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Constrained Equations with Impasse Points

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We give local normal forms for generic constrained equations of the form $f(z)\dot{z} = H(z)$ where f is a smooth real function defined on a manifold M and H is a smooth vector field on M.

1. INTRODUCTION

In this paper we will study constrained equations of the form

$$f(z)\dot{z} = H(z), \tag{1.1}$$

where f is a smooth (C^∞) real function on $\mathbb{R}^m$ and H is a smooth vector field on $\mathbb{R}^m$. Here all functions, fields, manifolds, maps, etc., will be C^∞.

In [1], Rabier posed the problem of studying equations of the form (1.1), because they can be useful to understand explicit systems of differential equations. In the same paper he described qualitative properties of the solutions of (1.1), in a neighborhood of certain points contained in the zero set of f (those satisfying $f(z) = 0$ and $df(z)H(z) \neq 0$).

Equation (1.1) is equivalent to a particular case of an equation of the form

$$A(z)\dot{z} = H(z), \tag{1.2}$$

where $A(z)$ is a linear map of $\mathbb{R}^m$ into itself. In [2] normal forms for (1.2) are given, in the case when $m = 2$.

In this context, our results extend those given in [1] in the sense that they contain generic cases of equations of the form (1.1), giving normal forms for their germs around a zero of f.

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Other kind of constrained equations are those of the form

$$\begin{aligned} F(x, y) &= 0 \\ \dot{x} &= G(x, y), \end{aligned} \tag{1.3}$$

where $F:\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $G:\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

The relation between this kind of constrained equations and those of the form (1.1) is that in the tangent space of $\mathbb{R}^m$, (1.1) is of the form

$$\begin{aligned} f(z)p - H(z) &= 0 \\ \dot{z} &= p. \end{aligned}$$

In [3] Takens studies Eq. (1.3) assuming that $F = (\partial U/\partial y_1, \dots, \partial U/\partial y_n)$ and some properties on the potential U. He defines in an adequate way discontinuous solutions and gives a general existence theorem; he also gives local normal forms for some low-dimensional cases.

Nevertheless, these results cannot be applied to our problem because one of the basic hypotheses in [3] is that $F^{-1}(0) \cap (K \times \mathbb{R}^n)$ is compact if K is a compact subset of $\mathbb{R}^m$. In our case this is satisfied, locally, just in trivial cases ($f(z_0) \neq 0$) or on not generic ones ($f(z_0) = 0$ and $H(z_0) = 0$).

The present paper is organized in the following form. In Section 2 we give some examples and definitions. In particular we give criteria about when two equations of the form (1.1) are equivalent; for this purpose we follow two approaches, the first takes into account the solutions, and the second just considers their images and the zero set of f.

In Section 3 we state our main results, which are applied generically, taking the product fine C^m topology for pairs (f, H) defined on a m-manifold.

Theorems 2 and 3 give, for each equivalence relation, normal forms for the germs, at the zero set of f, of pairs (f, H). In Proposition 1 we describe the invariant geometric properties of them.

Finally, in Section 4 we give the proofs, using Lemma 4 which is an analytic characterization of germs of pairs (f, H) that are equivalent to each normal form.

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2. DEFINITIONS AND EXAMPLES

DEFINITION 1. A *solution* of (1.1) is a mapping $z \in C^\infty(I, \mathbb{R}^m)$ (where I is an open interval) that satisfies $f(z(t))\dot{z}(t) = H(z(t))$. A *trajectory* is the graph of a solution, and a *phase curve* is the image of a solution (as a subset of $\mathbb{R}^m$).

DEFINITION 2. z is an *impasse point* of (1.1) if $f(z) = 0$ and the *impasse set* is $f^{-1}(0)$.

Remark. In a generic case, f and H do not take the value 0 at the same point, so the solutions cannot take the impasse points as values. Nevertheless, for each impasse point z_0, there exist solutions z, such that $\lim_{t \to t_0} z(t) = z_0$. In fact the phase curves of (1.1) are the integral curves of H excluding the impasse set.

It is also a generic property that in a neighborhood of an impasse point f is a submersion; so locally, the set of impasse points is a $(m-1)$-submanifold of $\mathbb{R}^m$.

It follows from these observations that, generically, in a neighborhood of an impasse point, the family of phase curves together with the impasse set is diffeomorphic to a family of parallel lines in $\mathbb{R}^m$ with a $(m-1)$-submanifold of $\mathbb{R}^m$.

EXAMPLE 1.

$$y\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

Using the diffeomorphism

$$\Psi(x, y) = \left(\frac{1}{\sqrt{2}} x, y - \frac{1}{2} x^2 \right)$$

one gets that Ψ sends the impasse set and the phase curves to the respective curves of the equation

$$(x^2 + y)\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In both cases the closure of the phase curve that passes through 0 is tangent to the impasse set and the others do not touch it or are transversal to it. (See Fig. 1).

EXAMPLE 2.

$$x\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad -x\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

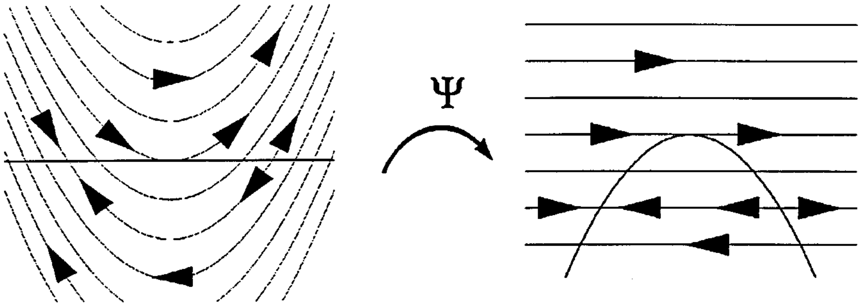

FIG. 1. The impasse sets and the phase curves of two equivalent constrained equations.

In these examples, the impasse set and the phase curves of each equation coincide, nevertheless their dynamics are different: in the first case the solutions move away from the impasse set and in the second they get closer to it (Fig. 2).

EXAMPLE 3.

$$\left(x^3 + xy + z\right)\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}.$$

Here the closure of each phase curve is an horizontal line, the one that passes through 0 has a 3rd order-contact with the impasse set, S, while the others have 2nd order-contact (if they pass through a certain curve, S^1, contained in the impasse set); almost every line is transversal to the impasse set (Fig. 3).

We will give two equivalence relations for germs of equations of the form (1.1). The first takes into account the trajectories of the equations, while the second considers their phase curves.

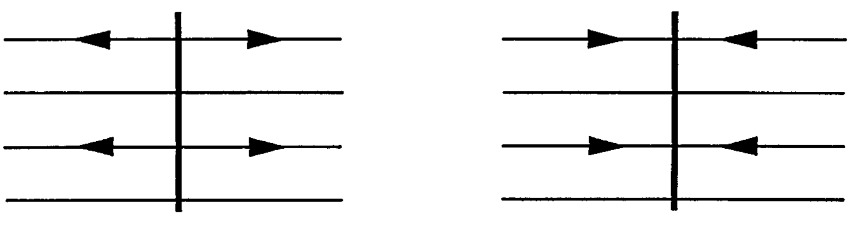

FIG. 2. Two equations with the same impasse sets and phase curves which have different dynamics.

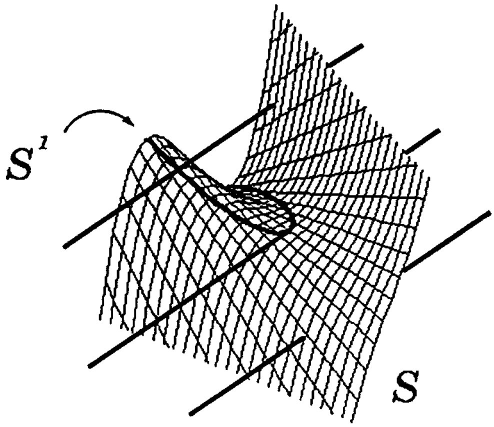

FIG. 3. The phase curves can touch the impasse set with different contact orders.

DEFINITION 3. We will say that the two triples (f, H, z) and $(\hat{f}, \hat{H}, \hat{z})$ are *path-equivalent* $[(f, H, z) \approx (\hat{f}, \hat{H}, \hat{z})]$ if there exist a diffeomorphism $\Psi : (\mathbb{R}^m, z) \to (\mathbb{R}^m, \hat{z})$ such that its germ at z satisfies

$$\hat{f}\,\Psi_* H = \left(f \circ \Psi^{-1}\right)\hat{H}, \tag{2.1}$$

where $\Psi_* H$ is the vector field of $\mathbb{R}^m$ induced from H by the differential of Ψ.

DEFINITION 4. We will say that (f, H, z) and $(\hat{f}, \hat{H}, \hat{z})$ are *phase-equivalent* $[(f, H, z) \sim (\hat{f}, \hat{H}, \hat{z})]$ if their germs satisfy

$$\Psi_* H = \lambda \hat{H}$$
$$f \circ \Psi^{-1} = \alpha \hat{f},$$

where $\Psi : (\mathbb{R}^m, z) \to (\mathbb{R}^m, \hat{z})$ is a diffeomorphism and λ and α are two real smooth functions on $\mathbb{R}^m$ that do not vanish at $\hat{z}$.

Remark. With generic hypothesis ($H \neq 0$ at the impasse points), path-equivalence means that, in a neighborhood of z_0, $t \to z(t)$ is a solution of $f(z)\dot{z} = H(z)$ if and only if $t \to \Psi(z(t))$ is a solution of $\hat{f}(z)\dot{z} = \hat{H}(z)$ (notice that in the complement of the impasse set, Eq. (2.1) can be written as $\Psi_*(H/f) = (\hat{H}/\hat{f})$), while if 0 is a regular value of f and $\hat{f}$, phase-equivalence means that, locally, Ψ sends the integral curves of H to the integral curves of $\hat{H}$ and the zero set of f to the zero set of $\hat{f}$.

EXAMPLE 4. The two equations mentioned on Example 1 are phase-equivalent, and in spite of the fact that Ψ does not satisfy Definition 3, they are also path-equivalent. (Take $\tilde{\Psi}(x, y) = (2^{-1/3}x, -2^{-2/3}x^2 + 2^{1/3}y)$.) The equations on Example 2 are phase-equivalent but they are not path-equivalent.

3. RESULTS

Our main results are Theorems 2 and 3 given below; we give normal forms for the germs at impasse points of generic pairs (f, H).

In the case of phase equivalence, the normal forms are $(f_k, \partial/\partial z_1, 0)$, where $k \leq m$ and

$$f_k(z_1, \dots, z_m) = z_1^k + z_1^{k-2}z_2 + z_1^{k-1}z_3 + \cdots + z_1 z_{k-1} + z_k.$$

DEFINITION 5. We will say that z is a *k-regular impasse point* of (f, H) if $(f, H, z) \sim (f_k, \partial/\partial z_1, 0)$, and that z is a *regular impasse point* if it is a k-regular impasse point for some $k \leq m$.

The following proposition gives a description of the geometry of the pairs (f, H) in a neighborhood of a k-regular impasse point. Here the sets S^l, defined below, play an important role

$$S^0 = f^{-1}(0)$$

$$S^l = \left\{ z \in S^{l-1} \mid H(z) \in T_z S^{l-1} \right\}.$$

EXAMPLE 5. In Example 1, $S^1 = \{0\}$ and for $l > 1$, $S^l = \phi$; in Example 2, $S^l = \phi$ for $l > 0$ and in Example 3, S^1 is the curve where the closures of the phase curves have contact of order 2 with the impasse set, $S^2 = \{0\}$ and $S^l = \phi$ for $l > 2$.

PROPOSITION 1. *If z is a k-regular impasse point, then in a neighborhood of z:*

(a) *For $l < k$, S^l is a submanifold of $\mathbb{R}^m$ of codimension $l + 1$ and $S^0 \supset S^1 \supset \cdots \supset S^{k-1}$.*

(b) *For $l \geq k$, $S^l = \phi$.*

(c) *Every impasse point is regular and the set of l-regular impasse points is $S^{l-1} \setminus S^l$.*

The next theorem guarantees that the pairs (f, H) satisfying that all of its impasse points are regular are generic. This is true if we take the domain of the pairs (f, H) to be any paracompact manifold M. The concept of a regular impasse point can be defined in a manifold in the usual way because it is independent of the coordinate system.

THEOREM 2. *Let M be a paracompact manifold and Λ be the subset of $C^\infty(M,\mathbb{R}) \times X^\infty(M)$*[1] *made of the pairs (f, H) that satisfy that every impasse point is a regular impasse point.*

Λ is an open and dense set with Whitney's C^m fine topology.

In general, phase-equivalence does not imply path-equivalence, but in the case of regular impasse points these two properties are almost the same, except that one has to consider the directions in which the solutions of the equations run through its phase curves. More precisely we have the next theorem

THEOREM 3. *Let z be a k-regular impasse point of (f, H).*

If k is odd, then

$$(f, H, z) \approx \left(f_k, \frac{\partial}{\partial z_1}, 0\right) \qquad or \qquad (f, H, z) \approx \left(-f_k, \frac{\partial}{\partial z_1}, 0\right)$$

and if k is even, then

$$(f, H, z) \approx \left(f_k, \frac{\partial}{\partial z_1}, 0\right).$$

4. PROOFS

Before starting, we will define m operators which will be useful to characterize the k-regular impasse points of a pair (f, H). This characterization will be stated on Lemma 4.

DEFINITION 6. For each $k \leq m$, let S_k be the operator

$$S_k : C^\infty(\mathbb{R}^m, \mathbb{R}) \times X^\infty(M) \to C^\infty(\mathbb{R}^m, \mathbb{R}^k)$$
$$S_k(f, H) = \left(f, D_H f, \ldots, D_H^{k-1} f\right),$$

where $D_H^i f$ denotes the ith derivative of f in the direction of H, i.e., $D_H f(z) = df(z)H(z)$ and $D_H^i f(z) = d(D_H^{i-1} f)(z)H(z)$.

Remark. The operators S_k are independent of the coordinate system in the sense that for any diffeomorphism φ,

$$S_k(f, H) \circ \varphi^{-1} = S_k\left(f \circ \varphi^{-1}, \varphi_* H\right)$$

so these operators are well defined if instead of $\mathbb{R}^m$ we take any manifold.

[1] $X^\infty(M)$ denotes the set of smooth vector fields on M.

LEMMA 4. *z is a k-regular impasse point of (f, H) if and only if the following three conditions hold*:

$$
\begin{aligned}
S_k(f, H)(z) &= 0 \\
D_H^k f(z) &\neq 0 \qquad (4.1)\\
S_k(f, H) &\text{ is a submersion at } z.
\end{aligned}
$$

To prove Lemma 4, we will use the two following lemmas.

LEMMA 5. *Let f, g, and α be real functions on $\mathbb{R}^m$, with $\alpha(0) \neq 0$, and $\Psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is a diffeomorphism such that $\Psi_*(\partial/\partial x) = \lambda(\partial/\partial x)$, where $\lambda(0) \neq 0$.*

(a) *If $g = \alpha f_k$ then*

(i) $g(0) = (\partial g/\partial x)(0) = \cdots = (\partial^{k-1} g/\partial x^{k-1})(0) = 0$ *and* $(\partial^k g/\partial x^k)(0) \neq 0$

(ii) *$S_k(g, \partial/\partial x)$ is a submersion at* 0.

(b) *If $f = g \circ \Psi$ then f satisfies properties* (i) *and* (ii) *if and only if g satisfies them.*

Proof. (a) Let $g = \alpha f_k$; clearly, (i) is valid. The Jacobian matrix of $S_k(g, \partial/\partial x)$ at 0 is of the form

$$
\begin{pmatrix}
0 & \cdots & ..0 & 0!\alpha(0) & 0 & \cdots & 0 \\
\vdots & & \cdot & * & \vdots & \cdots & \vdots \\
0 & (k-2)!\alpha(0) & 0.. & * & 0 & \cdots & 0 \\
k!\alpha(0) & * & \cdots & * & 0 & \cdots & 0
\end{pmatrix}
$$

so $S_k(g, \partial/\partial x)$ is a submersion at 0.

(b) Let $f = g \circ \Psi$ and λ such that $\Psi_*(\partial/\partial x) = \lambda(\partial/\partial x)$. We have that

$$
\frac{\partial^l f}{\partial x^l} = \sum_{i=1}^{l} B_i^l\left(\lambda, \ldots, \frac{\partial^{l-1}\lambda}{\partial x^{l-1}}\right) \frac{\partial^i g}{\partial x^i} \circ \Psi,
$$

where each B_i^l is an homogenous polynomial and $B_l^l(\lambda_1, \ldots, \lambda_l) \equiv \lambda_1^l$.

Using this identity and the fact that $\lambda(0) \neq 0$ one gets the first part of the statement.

One also gets that $S_k(f, \partial/\partial x)$ has the form

$$
S_k\left(f, \frac{\partial}{\partial x}\right) = S_k\left(g, \frac{\partial}{\partial x}\right) \circ \Psi\, M,
$$

where M is the triangular matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & B_1^1 & B_1^2 & \cdots & B_1^{k-1} \\ 0 & 0 & B_2^2 & \cdots & B_2^{k-1} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & B_{k-1}^{k-1} \end{pmatrix}.$$

Differentiating at zero and using that $S_k(g, \partial/\partial x)(0) = 0$ we get

$$D\left(S_k\left(f, \frac{\partial}{\partial x}\right)\right)(0) = {}^tM(0)\, D\left(S_k\left(g, \frac{\partial}{\partial x}\right) \circ \Psi\right)(0),$$

where $D(\cdot)$ is the Jacobian matrix of the respective map.

$M(0)$ is not singular, since $\lambda(0) \neq 0$ and Ψ is a diffeomorphism, so $S_k(g, \partial/\partial x)$ is a submersion at 0, if and only if $S_k(f, \partial/\partial x)$ is also a submersion. ∎

LEMMA 6. *Let* $\varphi : (\mathbb{R}, 0) \to \mathbb{R}$ *be a function of order* k *at* 0 *and* f *a deformation of* φ.

f is r-versal if and only if $S_k(f, \partial/\partial x)$ is a submersion at 0.

Proof. A deformation is versal if and only if it is infinitesimally versal. (See [4].) In the case of right-equivalences, this means that each germ α, can be represented as

$$\alpha = \dot{\varphi} h + \Sigma c_i \dot{f}_i,$$

where h is a function germ and each c_i is a real number, and $\dot{f}_i = (\partial f/\partial \mu_i)(\cdot, 0)$.

In a neighborhood of 0 we can write

$$\dot{f}_i(x) = P_i(x) + x^{k-1} r_i(x)$$
$$\alpha(x) = P(x) + x^{k-1} r(x)$$
$$\dot{\varphi}(x) = x^{k-1} \varphi_1(x),$$

where P_i and P are polynomials of degree not greater than $k - 2$.

Using this notation, we have that f is r-versal if and only if for each P and r there exist real numbers, c_i, and a function h such that in a neighborhood of 0,

$$P = \Sigma c_i P_i$$
$$r = \varphi_1 h + \Sigma c_i r_i.$$

For every collection $\{c_i\}$, the second equation is always true because $\varphi_1(0) \neq 0$, so the r-versality of f is equivalent to the fact that $\{P_i\}$ generates the space of polynomials of degree not greater than $k-2$, which means that the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial \mu_1}(0) & \cdots & \dfrac{\partial f}{\partial \mu_m}(0) \\ \vdots & & \vdots \\ \dfrac{\partial^{k-1} f}{\partial \mu_1 \partial x^{k-2}}(0) & \cdots & \dfrac{\partial^{k-1} f}{\partial \mu_m \partial x^{k-2}}(0) \end{pmatrix}$$

must have rank $k-1$.

This means that $S_k(f, \partial/\partial x)$ is a submersion at 0 because φ is of order k at 0. ∎

Proof of Lemma 4. Without loss of generality $z = 0$ and $H = \partial/\partial x$. This is because the operator S_k and the concept of a k-regular impasse point do not depend on the coordinate system.

If 0 is a k-regular impasse point, in a neighborhood of it, $f \circ \Psi^{-1} = \alpha f_k$, for α and Ψ satisfying the hypothesis of Lemma 5, using this fact, we have that the triple $(f, \partial/\partial x, 0)$ satisfies (4.1).

To see that (4.1) are sufficient conditions for 0 being a k-regular impasse point, we will use the fact that two r-versal deformations with the same number of parameters, say $m+1$, f, and $\hat{f}$, of two r-equivalent germs satisfy

$$f = \hat{f} \circ \Psi$$

for a diffeomorphism Ψ having the form

$$\Psi(x, \mu) = (\xi(x, \mu), \varphi(\mu)),$$

where $\xi : (\mathbb{R} \times \mathbb{R}^{m-1}, 0) \to \mathbb{R}$ and $\varphi : (\mathbb{R}^{m-1}, 0) \to (\mathbb{R}^m, 0)$ are smooth. (In [4] is given a proof in the case of V-versatility, in our case the proof is analogous.)

If $(f, \partial/\partial x, 0)$ satisfies (4.1) then $f(\cdot, 0)$ has a zero of order k at 0, so is r-equivalent to x^k or $-x^k$. Also, by looking at the germ of f as a deformation, Lemma 6 guarantees that f is r-versal. Using the result given above, we have that $f = \pm f_k \circ \Psi$ where $\Psi_*(\partial/\partial x) = (\partial \xi/\partial x)(\partial/\partial x)$, so 0 is k-regular impasse point. ∎

Proof of Proposition 1. Take a neighborhood of z such that $S_k(f, H)$ is a submersion on it. Notice that for $l < k$, $S_{l+1}(f, H)$ is also a submersion because it is a projection of S_k.

In this situation, we can show inductively that

$$S^l = \left[S_{l+1}(f, H)\right]^{-1}(0). \tag{4.2}$$

From this follows (a).

Intersecting with another neighborhood such that $D_H^k f$ has no zeros on it, we get $S^k = \phi$ and every impasse point is l-regular for some $l \leq k$, so using (4.2) we get (c). ∎

To prove Theorem 2, we will use the following

LEMMA 7. *Let V be the set of functions $g \in C^\infty(\mathbb{R} \times \mathbb{R}^{m-1}, \mathbb{R})$ such that the germ of g at any point $(x_0, \mu_0) \in g^{-1}(0)$ is a r-versal deformation of $g(\cdot, \mu_0)$.*

V is an open and dense subset of $C^\infty(\mathbb{R}^m, \mathbb{R})$ with the Whitney fine C^m topology.

Proof. For $k \leq m$ let V_k be the set of functions g such that 0 is a regular value of $S_k(g, \partial/\partial x)$, and let V_{m+1} be the set whose elements are the functions g such that $S_{m+1}(g, \partial/\partial x)$ is always different from 0.

Using Lemma 6, one gets that

$$V = \bigcap_{k=1}^{m+1} V_k.$$

Each V_k is an open set; we do not prove it because it is a particular case of the next theorem.

To show that V_k is dense we will use the Thom Transversality Theorem (see [4]).

Take coordinates of the space of m jets $J^m(\mathbb{R}^m, \mathbb{R})$ such that $j^m f(z)$ is represented by

$$\left(z_1, \ldots, z_m, p^0, \ldots, p^l_{(i_1, \ldots, i_m)}, \ldots, p^m_{(0, \ldots, 0, m)}\right),$$

where $p^l_{(i_1, \ldots, i_m)} = (\partial^l f / \partial z_1^{i_1} \cdots \partial z_m^{i_m})(z)$ and $l = i_1 + \cdots + i_m$.

For each $k \in \{1, \ldots, m\}$ let C_k be the submanifold of $J^m(\mathbb{R}^m, \mathbb{R})$ given by equations

$$p^0 = 0, p^1_{(1,0,\ldots,0)} = 0, \ldots, p^{k-1}_{(k-1,0,\ldots,0)} = 0$$

and C_{m+1} given by

$$p^1_{(1,0,\ldots,0)} = 0, \ldots, p^m_{(m,0,\ldots,0)} = 0.$$

We have that, for $k \leq m$ the set of functions such that $j^m f$ is transversal to C_k is V_k, and for $k = m+1$ it is contained in V_{m+1}, so V is dense. ∎

Proof of Theorem 2. *Openness*. We claim that Λ is a finite intersection of open sets with the C^m fine topology of $C^\infty(M, \mathbb{R}) \times X^\infty(M)$.

Using Lemma 4 we get that

$$\Lambda = \bigcap_{k=1}^{m+1} \Lambda_k,$$

where Λ_{m+1} is the set of pairs (f, H) such that $S_{m+1}(f, H)$ has no zeros, and for $k \in \{1, \ldots, m\}$ the elements of Λ_k are the pairs (f, H) such that 0 is a regular value of $S_k(f, H)$.

Using that M is paracompact, one gets that M is the locally finite union of a family of compact sets $\{Q_\alpha\}$ such that each Q_α is contained in a coordinate system.

Λ_k is open if and only if the restriction of each pair $(f, H) \in \Lambda_k$ to Q_α forms an open set with the C^m topology for pairs (f, H) defined on Q_α.

For $k \leq m$, this set is open because it is the inverse image of the set of functions greater than zero under the continuous operator

$$\mathsf{O}_k : C^\infty(Q_\alpha, \mathbb{R}) \times X^\infty(Q_\alpha) \to C^\infty(Q_\alpha, \mathbb{R})$$

$$(f, H) \mapsto \sum_{i=0}^{k-1} \left(D_H^i f\right)^2 + \sum \left(\frac{\partial(s_k(f, H))}{\partial(z_{i_1}, \ldots, z_{i_k})} \right)^2,$$

where

$$\frac{\partial(G_1, \ldots, G_k)}{\partial(z_{i_1}, \ldots, z_{i_k})} = \det\left[\left(\frac{\partial G_l}{\partial z_{i_j}} \right)_{l \in \{1, \ldots, k\},\, j \in \{1, \ldots, k\}} \right].$$

For $k = m + 1$, the proof is analogous.

Density. First we will prove that if M is a manifold that admits vector fields with no zeroes and H is one of those vector fields, then the set

$$B_H^M = \{ f \in C^\infty(M, \mathbb{R}) \mid (f, H) \in \Lambda \}$$

is dense in $C^\infty(M, \mathbb{R})$.

For each $z \in M$, take a chart φ of M at z defined on a neighborhood V such that $\varphi_* H = \partial/\partial x$.

Notice that $f \in C^\infty(V, \mathbb{R})$ is in B_H^V if and only if the germ of $f \circ \varphi^{-1}$ at each point (x_0, μ_0) of $(f \circ \varphi^{-1})^{-1}(0)$ is a r-versal deformation of $f \circ \varphi^{-1}(\cdot, \mu_0)$.

Consider the operator

$$\mathsf{O} : C^\infty(V, \mathbb{R}) \to C^\infty(\varphi(V), \mathbb{R})$$

$$f \mapsto f \circ \varphi^{-1}.$$

This operator is an homeomorphism with the Whitney fine topology (see [5]), so it preserves density. It follows from Lemma 7 that B_H^V is a dense subset of $C^\infty(V, \mathbb{R})$.

With the usual techniques and using the fact that B_H^M is open, we can extend this result to M. To do this, first check that for every compact subset K of M, the functions whose restrictions to K are in B_H^K form a dense subset of $C^\infty(M, \mathbb{R})$. This implies that B_H^M is dense.

Now we will prove that for any manifold M, Λ is dense. Take $\mathcal{N}$ and $\mathcal{M}$ open sets of $C^\infty(M, \mathbb{R})$ and $X^\infty(M)$, respectively. Let $H \in \mathcal{M}$ be a vector field on M with isolated zeros given by $\{z_i\}$, and f a function in $\mathcal{N}$ such that $f(z_i) \neq 0$.

Let $\overline{M} = M \setminus \{z_i\}$. Since H has no zeros on $\overline{M}$, we can take $g \in C^\infty(\overline{M}, \mathbb{R}) \cap B_H^{\overline{M}}$ arbitrarily close to $f|_{\overline{M}}$, we can suppose also that z_i is not an accumulation point of $g^{-1}(0)$.

Notice that the condition that $g \in B_H^{\overline{M}}$ just depends on the germs of g at the points in $g^{-1}(0)$, so we can perturb g in a family $\{V_i\}$ of disjoint neighborhoods of each z_i, that does not intersect $g^{-1}(0)$. This perturbation can be chosen in such a way that it can be extended to a function $\tilde{f}$ on M which is in $\mathcal{N}$ and has no zeroes in $\bigcup V_i$. Clearly, in this situation $\tilde{f} \in B_H^M \cap \mathcal{N}$.

The technique to carry this out is to choose, for each i, a function $\lambda_i \in C^\infty(M, [0,1])$ with support contained in V_i, which is equal to 1 in a neighborhood of z_i, and take

$$\tilde{f} = \Sigma \lambda_i f + (1 - \lambda_i) g.$$

The fact that g is sufficiently close to f guarantees that $\tilde{f}^{-1}(0) = g^{-1}(0)$, so $(\tilde{f}, H) \in (\mathcal{N} \times \mathcal{M}) \cap \Lambda$. ∎

Proof of Theorem 3. Before starting the proof, we will give a characterization of the fact that two triples on $\mathbb{R}^m$, $(f, \partial/\partial x, 0)$ and $(\tilde{f}, \partial/\partial x, 0)$, are path-equivalent.

Let

$$t_f(x, y) = \int_0^x f(s, y)\, ds,$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{m-1}$

t_f can be interpreted as the time that takes the point $(0, y)$ to go to the point (x, y) by the flow of the vector field $(1/f)(\partial/\partial x)$ (at least if f does not change sign).

It is obtained immediately that $(f, \partial/\partial x, 0) \approx (\tilde{f}, \partial/\partial x, 0)$ if and only if, in a neighborhood of 0,

$$t_f(x, y) = t_{\tilde{f}}(\Psi(x, y)) + h(y),$$

where $\Psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is a diffeomorphism such that $\Psi_*(\partial/\partial x) = \lambda(\partial/\partial x)$ and h is a function defined on $\mathbb{R}^{m-1}$.

With this observation, we will begin the proof.

Let z be a k-regular impasse point of (f, H). Without loss of generality, we can suppose that $M = \mathbb{R}^m$, $z = 0$, $H = \partial/\partial x$ and $f = \alpha f_k$ for a function α with $\alpha(0) \neq 0$.

Let us denote by T the function of $\mathbb{R}^{m+1}$ defined by

$$T(x, y, s) = t_{\alpha f_k}(x, y) + s.$$

T satisfies that $T(\cdot, 0, 0)$ has a zero of order k at 0 and $S_{k+1}(T, \partial/\partial x)$ is a submersion at 0. To prove the last fact, observe that

$$D\left(S_{k+1}\left(T, \frac{\partial}{\partial x}\right)\right)(0) = \begin{bmatrix} Dt_{\alpha f_k}(0) & 1 \\ DS_k\left(\alpha f_k, \dfrac{\partial}{\partial x}\right)(0) & 0 \end{bmatrix}$$

and we know, by Lemma 5, that $S_k(\alpha f_k, \partial/\partial x)$ is a submersion at 0, so applying Lemma 4 we have that the germ of T is a r-versal deformation of $T(\cdot, 0, 0)$ which is r-equivalent to x^{k+1} or $-x^{k+1}$ (depending on the sign of $\alpha(0)$).

The germ at 0 of the function

$$T_k(x, y, s) = t_{f_k}(x, y) + s$$

is also a r-versal deformation of $x^{k+1}/(k+1)$, so there exists a diffeomorphism $\Psi : (\mathbb{R}^{m+1}, 0) \to (\mathbb{R}^{m+1}, 0)$ such that $\Psi_*(\partial/\partial x) = \lambda(\partial/\partial x)$, and locally

$$T_k = T \circ \tilde{\Psi} \qquad \text{or} \qquad -T_k = T \circ \tilde{\Psi} \tag{4.3}$$

depending on the sign of $\alpha(0)$.

Now, let Ψ be defined by

$$\Psi(x, y) = \left(\tilde{\Psi}_1(x, y, 0), \tilde{\Psi}_2(y, 0), \ldots, \tilde{\Psi}_m(y, 0)\right),$$

where $\tilde{\Psi}_i$ is the ith coordinate function of $\tilde{\Psi}$.

Ψ is a diffeomorphism at 0 because $\tilde{\Psi}$ is one and $(\partial \tilde{\Psi}_{m+1}/\partial y_i)(0) = 0$ and $(\partial \tilde{\Psi}_{m+1}/\partial x)(0) = 0$ (this can be obtained by differentiating (4.3)).

Also, the germ of Ψ at 0 satisfies that

$$t_{\pm f_k}(x, y) = t_{\alpha f_k} \circ \Psi(x, y) + \tilde{\Psi}_{m+1}(y, 0)$$

so the theorem is proved. ∎

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