{r})^{-1}\mathbf{r} = (\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s})^*$, which readily implies that

$$\beta = -2\det A(\mathbf{r})\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s}.$$

Then

$$\begin{aligned}\beta^2 - 4\alpha\gamma &= 4\det A(\mathbf{r})^2\left(\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s}\right)^2\\ &\quad + 4\det A(\mathbf{0})\det A(\mathbf{r})\left(1 - \mathbf{s}^*\mathbf{s} - \mathbf{s}^*Q^*A(\mathbf{r})^{-1}Q\mathbf{s}\right).\end{aligned}$$

Therefore, it suffices to prove that

$$\det A(\mathbf{r})\left(\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s}\right)^2 + \det A(\mathbf{0})\left(1 - \mathbf{s}^*\mathbf{s} - \mathbf{s}^*Q^*A(\mathbf{r})^{-1}Q\mathbf{s}\right) = \det B(\mathbf{s}). \tag{2.5}$$

From (1.2) and Lemma 2.1(i), we see that

$$\det B(\mathbf{s}) = \det A(\mathbf{0})\left(1 - \mathbf{s}^*\mathbf{s} - \mathbf{s}^*Q^*A(\mathbf{0})^{-1}Q\mathbf{s}\right).$$

Thus, to establish (2.5) it is enough to prove the identity

$$\begin{aligned}&\mathbf{s}^*Q^*A(\mathbf{r})^{-1}\mathbf{r}\det A(\mathbf{r})\mathbf{r}^*A(\mathbf{r})^{-1}Q\mathbf{s} - \det A(\mathbf{0})\left(\mathbf{s}^*Q^*A(\mathbf{r})^{-1}Q\mathbf{s}\right)\\ &\qquad = -\det A(\mathbf{0})\left(\mathbf{s}^*Q^*A(\mathbf{0})^{-1}Q\mathbf{s}\right);\end{aligned} \tag{2.6}$$

equivalently,

$$\begin{aligned}&Q^*A(\mathbf{r})^{-1}\mathbf{r}\det A(\mathbf{r})\mathbf{r}^*A(\mathbf{r})^{-1}Q - \det A(\mathbf{0})\left(Q^*A(\mathbf{r})^{-1}Q\right)\\ &\qquad = -\det A(\mathbf{0})\left(Q^*A(\mathbf{0})^{-1}Q\right).\end{aligned} \tag{2.7}$$

To prove (2.7), it suffices to see that

$$A(\mathbf{r})^{-1}\mathbf{r}\det A(\mathbf{r})\mathbf{r}^*A(\mathbf{r})^{-1} - \det A(\mathbf{0})\,A(\mathbf{r})^{-1} = -\det A(\mathbf{0})\left(A(\mathbf{0})^{-1}\right),$$

which in turn reduces to showing that

$$\det A(\mathbf{r})\,A(\mathbf{r})^{-1}\mathbf{r}\mathbf{r}^*A(\mathbf{r})^{-1} = \det A(\mathbf{0})\left[A(\mathbf{r})^{-1} - A(\mathbf{0})^{-1}\right]. \tag{2.8}$$

If we apply Lemma 2.2 and Lemma 2.1(i) to the matrix $\begin{pmatrix} A(\mathbf{0}) & \mathbf{r} \\ \mathbf{r}^* & 1 \end{pmatrix}$, we see that

$$\det A(\mathbf{r}) = \det(A(\mathbf{0}) - \mathbf{r}\mathbf{r}^*) = \det\begin{pmatrix} A(\mathbf{0}) - \mathbf{r}\mathbf{r}^* & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} = \det\begin{pmatrix} A(\mathbf{0}) & \mathbf{r} \\ \mathbf{r}^* & 1 \end{pmatrix}$$
$$= \det A(\mathbf{0})\left(1 - \mathbf{r}^* A(\mathbf{0})^{-1}\mathbf{r}\right),$$

and (2.8) then becomes

$$\left(1 - \mathbf{r}^* A(\mathbf{0})^{-1}\mathbf{r}\right) A(\mathbf{r})^{-1}\mathbf{r}\mathbf{r}^* A(\mathbf{r})^{-1} = A(\mathbf{r})^{-1} - A(\mathbf{0})^{-1}. \qquad (2.9)$$

Now recall that the classical Resolvent Identity states that

$$A(\mathbf{r})^{-1} - A(\mathbf{0})^{-1} = A(\mathbf{r})^{-1}\mathbf{r}\mathbf{r}^* A(\mathbf{0})^{-1}.$$

Identity (2.9) is now

$$A(\mathbf{r})^{-1}\mathbf{r}\mathbf{r}^* A(\mathbf{r})^{-1} - A(\mathbf{r})^{-1}\mathbf{r}\left(\mathbf{r}^* A(\mathbf{0})^{-1}\mathbf{r}\right)\mathbf{r}^* A(\mathbf{r})^{-1} = A(\mathbf{r})^{-1}\mathbf{r}\mathbf{r}^* A(\mathbf{0})^{-1}. \qquad (2.10)$$

An inspection of (2.10) reveals that it is sufficient to establish that

$$A(\mathbf{r})^{-1} - A(\mathbf{0})^{-1}\mathbf{r}\mathbf{r}^* A(\mathbf{r})^{-1} = A(\mathbf{0})^{-1},$$

or that

$$I - A(\mathbf{0})^{-1}\mathbf{r}\mathbf{r}^* = A(\mathbf{0})^{-1} A(\mathbf{r}).$$

which obviously follows from the identity $A(\mathbf{0}) - \mathbf{r}\mathbf{r}^* = A(\mathbf{r})$. The proof of Corollary 1.4 is now complete. ▮

Proof of Corollary 1.5. Since $\alpha = -\det(I - QQ^*) < 0$ and $\beta^2 - 4\alpha\gamma = 4\det A(\mathbf{r})\det B(\mathbf{s}) \geq 0$, it is clear that the graph of $\det P$ as a function of x is a downward parabola with non-negative discriminant, assuring the existence of *real* x-intercepts $x_l \leq x_r$. Any value of x between x_l and x_r makes $\det P \geq 0$, which in turn guarantees that T is a contraction. ▮

In connection with the study of the 4×4 case, we shall present a bit later the Derivative Method, which will allow us to decide on the relative position of the two parabolas mentioned in Section 1 right after Corollary 1.5. We would like to end this section, however, with a description of the 2×2 and 3×3 cases, to exhibit their simplicity in light of Theorem 1.3 and Corollaries 1.4 and 1.5.

2.1. *The* 2×2 *Case*. Two cases arise.

(i) $A(\mathbf{r}) = 1 - a^2 - b^2 > 0$. The solution set is the closed interval determined by the roots of $\det P = 0$, namely,

$$x_l = \frac{-1 - a + b^2}{1 + a}, \qquad x_r = \frac{-1 + a + b^2}{-1 + a};$$

of course, $-1 \le x_l \le x_r \le 1$. (This is identical to the solution set found following the recipe in [ArG, DKW, FoF].)

(ii) $1 - a^2 - b^2 = 0$ (extremal case). Lemma 2.1(iii) applied to (2.10) shows that we must necessarily have $-ab - bx = 0$. If $b \neq 0$, then $x = -a$, and if $b = 0$, then $|a| = 1$, and we may therefore take any x such that $|x| \le 1$. In either case, the resulting P is positive, so H is a contraction.

As an illustration of what we do in Section 5, let us show here another approach to the extremal case. In order to reduce this case to (the non-extremal) case (i), we multiply by $0 < t < 1$ each of the given known entries of H, so that P has now the form

$$P_t := \begin{pmatrix} 1 - t^2a^2 - t^2b^2 & -t^2ab - tbx \\ -t^2ab - tbx & 1 - t^2b^2 - x^2 \end{pmatrix},$$

which leads to

$$x_l(t) = \frac{-1 - at + b^2t^2}{1 + at}, \qquad x_r(t) = \frac{-1 + at + b^2t^2}{-1 + at}.$$

We observe that

$$\frac{dx_l}{dt} = \frac{b^2t(2 - at)}{(1 - at)^2} > 0 \qquad \text{and} \qquad \frac{dx_r}{dt} = -\frac{b^2t(2 + at)}{(1 + at)^2} < 0,$$

which implies that x_l is an increasing function of t, while x_r is a decreasing function of t. Moreover,

$$x_r(t) - x_l(t) = \frac{2\left[1 - (a^2 + b^2)t^2\right]}{1 - a^2t^2} = \frac{2(1 - t^2)}{1 - a^2t^2}.$$

If $|a| = 1$ then $b = 0$, and $x_r(t) - x_l(t) = 2$, forcing $x_l(t) = -1$ and $x_r(t) = 1$; that is, any x with $|x| \le 1$ works. If $|a| < 1$, then $x_r(t) - x_l(t) \to 0$ as $t \to 1^-$, and $x = -a$ is the only solution.

Before we discuss the 3×3 case, we need some notation. Given $H(a_1, \dots, a_n; x_1, \dots, x_{n-1})$, we let H_{ij} denote the upper-left submatrix of H of size $i \times j$ and we let

$$P_{ij} := I - H_{ij} H_{ij}^* \qquad (1 \le i, j \le n).$$

Thus, for example, $H_{nn} = H$ and $P_{nn} = P$.

2.2. *The* 3×3 *Case*. Here three submatrices are completely determined by the given data, namely,

$$H_{13} = \begin{pmatrix} a & b & c \end{pmatrix}, \qquad H_{22} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad H_{31} (= H_{13}^*).$$

We shall see that Problem 1.1 admits a solution precisely when it is well-posed. By hypothesis, $1 - a^2 - b^2 - c^2 = \det P_{13} \ge 0$ and $1 - a^2 - 2b^2 + b^4 - 2ab^2c - c^2 + a^2c^2 = \det P_{22} \ge 0$; two cases arise.

- If $\mathcal{A}(\mathbf{r}) = 1 - a^2 - b^2 - c^2 > 0$, Theorem 1.3, taking $T = H_{23}$, states that we can always find x making $P_{23} \ge 0$. For, $\det P_{23} = \alpha x^2 + \beta x + \gamma$, where $\alpha = -\det(I - QQ^*) = -1 + a^2 + b^2$ and $\beta^2 - 4\alpha\gamma = 4 \det \mathcal{A}(\mathbf{r}) \det B(\mathbf{s}) = 4 \det P_{13} \det P_{22} \ge 0$. Having found x, a new application of Theorem 1.3 gives the interval for y.

- If $1 - a^2 - b^2 - c^2 = 0$ (extremal case), then

$$P \equiv P_{33} = I - H_{33} H_{33}^*$$
$$= \begin{pmatrix} 1 - a^2 - b^2 - c^2 & -ab - bc - cx & -ac - bx - cy \\ -ab - bc - cx & 1 - b^2 - c^2 - x^2 & -bc - cx - xy \\ -ac - bx - cy & -bc - cx - xy & 1 - c^2 - x^2 - y^2 \end{pmatrix} \quad (2.11)$$

with $p_{11} = 0$. Since both p_{12} and p_{13} must then be zero, we obtain

$$-ab - bc - cx = 0$$
$$-ac - bx - cy = 0.$$

If $c = 0$, then $a^2 + b^2 = 1$ and $ab = 0$, and we can easily find the values of x and y. If $c \neq 0$, we have

$$x = \frac{-ab - bc}{c} \qquad \text{and} \qquad y = \frac{ab^2 + b^2c - ac^2}{c^2} \qquad (2.12)$$

as the only possible candidates for a solution. Substituting these values and $c = \pm\sqrt{1 - a^2 - b^2}$ in (2.11), we get zeros in the first column and first row of P, and

$$p_{22} = \frac{a^2 - a^4 - b^2 - a^2b^2 + b^4 \mp 2ab^2\sqrt{1 - a^2 - b^2}}{1 - a^2 - b^2}. \qquad (2.13)$$

In order to analyze the sign of p_{22}, we solve $p_{22} = 0$ and obtain

$$(a + a^2 + b + ab + b^2)(a + a^2 - b - ab + b^2)(-a + a^2 + b - ab + b^2)$$
$$(-a + a^2 - b + ab + b^2) = 0;$$

the four factors correspond to four ellipses which determine the regions where the sign of p_{22} is constant, and we can evaluate p_{22} at one point in each region to obtain the diagram in Figure 1 ($p_{22} > 0$ in the marked regions).

On the other hand, H_{22} is a contraction, so

$$P_{22} = I - H_{22}H_{22}^* = \begin{pmatrix} 1 - a^2 - b^2 & -ab - ac \\ -ab - ac & 1 - b^2 - c^2 \end{pmatrix}$$

is positive, and its determinant must therefore be non-negative. But

$$\det P_{22} = 1 - a^2 - 2b^2 + b^4 - 2ab^2c - c^2 + a^2c^2,$$

which, upon the substitution $c = \sqrt{1 - a^2 - b^2}$, becomes

$$\det P_{22} = a^2 - a^4 - b^2 - a^2b^2 + b^4 \mp 2ab^2\sqrt{1 - a^2 - b^2}. \quad (2.14)$$

We observe that the right-hand side of (2.14) is precisely the numerator of (2.13), so $p_{22} \geq 0$. (The assumption $P_{22} \geq 0$ translates into requiring that

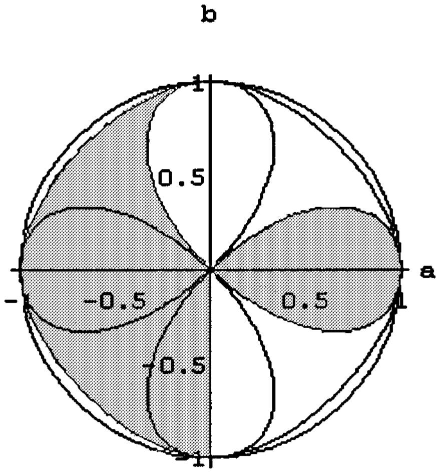


FIG. 1. Solution of the 3×3 extremal case.

the values of a and b lie in the white regions, but once we are inside those regions, we can guarantee that $p_{22} \geq 0$.)

With a direct calculation we can now see that

$$\det\begin{pmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{pmatrix} = 0;$$

thus, under the assumption that $p_{22} > 0$, Lemma 2.1(ii-a) implies that $P_{33} \geq 0$, so the choices of x and y made in (2.12) solve the problem. Using (2.12), we observe that

$$cp_{23} + bp_{22} = b(1 - a^2 - b^2 - c^2) = 0;$$

when $p_{22} = 0$ (which implies that the pair (a, b) belongs to one of the four ellipses), and if we recall that $c \neq 0$, we obtain $p_{23} = 0$, and the positivity of P_{33} is controlled by the entry p_{33}. A direct calculation, however, shows that

$$c^2 p_{33} - b^2 p_{22} = (c^2 - b^2)(1 - a^2 - b^2 - c^2) = 0,$$

from which we see that $p_{33} = 0$, too. We see, therefore, that with the values of x and y from (2.12) and $p_{22} = 0$, $P_{33} = 0$.

Remark 2.3. It is worth mentioning that the roles of a and c in the previous discussion are interchangeable; this will be helpful in Section 4.

3. THE 4×4 CASE

Let $H(a, b, c, d; x, y, z)$ be a partial contraction. We must find x such that H remains a partial contraction, so we need the maximal completely determined submatrices H_{24} and H_{33} to be contractions. (As with the 3×3 case, once x has been found, it is straightforward to obtain y, and a fortiori z. For, with x given, we let $Q := \left(\begin{smallmatrix} a & b & c \\ b & c & d \end{smallmatrix}\right)$, $\mathbf{r} := \left(\begin{smallmatrix} d \\ x \end{smallmatrix}\right)$, and $\mathbf{s}^* := (c\ d\ x)$, and we apply Corollary 1.5 (if $A(\mathbf{r})$ is invertible) or Theorem 5.8 (if $A(\mathbf{r})$ is singular). In either case, we see that there exists y such that $H(a, b, c, d; x, y)$ is partially contractive. Again, with x and y now given, we repeat the above procedure to find z, this time letting

$$Q := \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & x \end{pmatrix}, \qquad \mathbf{r} := \begin{pmatrix} d \\ x \\ y \end{pmatrix}$$

and $\mathbf{s} := \mathbf{r}$.)

Each of the two previous problems can be solved separately using the results in Section 2. Thus, the first one requires the analysis of the positivity of $P_{24} := I - H_{24}H_{24}^*$, while the second one requires the use of $P_{33} := I - H_{33}H_{33}^*$. The crux of the matter lies in that the values of x solving each problem must be compatible, i.e., the intersection of the two intervals associated with each problem must be non-empty.

Our next task is to obtain a useful criterion to determine when two given parabolas are simultaneously non-negative. The following result is quite elementary, but extremely useful.

PROPOSITION 3.1. *Let $y = f(x)$ and $y = g(x)$ be two downward parabolas, with vertices (x_f, y_f) and (x_g, y_g), satisfying $y_f, y_g \geq 0$. Then f and g can behave in exactly one of the following ways* (*see Fig.* 2).

(i) *The graphs of f and g are identical.*

(ii) *The graphs of f and g intersect in a unique* double *point* (*and as a result, one of the parabolas is "inside" the other*).

(iii) *The graphs of f and g intersect in a unique* simple *point x_0, and two subcases arise*:

(a) $\operatorname{sgn}(f'(x_0)) = \operatorname{sgn}(g'(x_0))$; *and*

(b) $\operatorname{sgn}(f'(x_0)) \neq \operatorname{sgn}(g'(x_0))$.

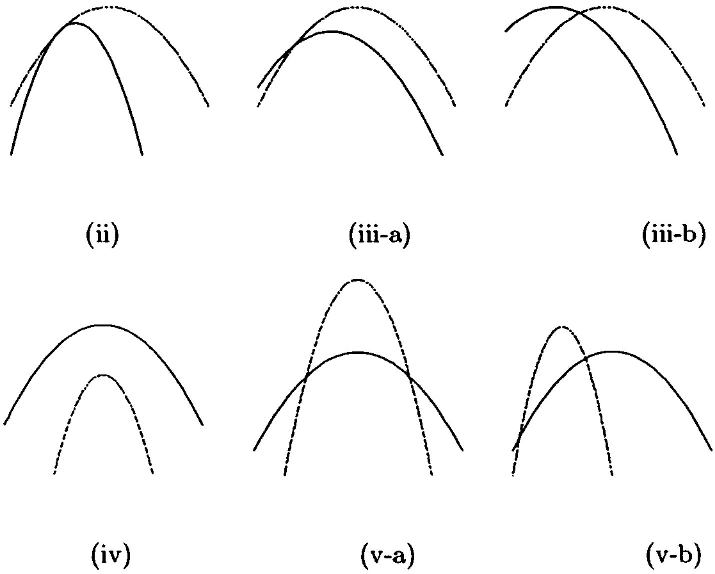


FIG. 2. Cases of intersection of two downward parabolas.

(iv) *The graphs of* f *and* g *do not intersect* (*and therefore one of the parabolas is properly "inside" the other*).

(v) *The graphs of* f *and* g *intersect in two simple points* x_1 *and* x_2, *and two subcases arise*:

(a) $\operatorname{sgn}(f'(x_1)) = \operatorname{sgn}(g'(x_1))$ *and* $\operatorname{sgn}(f'(x_2)) = \operatorname{sgn}(g'(x_2))$; *and*

(b) *either* $\operatorname{sgn}(f'(x_1)) \neq \operatorname{sgn}(g'(x_1))$ *or* $\operatorname{sgn}(f'(x_2)) \neq \operatorname{sgn}(g'(x_2))$.

THEOREM 3.2. *Let* $f, g \in \mathbf{R}[x]$ *be the two quadratic polynomials associated with* H_{24} *and* H_{33}, *respectively, and assume that the associated matrices* P_{24} *and* P_{33}, *when written in the form given in* (2.1), *have* P_0 *positive and invertible*. *Then there exist* x, y, z, *making* H_{44} *a contraction whenever* f *and* g *intersect according to cases* (i), (ii), (iii-a), (iv), *and* (v-a).

Proof. Recall that $f(x) := \det(I - H_{24}H_{24}^*)$ and $g(x) := \det(I - H_{33}H_{33}^*)$. Observe that if f and g intersect according to cases (i), (ii), (iii-a), (iv), and (v-a), the positivity interval for one of f or g is a subset of the positivity interval for the other, and thus values of x making both f and g non-negative exist. ∎

It follows from Theorem 3.2 that the study of Problem 1.1 for H_{44} consists of the analysis of cases (iii-b) and (v-b) in Proposition 3.1. This is what we proceed to do now for a number of important instances. But first let us summarize conceptually our algorithm for solving Problem 1.1 for H_{44}.

3.1. *The Derivative Method*. The algorithm to test if a partial contraction can be completed is shown in Figure (3).

EXAMPLE 3.3. Let $a = -\frac{3}{5}$, $b = -\frac{1}{2}$, $c = \frac{3}{7}$, and $d = \frac{1}{3}$. Then

$$\det P_{24} = -\frac{692459}{77792400} - \frac{16}{105}x - \frac{1011}{4900}x^2$$

and

$$\det(P_{33}) = \frac{-256393463 + 967303512x - 813902985x^2}{3811827600}.$$

A calculation shows that $\det P_{24} = \det P_{33}$ on $[-1, 1]$ precisely when $x = x_1 := 2986/20727 \cong 0.144063$, that

$$\left(\frac{d \det P_{24}}{dx}\right)\left(\frac{d \det P_{33}}{dx}\right) < 0,$$

and that $\det P_{24}(x_1) < 0$. We then conclude that Problem 1.1 with data (a, b, c, d) admits no solution; see Figures 3 and 4.

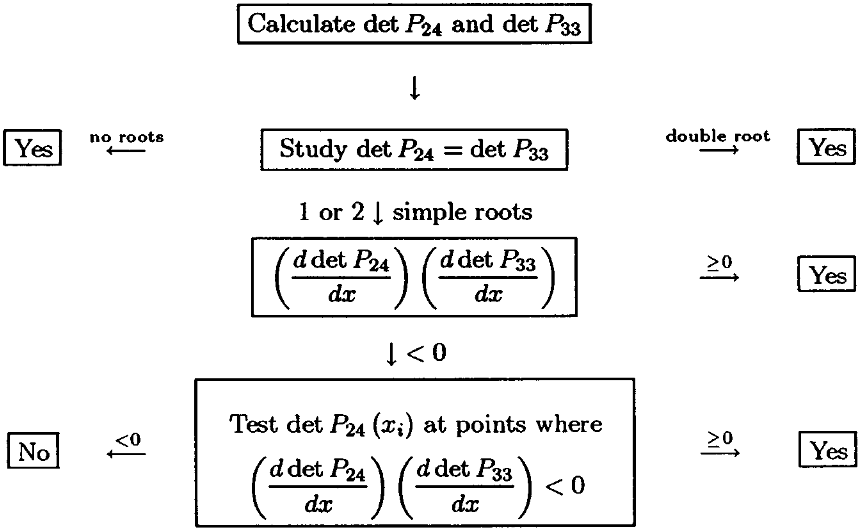


FIG. 3. The derivative method.

3.2. *Two Variables Equal to Zero.* Of the six possible cases, five are straightforward.

Case	Intersection	Solution
$a \neq 0$, $b \neq 0$, and $c = d = 0$	double point	yes
$a \neq 0$, $c \neq 0$, and $b = d = 0$	double point	yes
$a \neq 0$, $d \neq 0$, and $b = c = 0$	none	yes
$c \neq 0$, $d \neq 0$, and $a = b = 0$	none	yes
$b \neq 0$, $d \neq 0$, and $a = c = 0$	double point	yes

Case $b \neq 0$, $c \neq 0$, *and* $a = d = 0$. Here we have

$$\det(P_{24}) = 1 - 2b^2 + b^4 - 2c^2 + b^2c^2 + c^4 + (-1 + b^2 + c^2)x^2$$

and

$$\det(P_{33}) = 1 - 2b^2 + b^4 - 3c^2 + 2b^2c^2 + 3c^4 - c^6 - 2b^2c^3x + (-1 + 2b^2 - b^4 + c^2)x^2.$$

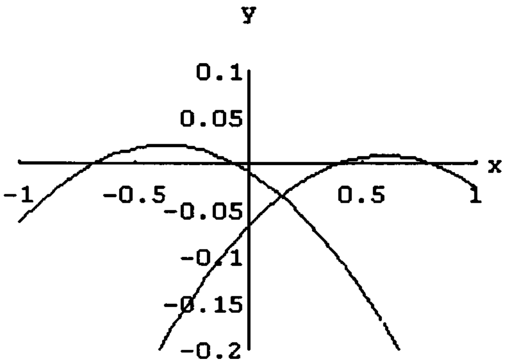


FIG. 4. Example 3×3.

Without loss of generality, we can assume that $b \neq \pm 1$, and these parabolas intersect at points whose x coordinates are given by

$$x_1 = \frac{c(1 + b - c^2)}{b^2 + b} \qquad \text{and} \qquad x_2 = \frac{c(-1 + b + c^2)}{b - b^2}.$$

By the Derivative Method, we can always assume that $x_1 \neq x_2$ (incidentally, $x_1 = x_2$ only when $b^2 + c^2 = 1$, the extremal case). Calculating the derivatives of $\det(P_{24})$ and $\det(P_{33})$, we obtain

$$\frac{d \det(P_{24})}{dx} = 2(-1 + b^2 + c^2)x$$

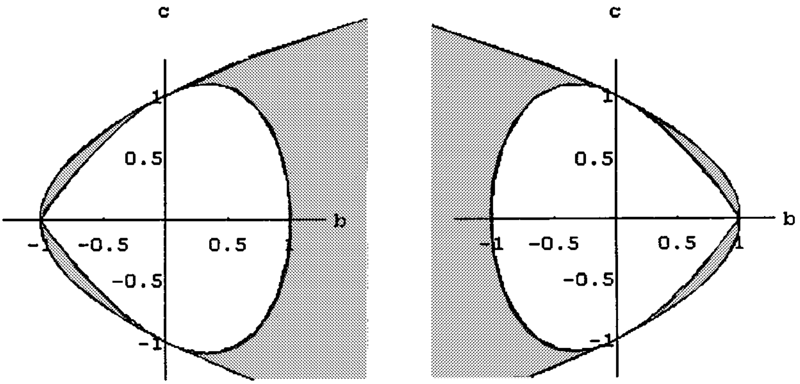
and

$$\frac{d \det(P_{33})}{dx} = -2b^2c^3 + 2(-1 + 2b^2 - b^4 + c^2)x.$$

Analysis of the Sign of the Derivatives. In order to know if the derivatives have the same sign at x_1, we first evaluate them at x_1, we then multiply these two values, and we finally analyze the sign of the product:

$$\frac{d \det(P_{24})}{dx} \frac{d \det(P_{33})}{dx}(x_1) = \left[\frac{4c^2(-1 + b^2 + c^2)^2}{b^2(1 + b)^2} \right] \times \left[(1 + b - c^2)(1 + b - b^2 - b^3 - c^2) \right].$$

The first factor is a square, so the sign of $((d \det(P_{24})/dx)$ $(d \det(P_{33})/dx))(x_1)$ depends on the remaining expression, which is the



$$\frac{d\det(P_{24})}{dx}\frac{d\det(P_{33})}{dx}(x_1) < 0 \qquad \frac{d\det(P_{24})}{dx}\frac{d\det(P_{33})}{dx}(x_2) < 0$$

FIG. 5. H_{44} is a partial contraction and the signs of the derivatives are different in the shaded regions.

product of two factors whose graphs (in the (b, c)-plane) are given by two algebraic curves. It follows at once that $((d\det(P_{24})/dx)(d\det(P_{33})/dx))(x_1)$ is negative precisely when the pair (b, c) is inside one of the curves and outside the other (see Fig. 5).

The analysis at x_2 is very similar, and yields

$$\frac{d\det(P_{24})}{dx}\frac{d\det(P_{33})}{dx}(x_2) = \left[\frac{4c^2(-1+b^2+c^2)^2}{b^2(-1+b)^2}\right] \times\left[(1-b-c^2)(1-b-b^2+b^3-c^2)\right].$$

The places where $((d\det(P_{24})/dx)(d\det(P_{33})/dx))(x_2) < 0$ are shown in Figure 5.

Now recall that H_{44} is a partial contraction, so H_{14} and H_{23} are contractions; we thus have the conditions

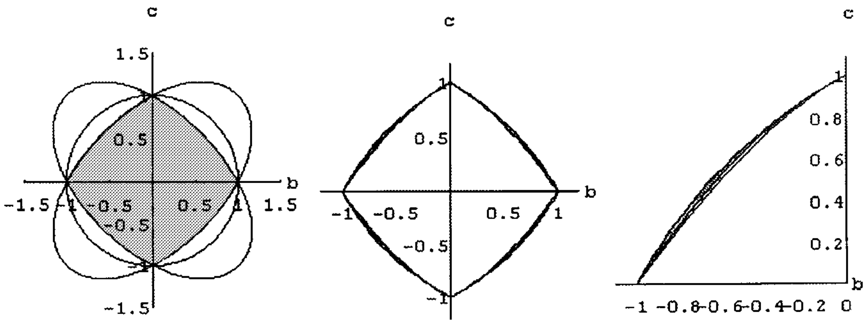
$$\det(P_{14}) = 1 - b^2 - c^2 \geq 0$$

and

$$\det(P_{23}) = (-1+b^2-bc+c^2)(-1+b^2+bc+c^2) \geq 0.$$

As a consequence, we obtain that the pair (b, c) must lie inside both ellipses in Figure 6(a).

Finally, the regions where H_{44} is a bona fide partial contraction and the signs of the derivatives in one of the intersecting points are different are the small regions shown in Figure 6(b). To determine the subregions for



det $P_{23} \geq 0$ inside both ellipses (a)

H_{44} is a partial contraction and signs of derivatives are different in the shaded regions (b)

No solution in the shaded subregion (c)

FIG. 6. Problem 1.1 for $H(0, b, c, 0; x, y, z)$.

which H_{44} does not admit a contractive completion, we study the zeros of det $P_{24}(x_1)$, given by

$$(-b - b^2 + b^3 + b^4 + c + 2bc - b^3c - c^2 + b^2c^2 - c^3 - bc^3 + c^4)$$

$$(-b - b^2 + b^3 + b^4 - c - 2bc + b^3c - c^2 + b^2c^2 + c^3 + bc^3 + c^4) = 0,$$

and the zeros of det $P_{24}(x_2)$, given by a similar equation.

Drawing the solutions for c, we find that each of the regions in Fig. 6(b) is bisected by one of these curves: on one side of them a solution exists, on the other side no solution exists. (See Fig. 6(c), the region in the second quadrant is bisected by the curve given by det $P_{24}(x_1)$.)

4. HANKEL EXTENSIONS

In this section we deal with Problem 1.7. We begin with the case $n = 2$.

4.1. *The Case* $n = 2$. Let

$$H_\Delta(a, b) = \begin{pmatrix} a & b \\ b & \end{pmatrix}$$

be a partial contraction; we wish to find x such that

$$H_\Delta(a, b, x) = \begin{pmatrix} a & b & x \\ b & x & \\ x & & \end{pmatrix}$$

is a partial contraction. Even in this simple case, Problem 1.7 cannot always be solved.

EXAMPLE 4.1. In $H_\Delta(1/\sqrt{2}, 1/\sqrt{2}, x)$, the submatrix $(1/\sqrt{2} \;\; 1/\sqrt{2} \;\; x)$ is contractive only if $x = 0$, while $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & x \end{pmatrix}$ admits $x = -1/\sqrt{2}$ as the only solution which makes it contractive.

Our ploy for solving Problem 1.7 in this case is as follows. Using the results of Section 2.1, Problem 1.1 for $H(a, b; x)$ always admits a solution, described by the interval $[x_l^{(1)}, x_r^{(1)}]$, where $x_l^{(1)} := (-1 - a + b^2)/(1 + a)$ and $x_r^{(1)} := (-1 + a + b^2)/(-1 + a)$. (Of course, the extremal case $a^2 + b^2 = 1$ must be dealt with separately, and we leave this to the reader.) On the other hand, the row matrix $(a \;\; b \;\; x)$ is a contraction if and only if $|x| \le \sqrt{1 - a^2 - b^2}$. It follows that Problem 1.7 (which is well-posed for (a,b) in the closed unit circle) admits a solution precisely when the "fibers" $(a, b) \times [x_l^{(1)}, x_r^{(1)}]$ have non-empty intersection with the closed unit ball in $\mathbf{R}^3$. A moment's reflection shows that this is equivalent to requiring that $x_l^{(1)} \le \sqrt{1 - a^2 - b^2}$ and that $x_r^{(1)} \ge -\sqrt{1 - a^2 - b^2}$. By projecting onto the (a, b)-plane, we see at once that Problem 1.7 admits *no* solution if and only if (a, b) is in the unit disk $a^2 + b^2 \le 1$ and in one of the following sets

- under the parabola $b = -1 + a^2$ and outside the ellipses e_1: $a^2 + ab + b^2 + a + b = 0$ and e_3: $a^2 - ab + b^2 - a + b = 0$,

- above the parabola $b = 1 - a^2$; and outside the ellipses e_2: $a^2 - ab + b^2 + a - b = 0$ and e_4: $a^2 + ab + b^2 - a - b = 0$.

The shaded region in Figure 7 shows the places where Problem 1.7 admits no solution. (Observe that the fact that Problem 1.7 with data $(1/\sqrt{2}, 1/\sqrt{2})$ admits no solution, or for that matter, any pair (a, b) on the unit circle with $a, b \neq 0$, is now rather trivial.)

Remark 4.2. The reader is asked to compare the previous analysis with the 3×3 extremal case of Problem 1.1, exemplified in Figure 1. For (a, b) in the shaded region inside the ellipses of Figure 1, Problem 1.1 with data $(a, b, \sqrt{1 - a^2 - b^2})$ is *not* well-posed, while Problem 1.7 with data (a, b) *is* well-posed but does *not* have a solution.

4.2. *The Case* $n = 3$. We assume that $H_\Delta(a, b, c)$ is a partial contraction, and we wish to find x such that $H_\Delta(a, b, c, x)$ is still a partial contraction. Three submatrices are completely determined: H_{13}, H_{13}^*, and H_{22}.

One can attempt to solve this case by imitating the method used in Section 4.1. Unlike the situation there, however, we do not have at our disposal explicit formulas for the solution interval $[x_l, x_r]$ of Problem 1.1

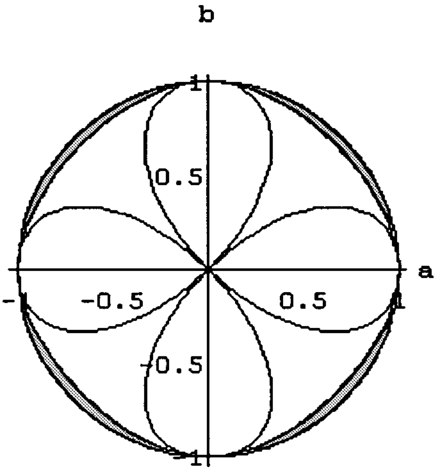


FIG. 7. Analysis of $H_\Delta(a, b)$.

for $H(a, b, c; x, y)$. We are forced, therefore, to proceed differently. By hypothesis, $P_{13} = I - H_{13}H_{13}^*$ and $P_{22} = I - H_{22}H_{22}^*$ are positive, so

$$\det P_{13} = 1 - a^2 - b^2 - c^2 \geq 0$$

and

$$\det P_{22} = 1 - a^2 - 2b^2 + b^4 - 2ab^2c - c^2 + a^2c^2 \geq 0. \tag{4.1}$$

We must find x such that

$$H_{23} = \begin{pmatrix} a & b & c \\ b & c & x \end{pmatrix} \quad \text{and} \quad H_{14} = (a\ b\ c\ x)$$

are contractions. We calculate

$$\det P_{14} = 1 - a^2 - b^2 - c^2 - x^2 \tag{4.2}$$

and

$$\begin{aligned} \det P_{23} = 1 - a^2 - 2b^2 + b^4 - 2ab^2c - 2c^2 + a^2c^2 + b^2c^2 + c^4 \\ + (-2abc - 2bc^2)x + (-1 + a^2 + b^2)x^2. \end{aligned} \tag{4.3}$$

To determine if the two associated parabolas intersect, we examine the discriminant of $\det P_{14} - \det P_{23}$, which equals

$$4\left[(b^4 + a^2c^2)(1 - a^2 - b^2 - c^2) + b^2\left(a^2 + 2ac(a^2 + b^2 + c^2) + c^2\right)\right]. \tag{4.4}$$

If $ac \geq 0$, the expression in (4.4) is clearly non-negative; if $ac < 0$ then

$$a^2 + 2ac\left(a^2 + b^2 + c^2\right) + c^2 \geq a^2 + 2ac + c^2 = (a + c)^2$$

(recall that $a^2 + b^2 + c^2 \leq 1$ by hypothesis), so (4.4) is again non-negative. It follows that the parabolas do indeed meet. Moreover, a double point occurs precisely when the expression in (4.4) equals zero, and two cases arise.

Case 1. $ac \geq 0$. Here we must have ($b = 0$ or $a^2 + b^2 + c^2 = 1$) and ($a = 0$ or $c = 0$ or $a^2 + b^2 + c^2 = 1$) and ($b = 0$ or $a = c = 0$).

Case 2. $ac < 0$. Hence we must have $a^2 + b^2 + c^2 = 1$ and ($b = 0$ or $a = -c$).

In Case 1, $b \neq 0$ forces $a = c = 0$, which a fortiori implies that $b = \pm 1$, and the only solution for Problem 1.7 is $x = 0$. If $b = 0$ and $a \neq 0$ then $c = 0$ or $a^2 + c^2 = 1$, with $|x| \leq \sqrt{1 - a^2}$ and $x = 0$ providing the solutions, respectively. If $b = a = 0$, then $|x| \leq \sqrt{1 - c^2}$ solves Problem 1.7.

In Case 2, $b \neq 0$ forces $c = -a$ and $b = \pm\sqrt{1 - 2a^2}$, and again $x = 0$ is the only solution. If $b = 0$, then $c = \pm\sqrt{1 - a^2} \neq 0$; therefore $x = 0$ is the only solution.

In terms of determining whether a solution to Problem 1.7 exists, the discussion of Cases 1 and 2 was unnecessary, since we know by the Derivative Method that double points always give rise to a solution. However, to obtain *quantitative* information, the additional analysis is required.

When the discriminant of $\det P_{14} - \det P_{23} > 0$ (which of course leads to two simple roots $x_1 \neq x_2$), the analysis using the Derivative Method is far more complicated, and we content ourselves with studying the situation when one of a, b, c equals zero.

Case $c = 0$. Here

$$\det P_{13} = 1 - a^2 - b^2 \geq 0$$

and

$$\det P_{22} = (-1 - a + b^2)(-1 + a + b^2) \geq 0,$$

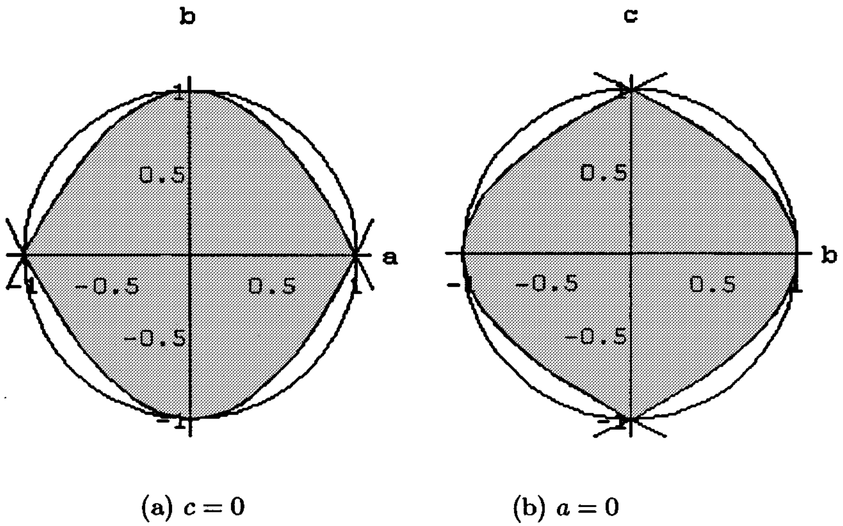


FIG. 8. Problem 1.7 is well-posed and has a solution inside the parabolas.

so $H_\Delta(a, b, 0)$ is a partial contraction whenever (a, b) is inside the unit circle and inside the parabolas $a = b^2 - 1$ and $a = 1 - b^2$ (see Fig. 8(a)).

Now we substitute $c = 0$ in (4.2) and (4.3),

$$\det P_{14} = 1 - a^2 - b^2 - x^2$$

and

$$\det P_{23} = 1 - a^2 - 2b^2 + b^4 + (-1 + a^2 + b^2)x^2,$$

whose intersecting points have x-coordinates given by

$$x_1 = \frac{b\sqrt{1 - b^2}}{\sqrt{a^2 + b^2}} \qquad \text{and} \qquad x_2 = -\frac{b\sqrt{1 - b^2}}{\sqrt{a^2 + b^2}}.$$

Then

$$\left(\frac{d \det P_{14}}{dx} \frac{d \det P_{23}}{dx}\right)(x_i) = \frac{4b^2(-1 + b^2)(-1 + a^2 + b^2)}{a^2 + b^2} \geq 0$$

for $i = 1, 2$, so $H_\Delta(a, b, 0)$ admits a partially contractive extension $H_\Delta(a, b, 0, x)$ whenever $H_\Delta(a, b, 0)$ is a partial contraction. Therefore, Problem 1.7 admits a solution if and only if it is well-posed.

Case $b = 0$. We have

$$\det P_{13} = 1 - a^2 - c^2 \geq 0 \qquad \text{and} \qquad \det P_{22} = (1 + a^2)(1 - c^2) \geq 0$$

so, $H_\Delta(a, 0, c)$ is a partial contraction whenever (a, c) is inside the unit circle; in other words, Problem 1.7 is always well-posed in this case. If we substitute $b = 0$ in (4.2) and (4.3), we obtain

$$\left(\frac{d \det P_{14}}{dx} \frac{d \det P_{23}}{dx}\right)(x_i) = \frac{4c^2(-1 + a^2)(-1 + a^2 + c^2)}{a^2} \geq 0$$

for $i = 1, 2$, so Problem 1.7 always admits a solution.

Case $a = 0$. Observe that

$$\det P_{13} = 1 - b^2 - c^2 \qquad \text{and} \qquad \det P_{22} = (-1 + b^2 - c)(-1 + b^2 + c),$$

so $H_\Delta(0, b, c)$ is a partial contraction whenever (a, c) is inside the unit circle and inside the parabolas $c = b^2 - 1$ and $c = 1 - b^2$ (see Fig. 8(b)).

We now substitute $a = 0$ in (4.2) and (4.3) to obtain

$$\det P_{14} = 1 - b^2 - c^2 - x^2$$

and

$$\det P_{23} = 1 - 2b^2 + b^4 - 2c^2 + b^2c^2 + c^4 - 2bc^2x + (-1 + b^2)x^2,$$

whose intersecting points have x-coordinates given by

$$x_1 = \frac{c^2 + \sqrt{(1 - b^2)(b^2 + c^2)}}{b}$$

and

$$x_2 = \frac{c^2 - \sqrt{(1 - b^2)(b^2 + c^2)}}{b}.$$

Therefore

$$\left(\frac{d \det P_{14}}{dx} \frac{d \det P_{23}}{dx}\right)(x_1)$$
$$= \frac{4}{b^2}\left[(1 - b^2)^2(b^2 + c^2) + c^4 + c^2(2 - b^2)\sqrt{(1 - b^2)(b^2 + c^2)}\right] \tag{4.5}$$

and

$$\left(\frac{d \det P_{23}}{dx} \frac{d \det P_{14}}{dx}\right)(x_2)$$
$$= \frac{4}{b^2}\left[(1-b^2)^2(b^2+c^2)+c^4-c^2(2-b^2)\sqrt{(1-b^2)(b^2+c^2)}\right]. \tag{4.6}$$

Since the right-hand side of (4.5) is obviously non-negative, the Derivative Method implies that it suffices to focus attention on the right-hand side of (4.6). Before we proceed, let us remark that $\det P_{14}(x_2) < 0$ except on the ellipse E: $2b^2 + c^2 = 1$ (where it equals zero). For,

$$\det P_{14}(x_2) = \frac{c^2\left(2\sqrt{(1-b^2)(b^2+c^2)} - (1+c^2)\right)}{b^2},$$

and it follows easily that $\det P_{14}(x_2) = 0$ on E. Off E, $(-1+2b^2+c^2)^2 > 0$, which is equivalent to $4(1-b^2)(b^2+c^2) < (1+c^2)^2$, or $2\sqrt{(1-b^2)(b^2+c^2)} - (1+c^2) < 0$, from which it is apparent that $\det(P_{14}(x_2) < 0$.

Now set $M := (1-b^2)^2(b^2+c^2)+c^4$ and $N := c^2(2-b^2)\sqrt{(1-b^2)(b^2+c^2)}$; since both M and N are non-negative,

$$\left(\frac{d \det P_{23}}{dx} \frac{d \det P_{14}}{dx}\right)(x_2) \geq 0 \Leftrightarrow M^2 \geq N^2.$$

A calculation shows that

$$M^2 - N^2 = \left[b^2(1-b^2-c^2)+c^2(1-c^2)\right]\left[(b^2+c^2)(1-b^2)^3 - c^4\right]; \tag{4.7}$$

since the first factor of the right-hand side of (4.7) is clearly positive, it now suffices to prove that the expression $(b^2+c^2)(1-b^2)^3 - c^4$ is non-negative in the specified region. Changing variables to $s := 1-b^2$ and $t := c^2$, we see that it is enough to establish the non-negativity of $f(s,t) := (1-s+t)s^3 - t^2$ in the region determined by the inequalities $0 \leq s \leq 1$, $0 \leq t \leq s^2$. (Recall that $-1+b^2 \leq c \leq 1-b^2$, so $|c| \leq 1-b^2$, or $c^2 \leq (1-b^2)^2$, i.e., $t \leq s^2$). To study f, it is convenient to introduce yet one more variable, $r := t/s^2$ $(s > 0)$. If $g(r,s) := f(s, rs^2)$, then $f \geq 0$ inside the parabolas if and only if $f(0,0) \geq 0$ and $g \geq 0$ whenever $0 \leq r \leq 1$ and $0 < s \leq 1$. Since $f(0,0) = 0$, we focus on g. Observe that

$$h(r,s) := \frac{g(r,s)}{s^3} = 1 - s + rs^2 - r^2 s,$$

and that

$$\frac{\partial h}{\partial r}(r,s) = s^2 - 2rs \qquad \text{and} \qquad \frac{\partial h}{\partial s}(r,s) = -1 + 2rs - r^2.$$

It follows easily that $\nabla h \neq 0$ in the region $0 \leq r \leq 1, 0 < s \leq 1$, so $g \geq 0 \Leftrightarrow h \geq 0$ in the boundary of the region. But

$$\begin{aligned} h(r,0) &= 1 && (0 \leq r \leq 1) \\ h(1,s) &= (1-s)^2 && (0 \leq s \leq 1) \\ h(r,1) &= r(1-r) && (0 \leq r \leq 1) \\ h(0,s) &= 1-s && (0 \leq s \leq 1), \end{aligned}$$

which completes the proof.

We summarize our findings as follows.

THEOREM 4.3. *Assume that $a = 0$ or $b = 0$ or $c = 0$. Then*

(i) *$H_\Delta(a,b,c)$ admits a contractive extension $H_\Delta(a,b,c,x) \Leftrightarrow H_\Delta(a,b,c)$ is partially contractive.*

(ii-a) *$H_\Delta(0, b, c)$ is partially contractive* $\Leftrightarrow (-1 + b^2 - c)(-1 + b^2 + c) \geq 0$.

(ii-b) *$H_\Delta(a,0,c)$ is partially contractive* $\Leftrightarrow 1 - a^2 - c^2 \geq 0$.

(ii-c) *$H_\Delta(a,b,0)$ is partially contractive* $\Leftrightarrow (-1 - a + b^2)(-1 + a + b^2) \geq 0$.

We conclude this section with an observation linking Problems 1.1 and 1.7.

Remark 4.4. Let $a_1, a_2, \ldots, a_n$ be given real numbers, and assume that Problem 1.1 is well-posed and admits a solution $x, x_2, \ldots, x_{n-1}$. Then Problem 1.7 for $a_2, \ldots, a_n$ is well-posed and admits x as a solution.

EXAMPLE 4.5. Let b, c be real numbers such that (b,c) belongs to one of the crescent moons for which Problem 1.7 is well-posed but admits no solution (see Fig. 7). Then Problem 1.1 with data $(\pm\sqrt{1 - b^2 - c^2}, b, c)$ is not well-posed (because otherwise, it would have a solution, which contradicts Remark 4.4); cf. Figure 1 with a replaced by c.

5. EXTREMAL CASES

DEFINITION 5.1. Let $a_1, \ldots, a_n$ be given data for Problem 1.1 or Problem 1.7. We say that $\{a_1, \ldots, a_n\}$ is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$. Equivalently, using the notation in Section 2, if $1 - QQ^* - \mathbf{r}\mathbf{r}^* = 0$, where $Q = (a_1 \ \cdots \ a_{n-1})$ and $\mathbf{r} = a_n$.

The extremal cases considered in Section 2 always provided unique solutions for Problem 1.1, whenever $a_n \neq 0$. The next result shows that this was no accident.

PROPOSITION 5.2. *Let* $\{a_1, \dots, a_n\}$ *be extremal data for Problem* 1.1, *assume that* $a_n \neq 0$, *and suppose that Problem* 1.1 *admits a solution* $\{x_1, \dots, x_{n-1}\}$. *Then* $\{x_1, \dots, x_{n-1}\}$ *is the unique solution of Problem* 1.1 *with data* $\{a_1, \dots, a_n\}$.

Proof. We write

$$H = H(a_1, \dots, a_n; x_1, \dots, x_{n-1}) = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)^*,$$

where $\mathbf{r}_i$ denotes the ith row of the matrix $(i = 1, \dots, n)$, e.g., $\mathbf{r}_1 = (a_1 \ a_2 \ \cdots \ a_n)$. It follows that $P := I - HH^* = (\delta_{ij} - \mathbf{r}_i \mathbf{r}_j^*)$, where δ_{ij} is Kronecker's delta function. Since $1 - \mathbf{r}_1 \mathbf{r}_1^* = 0$, and since $P \geq 0$ by hypothesis, we must necessarily have $-\mathbf{r}_1 \mathbf{r}_2^* = \cdots = -\mathbf{r}_1 \mathbf{r}_n^* = 0$. Now, for $i = 1, \dots, n-1$,

$$\mathbf{r}_1 \mathbf{r}_{i+1}^* = a_1 a_{i+1} + \cdots + a_{n-i} a_n + a_{n-i+1} x_1 + \cdots + a_{n-1} x_{i-1} + a_n x_i,$$

and $a_n \neq 0$, so we get at once that

$$x_i = -\frac{1}{a_n}(a_1 a_{i+1} + \cdots + a_{n-i} a_n + a_{n-i+1} x_1 + \cdots + a_{n-1} x_{i-1}) \qquad (i = 1, \dots, n-1),$$

thus showing that $\{x_1, \dots, x_{n-1}\}$ is unique. ∎

We saw in Section 2 that the extremal case of Problem 1.1 for $H(a, b, c; x, y)$ always admits a solution, which is often unique (e.g., when $c \neq 0$). The existence proof for $c \neq 0$ we gave there (by expressing p_{23} and p_{33} in terms of p_{22}) might lead one to believe that a generalization to the case of $H(a, b, c, d; x, y, z)$ is at hand. Surprisingly, the next example shows that this is not possible.

EXAMPLE 5.3. Consider Problem 1.1 for $H(a, a, 0, d; x, y, z)$, where $2a^2 + d^2 = 1$, $d \neq 0$. One checks easily that Problem 1.1 is well-posed, and that the only possible candidate for a solution is $x = -a^2/d$, $y = -a$, and $z = (a/d^2)(a^2 - d^2)$. With these values, $p_{22} = a^2(1-3a^2)/(1-2a^2)$, which can be made positive provided that $a^2 < \frac{1}{3}$. However,

$$\det\begin{pmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{pmatrix} = \frac{a^5}{(1-2a^2)^2}\left(7a^3 - 3a + (2-4a^2)\sqrt{1-2a^2}\right),$$

so choosing, for instance, $a = -\frac{1}{2}$, it follows that $p_{22} = \frac{1}{8}$ and

$$\det\begin{pmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{pmatrix} \cong -0.166513 < 0.$$

Thus, despite being well-posed, Problem 1.1 has no solution.

Next, we prove an elementary lemma.

LEMMA 5.4. *Assume that Problem* 1.1 *is well-posed. If* $\det(I - QQ^*) = 0$, *then* $\det A(\mathbf{r}) = \det B(\mathbf{s}) = 0$.

Proof. Let $D := \left(\begin{smallmatrix} I - QQ^* & \mathbf{r} \\ \mathbf{r}^* & 1 \end{smallmatrix}\right)$. By Lemma 2.1(ii-b), $D \geq 0$, and by Lemma 2.2,

$$\det D = \det\begin{pmatrix} I - QQ^* - \mathbf{r}\mathbf{r}^* & \mathbf{r} \\ 0 & 1 \end{pmatrix} = \det\left(I - (Q \quad \mathbf{r})(Q \quad \mathbf{r})^*\right).$$

On the other hand, the classical theorem of Frobenius and Gundelfinger [Ioh, p. 34, Corollary, and Theorem I.6.1] asserts that $\det(I - QQ^*) = 0 \Rightarrow \det D = 0$. It follows that $\det A(\mathbf{r}) = 0$. The proof for $\det B(\mathbf{s})$ is entirely similar. ∎

An obvious consequence of Lemma 5.4 is that whenever $A(\mathbf{r})$ is invertible, the degree of $\det P$ in x is always 2, as we observed in Sections 2 and 3. The *coefficient* of $\det P$ is always $-\det(I - QQ^*)$, as we now see.

THEOREM 5.5. *Let T as in* (1.1) *be a partial contraction, and let* $P := I - TT^*$. *Then* $\det P = \alpha x^2 + \beta x + \gamma$, *where* $\alpha = -\det(I - QQ^*)$ *and* $\beta^2 - 4\alpha\gamma = 4\det A(\mathbf{r})\det B(\mathbf{s})$. *Moreover*

$$\gamma = \det\begin{pmatrix} I - QQ^* - \mathbf{r}\mathbf{r}^* & -Q\mathbf{s} \\ -\mathbf{s}^*Q^* & 1 - \mathbf{s}^*\mathbf{s} \end{pmatrix}.$$

Proof. Let $0 < t < 1$, and consider

$$T^{(t)} := \begin{pmatrix} tQ & t\mathbf{r} \\ t\mathbf{s}^* & x \end{pmatrix}.$$

Then $A^{(t)}(\mathbf{r}) := I - t^2QQ^* - t^2\mathbf{r}\mathbf{r}^*$ is positive and invertible, so by Theorem 1.3, $\alpha^{(t)} = -\det(I - t^2QQ^*)$ and $(\beta^{(t)})^2 - 4\alpha^{(t)}\gamma^{(t)} = 4\det$

$A^{(t)}(\mathbf{r})\det B^{(t)}(\mathbf{s})$. Since $\det(I - t^2QQ^*) \to \det(I - QQ^*)$ and $\det A^{(t)}(\mathbf{r})\det B^{(t)}(\mathbf{s}) \to \det A(\mathbf{r})\det B(\mathbf{s})$, as $t \to 1^-$, the result follows. ∎

COROLLARY 5.6. *Assume that Problem* 1.1 *with data* $\{a_1, \ldots, a_n\}$ *is well-posed and extremal. Then the discriminant of* $\det P$ *is always zero.*

COROLLARY 5.7. *Let T and P be as in Theorem* 5.5, *and assume that* $\det(I - QQ^*) = 0$. *Then* $\alpha = \beta = 0$. *If, in addition,* $P \geq 0$ *for some x, then* $\gamma = 0$; *that is,* $\det P$ *degenerates to the constant zero.*

Proof. By Lemma 5.4, $\det A(\mathbf{r}) = 0$, so $\beta^2 = 4\alpha\gamma = 0$ (by Theorem 5.5). Thus $\alpha = \beta = 0$, and $\det P \equiv \gamma$. Recalling that P is given by (2.2), an application of the theorem of Frobenius and Gundelfinger shows that $\det P = 0$ when $P \geq 0$; thus $\gamma = 0$. ∎

A typical instance of Corollary 5.7 occurs when $a = 0$, $b = \frac{1}{2}$, $c = \frac{3}{4}$, and $d = -\frac{3}{8}$. Since $\det P \equiv 0$, one must resort to a variation of the proof of Theorem 5.5 to find a solution. This is formally established in the following result, which presents an algorithmic solution of Problem 1.1 in case $A(\mathbf{r})$ is singular, in particular in case $\{a_1, \ldots, a_n\}$ is extremal.

THEOREM 5.8. *Assume that* $T = \left(\begin{smallmatrix} Q & \mathbf{r} \\ \mathbf{s}^* & x \end{smallmatrix}\right)$, *with* $A(\mathbf{r}) = I - QQ^* - \mathbf{r}\mathbf{r}^*$ *positive and singular, and* $B(\mathbf{s}) \geq 0$. *Then there exists x such that T is a contraction.*

Proof. Let $T^{(t)}$ be as in the proof of Theorem 5.5. Since $A^{(t)}(\mathbf{r})$ is positive and invertible, Corollary 1.5 guarantees the existence of an interval $[x_l^{(t)}, x_r^{(t)}]$ consisting of solutions. Since $T^{(t)}$ converges in norm to T, and in view of Theorem 5.5 and Corollary 5.6, it follows that, as $t \to 1^-$, the intervals $[x_l^{(t)}, x_r^{(t)}]$ must shrink to a non-empty closed interval $[x_l, x_r]$; any x-value in this interval makes T contractive. ∎

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