THE $\alpha$-BOUNDIFICATION OF $\alpha$

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Abstract. A space $X$ is $< \alpha$-bounded if for all $A \subseteq X$ with $|A| < \alpha$, $\text{cl}_X A$ is compact. Let $B(\alpha)$ be the smallest $< \alpha$-bounded subspace of $\beta(\alpha)$ containing $\alpha$. It is shown that the following properties are equivalent: (a) $\alpha$ is a singular cardinal; (b) $B(\alpha)$ is not locally compact; (c) $B(\alpha)$ is $\alpha$-pseudocompact; (d) $B(\alpha)$ is initially $\alpha$-compact. Define $B^0(\alpha) = \alpha$ and $B^n(\alpha) = \{\text{cl}_{\beta(\alpha)} A : A \subseteq B^{n-1}(\alpha), |A| < \alpha\}$ for $0 < n < \omega$. We also prove that $B^2(\alpha) \neq B^3(\alpha)$ when $\omega = \text{cf}(\alpha) < \alpha$. Finally, we calculate the cardinality of $B(\alpha)$ and prove that, for every singular cardinal $\alpha$, $|B(\alpha)| = |\beta(\alpha)|^\alpha = |N(\alpha)|^{\text{cf}(\alpha)}$ where $N(\alpha) = \{p \in \beta(\alpha) : \text{there is } A \in p \text{ with } |A| < \alpha\}$. 

0. Introduction

In [15] O'Callaghan proved the following properties of the $\alpha$-boundification $B(\alpha)$ of the discrete space of cardinality $\alpha$ (for definitions see 1.3 and 1.4).

0.1. (a) If $\alpha$ is a regular cardinal, then $B(\alpha)$ is the set of nonuniform ultrafilters on $\alpha$.

(b) $\alpha$ is a singular cardinal if and only if $B(\alpha)$ contains a uniform ultrafilter.

(c) If we assume one of the following statements:

(i) GCH,

(ii) $\alpha$ is a strong limit cardinal,

(iii) $\alpha$ is a regular cardinal,

then $B(\alpha) \neq \beta(\alpha)$. Moreover, if (i) or (ii) holds, then $|B(\alpha)| \leq 2^\alpha$.

From 0.1 it follows that if $\alpha$ is regular, then $B(\alpha) = N(\alpha) = B^\xi(\alpha)$ for each $0 < \xi < \alpha^+$. Hence, $B(\alpha)$ is known when $\alpha$ is a regular cardinal. Thus, the following question, due to Comfort, appears natural.

0.2. Is $B^2(\alpha) \neq B^3(\alpha)$ whenever $\alpha$ is a singular cardinal?

In this paper we are principally concerned with singular cardinals. It is shown, in §2, that $B(\alpha) \neq \beta(\alpha)$ for every cardinal $\alpha$ (Corollary 2.4), and we will
also obtain some topological properties of $B(\alpha)$. In §3 we answer Comfort's question 0.2 in the affirmative when $\omega = \text{cf}(\alpha) < \alpha$. In these two sections Kunen's $\alpha$-good ultrafilters will play an important role. In the last section, the cardinality of $B(\alpha)$ is calculated and we will prove that $|B(\alpha)| = |B(\alpha)|^\alpha = |N(\alpha)|^{\text{cf}(\alpha)}$ for every singular cardinal.

1. Preliminaries

Throughout this paper, all spaces are assumed to be completely regular and Hausdorff. If $X$ is a space and $B \subseteq X$, $\text{cl}_X B$ denotes the closure of $B$ in $X$. For $x \in X$, $A(x)$ is the set of neighborhoods of $x$ in $X$. $\mathcal{P}(X)$ is the set of all subsets of a set $X$. The Greek letters stand for ordinal numbers; in particular, $\alpha$, $\kappa$, $\theta$ denote infinite cardinal numbers; $\gamma$, $\nu$, $\mu$ denote arbitrary cardinals; and $\delta$, $\xi$, $\lambda$, $\eta$ denote ordinal numbers. For a cardinal $\alpha$, we let $\alpha^+$ stand for the smallest cardinal greater than $\alpha$. For $\kappa$, $\gamma$ cardinals we set $[\kappa]^\gamma = \{M \subseteq \kappa : |M| = \gamma\}$ and $[\kappa]^{<\gamma} = \{M \subseteq \kappa : |M| < \gamma\}$.

We do not distinguish notationally between a cardinal number $\alpha$ and the discrete space whose underlying set is that cardinal. For a space $X$, $\beta X$ stands for the Stone-Čech compactification of $X$. If $f : X \to Y$ is a continuous function, we let $\bar{f} : \beta X \to \beta Y$ stand for the Stone extension of $f$. The remainder of $\beta X$ is $X^* = \beta X \setminus X$; in particular, $\alpha^* = \beta(\alpha) \setminus \alpha$. For $A \subseteq \alpha$ we have that (see [2, Chapter 2])

$$\hat{A} = \{p \in \beta(\alpha) : A \subseteq p\} = \text{cl}_{\beta(\alpha)} A \quad \text{and} \quad A^* = \hat{A} \setminus A.$$ 

We shall use the terminology and notation of Comfort and Negrepontis [2].

The notion of $\alpha$-bounded space was introduced in [7] and modified by Comfort as follows.

1.1. Definition. A space $X$ is $<\alpha$-bounded if for every $A \subseteq X$ of cardinality less than $\alpha$, $\text{cl}_X A$ is a compact set.

It is evident that every space is $<\omega$-bounded, and if $X$ is $<\alpha$-bounded, then $X$ is $<\gamma$-bounded for every $\gamma \leq \alpha$. The basic properties of $<\alpha$-bounded spaces are summarized in the following proposition (see, e.g., [7, 8]).

1.2. Proposition. Let $\alpha$ be a cardinal number. Then

(a) every compact space is $<\alpha$-bounded;
(b) every closed subset of a $<\alpha$-bounded space is $<\alpha$-bounded;
(c) the product of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded;
(d) the intersection of a set of $<\alpha$-bounded spaces is $<\alpha$-bounded;
(e) the continuous image of a $<\alpha$-bounded space is $<\alpha$-bounded.

Notice that (d) is a particular case of Lemma 2 of [8], and it is a consequence of (b) and (c).

1.3. For a $<\alpha$-bounded space $Z$ and $X \subseteq Z$, we set

$$B_\alpha(X, Z) = \bigcap \{Y : X \subseteq Y \subseteq Z \text{ and } Y \text{ is } <\alpha\text{-bounded}\}.$$ 

It follows from 1.2(d) that $B_\alpha(X, Z)$ is the smallest $<\alpha$-bounded space containing $X$ and is contained in $Z$. If $Z = \beta X$, then $B_\alpha(X, Z)$ will be denoted by $B_\alpha(X)$. In this case $B_\alpha(X)$ has the following extension property: For each
< $\alpha$-bounded space $Y$ and each continuous function $f: X \to Y$, there exists a continuous function $\hat{f}: B_\alpha(X) \to Y$ such that $\hat{f}|_X = f$ [5, 8]. $B_\alpha(X)$ is called the $\alpha$-boundification of $X$.

1.4. **Notation.** For a space $X$, we define

- $B_0^\alpha(X) = X$;
- $B_\xi^{\alpha+1}(X) = \bigcup\{c_{B_\xi X} A : A \subseteq B_\xi^\alpha(X) \text{ and } |A| < \alpha\}$ for each ordinal $\xi < \alpha^+$; and
- $B_\xi^\alpha(X) = \bigcup_{\xi < \xi} B_\xi^\alpha(X)$ if $\xi$ is a limit ordinal.

Notice that $B_\alpha(X) = \bigcup_{\xi < \alpha^+} B_\xi^\alpha(X)$.

Let $p \in \beta(\alpha)$. The norm of $p$ is $\|p\| = \min\{|A| : A \in p\}$, and $p$ is $\kappa$-uniform if $\|p\| \geq \kappa$. If $p$ is $\alpha$-uniform, then $p$ is said to be uniform. We set $U(\alpha) = \{p \in \beta(\alpha) : \|p\| = \alpha\}$; $N(\alpha) = \{p \in \beta(\alpha) : \|p\| < \alpha\}$; and $W_\kappa(\alpha) = \{p \in \beta(\alpha) : \|p\| = \kappa\}$ ($\kappa < \alpha$). We write $B(\alpha)$ and $B_\xi^\alpha(\alpha)$ instead of $B_\alpha(\alpha)$ and $B_\xi^\alpha(\alpha)$, for $\xi < \alpha^+$, respectively. Note that $N(\alpha) = B_1^\alpha(\alpha)$.

2. **Some topological properties of $B(\alpha)$**

In this section, we observe that $B(\alpha) \neq \beta(\alpha)$ for every cardinal $\alpha$ (Corollary 2.4), and we prove that some topological conditions are equivalent to the singularity of $\alpha$ (Theorem 2.10). We will first give some definitions and preliminary results.

2.1. **Definition.** Let $X$ be a space, $B \subseteq X$, and $p \in cl_X B$.

- (a) $a(p, B) = \min\{|M| : M \subseteq B \text{ and } p \in cl_X M\}$;
- (b) $p$ is said to be a weak $P_\alpha$-point of $X$ if $a(p, X\{p\}) \geq \alpha$. A weak $P$-point is a weak $P_{\omega_1}$-point.

2.2. **Definition.** (a) Let $\gamma$ be a cardinal. A function $h: [\gamma]^{<\omega} \to \mathcal{P}(X)$ is called monotone if $h(A) \subseteq h(B)$ for $A, B \in [\gamma]^{<\omega}$ and $B \subseteq A$, and $f$ is said to be multiplicative if $h(A \cup B) = h(A) \cup h(B)$ for $A, B \in [\gamma]^{<\omega}$.

- (b) $p \in X$ is $\alpha$-good if for each $\kappa < \alpha$ and each monotone function $f: [\kappa]^{<\omega} \to \mathcal{M}(p)$ there is a multiplicative function $g: [\kappa]^{<\omega} \to \mathcal{N}(p)$ which refines $f$ (i.e., $g(A) \subseteq f(A)$ for all $A \in [\kappa]^{<\omega}$).

2.3. **Proposition.** Let $X$ be a space.

- (a) If $X$ is $< \alpha$-bounded and $p$ is a weak $P_\alpha$-point of $X$, then $X\{p\}$ is $< \alpha$-bounded.
- (b) (Kunen) If $X$ is a compact 0-dimensional space and $p \in X$ is $\alpha^{++}$-good, then $p$ is a weak $P_\alpha$-point (for a proof see [4, 4.8]).
- (c) [14] There are $2^{2^{\alpha}}$ countably incomplete uniform ultrafilters in $\beta(\alpha)$ which are $\alpha^+$-good.

As an immediate consequence of Proposition 2.3 we have:

2.4. **Corollary.** (a) For every limit cardinal $\alpha$, there are $2^{2^{\alpha}}$ countably incomplete weak $P_\alpha$-points in $U(\alpha)$.

- (b) $|\beta(\alpha) \setminus B(\alpha)| = 2^{2^{\alpha}}$ for every $\alpha$. In particular, we have that $B(\alpha) \neq \beta(\alpha)$ for every cardinal $\alpha$ (see 0.1(c)).

In the next theorem, we are going to give some topological properties of $B(\alpha)$ when $\alpha$ is a singular cardinal. The concepts and results that follow are needed.
2.5. **Definition** (Saks-Woods). Let $M \subseteq \beta(\alpha)$. A space $X$ is said to be $M$-compact if for every function $f: \alpha \rightarrow X$ we have that $\overline{f(p)} \in X$ for each $p \in M$.

The definition of $p$-compactness for a point $p \in \beta(\alpha)$ was given initially by Bernstein [1]. For other results on spaces required to be $p$-compact simultaneously for various $p$, see Woods [18] and Saks [17].

In [10] the topological properties which are productive, closed hereditary, and surjective are characterized in terms of ultrafilters as follows.

2.6. **Proposition.** Let $P$ be a topological property which is productive, closed hereditary, and surjective. A space $X$ of cardinality $\alpha$ has $P$ if and only if $X$ is $P(\alpha)$-compact, where $P(\alpha)$ is the maximal $P$-reflection of $\alpha$. In particular, a space $X$ is $<\alpha$-bounded iff $X$ is $B(\alpha)$-compact.

2.7. **Definition.** Let $X$ be a space and $\omega \leq \alpha$.

(a) $X$ is said to be a $\alpha$-pseudocompact if every continuous image of $X$ in $\mathbb{R}^\alpha$ is compact.
(b) $X$ is initially $\alpha$-compact if every open cover $\mathcal{U}$ of $X$, with $|\mathcal{U}| \leq \alpha$, has a finite subcover.

The following lemma is due to Retta [16].

2.8. **Lemma.** Let $X$ be a space. Then $X$ is $\alpha$-pseudocompact if and only if every cozero cover of $X$ of cardinality $\leq \alpha$ has a finite subcover.

2.9. **Lemma** [6]. If $\alpha$ is singular, then every $<\alpha$-bounded space is initially $\alpha$-compact.

Now we will prove the main result of this section.

2.10. **Theorem.** The following conditions are equivalent.

(a) $\alpha$ is a singular cardinal.
(b) $B(\alpha)$ is not locally compact.
(c) $B(\alpha)$ is $\alpha$-pseudocompact.
(d) Every $<\alpha$-bounded space is $\alpha$-pseudocompact.
(e) $B(\alpha)$ is initially $\alpha$-compact.
(f) Every $<\alpha$-bounded space is initially $\alpha$-compact.

**Proof.** (d) $\Rightarrow$ (c) and (f) $\Rightarrow$ (e) are trivial, and (a) $\Rightarrow$ (f), (a) $\Rightarrow$ (d), and (e) $\Rightarrow$ (c) are direct consequences of Lemmas 2.8 and 2.9. In order to complete the proof we will show (a) $\iff$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose that $\alpha$ is singular and $B(\alpha)$ is locally compact. Then $B(\alpha) \cap U(\alpha)$ is a nonempty open subset of $U(\alpha)$. Fix an arbitrary $p \in U(\alpha)$. We will show that $p \in B(\alpha)$. Indeed, since the type $T(p) = \{q \in \beta(\alpha) :$ there is a permutation $h$ of $\alpha$ such that $h(p) = q\}$ of $p$ is a dense subset of $U(\alpha)$ (see [2] for a proof), there is $q \in T(p) \cap B(\alpha) \cap U(\alpha)$. Choose a permutation $f$ of $\alpha$ such that $\overline{f(q)} = p$. According to Proposition 2.6, we have that $B(\alpha)$ is $B(\alpha)$-compact and so $\overline{f(q)} = p \in B(\alpha)$; thus, $p \in B(\alpha)$. But this implies that $\beta(\alpha) = B(\alpha)$, a contradiction to Corollary 2.4(b).

(b) $\Rightarrow$ (a) Suppose that $\alpha$ is a regular cardinal. From 0.1(a) it follows that $B(\alpha) = N(\alpha)$. Since $N(\alpha)$ is open in $\beta(\alpha)$, we have that $B(\alpha)$ is locally compact.
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(c) $\Rightarrow$ (a) If $\alpha$ is a regular cardinal and $\mathcal{F} = \{\text{cl}_{\beta(\alpha)} \kappa : \kappa < \alpha\}$, then $\mathcal{F}$ is a cozero cover of $N(\alpha) = B(\alpha)$ of cardinality $\alpha$ without a finite subcover. Now Retta's result (Lemma 2.8) implies that $B(\alpha)$ is not $\alpha$-pseudocompact.

3. The sets $B^2(\alpha)$ and $B^3(\alpha)$

Our main goal here is to give an answer to Comfort's question 0.2 in the affirmative when $\omega = \text{cf}(\alpha) < \alpha$ (see 3.5, 3.7, and 3.8). Because of 0.1, we will only be concerned with singular cardinals. Thus, throughout this section, $\alpha$ will denote a singular cardinal.

3.1. Definition (Keisler [11]). For $p \in X$, let

$$G(p) = \min\{\gamma : \gamma \text{ is a cardinal number and } p \text{ is not } \gamma^+\text{-good}\}.$$  

$G(p)$ is called the degree of goodness of $p$.

3.2. We point out that $B^2(\alpha) \setminus B^1(\alpha)$ is dense in $U(\alpha)$. Indeed, let $\{\kappa_\xi : \xi < \text{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to $\alpha$. Let $A \in [\alpha]^{\omega}$. Choose $p_\xi \in \mathcal{W}_{\kappa_\xi}(\alpha) \setminus \tilde{A}$ for each $\xi < \text{cf}(\alpha)$. If $p \in \beta(\alpha)$ is a complete accumulation point of $\{p_\xi : \xi < \text{cf}(\alpha)\}$, then $p \in (B^2(\alpha) \setminus B^1(\alpha)) \setminus \tilde{A}$.

We will prove in 3.5 that, for each $\kappa < \alpha$, the subset of $B^2(\alpha) \setminus B^1(\alpha)$ of all ultrafilters of degree of goodness equal to $\kappa^+$ is dense in $U(\alpha)$. For our purpose we need the following two lemmas (they are Theorems 10.5 and 10.6 of [2], respectively).

3.3. Lemma (Keisler [12]). Let $\omega \leq \gamma \leq \kappa$, let $r$ be a function from $\kappa$ onto $\gamma$, and let $e : \beta(\gamma) \rightarrow \beta(\kappa)$ be a continuous function such that $\tilde{r} \circ e$ is the identity function on $\beta(\gamma)$. If $q \in \beta(\gamma)$ is countably incomplete and $p = e(q) \in \beta(\kappa)$, then $p$ and $q$ have the same degree of goodness.

Let $\kappa$ be a cardinal number. A family $\mathcal{F}$ of subsets of $\kappa$ is said to have the uniform finite intersection property if $\mathcal{F} \neq \emptyset$ and $|\bigcap_{k \leq n} A_k| = \kappa$ whenever $n < \omega$ and $A_k \in \mathcal{F}$ for every $k \leq n$.

3.4. Lemma. Let $\omega \leq \gamma \leq \kappa$. Every family of subsets of $\kappa$ with the uniform finite intersection property and of cardinality at most $\kappa$ is contained in $2^\kappa$ distinct uniform ultrafilters each of which is countably incomplete and has degree of goodness equal to $\gamma^+$.

By an $\alpha$-partition of $\alpha$ we mean a collection $\mathcal{F}$ of subsets of $\alpha$ such that:

(a) $\alpha = \bigcup \mathcal{F}$; (b) $|A| = \alpha$ for every $A \in \mathcal{F}$; and (c) $A \cap B = \emptyset$ whenever $A$ and $B$ are distinct elements of $\mathcal{F}$.

3.5. Theorem. For every $\kappa < \alpha$, the set

$$\{p \in \beta(\alpha) : p \in B^2(\alpha) \cap U(\alpha) \text{ and } G(p) = \kappa^+\}$$

is dense in $U(\alpha)$.

Proof. Let $A \in [\alpha]^{\omega}$, $\{A_\xi : \xi < \kappa\}$ be an $\alpha$-partition of $A$, and $\{\alpha_\eta : \eta < \text{cf}(\alpha)\}$ be a strictly increasing sequence of cardinals converging to $\alpha$. For every $(\xi, \eta) \in \kappa \times \text{cf}(\alpha)$, pick $p(\xi, \eta) \in A_\xi \cap \mathcal{W}_{\alpha_\eta}(\alpha)$. For every $\xi < \kappa$ we choose a complete accumulation point $p_\xi$ of $\{p(\xi, \eta) : \eta < \text{cf}(\alpha)\}$. It is not difficult to
see that \( p_\xi \in \hat{A}_\xi \cap U(\alpha) \cap B^2(\alpha) \) for each \( \xi < \kappa \). Let \( f : \kappa \to \beta(\alpha) \) be defined by \( f(\xi) = p_\xi \) for \( \xi < \kappa \). According to Lemma 3.4, we can take a countably incomplete ultrafilter \( q \in \beta(\kappa) \) with \( G(q) = \kappa^+ \). Then \( \overline{\int}(q) \in B^2(\alpha) \cap U(\alpha) \cap \hat{A} \) and, by Lemma 3.3, \( G(\overline{\int}(q)) = \kappa^+ \).

The following theorem answers question 0.2 in the affirmative when \( \omega = \text{cf}(\alpha) < \alpha \) (see Corollary 3.8). We need the following lemma; its proof is standard in showing that regular Lindelöf spaces are normal.

3.6. Lemma. Let \( X \) be a normal space. Let \( E = \bigcup_{n<\omega} E_n \) and \( D = \bigcup_{n<\omega} D_n \) be subsets of \( X \) such that \( \text{cl}_X(E_n) \cap \text{cl}_X(D) = \text{cl}_X(D_n) \cap \text{cl}_X(E) = \emptyset \) for every \( n < \omega \). Then there are two disjoint cozero sets \( S, T \subseteq X \) satisfying \( E \subseteq S \) and \( D \subseteq T \).

3.7. Theorem. Assume that \( \text{cf}(\alpha) = \omega \). For each \( n < \omega \), let \( p_n \in U(\alpha) \) with \( G(p_n) = \kappa_n^+ \) where \( \kappa_n \not< \alpha \). If \( p \) is an accumulation point of \( D = \{ p_n : n < \omega \} \), then \( a(p, N(\alpha)) = a \).

Proof. Let \( A \subseteq N(\alpha) \) be of cardinality \( \gamma < \alpha \), and let \( A_n = \{ x \in A : \kappa_n < \|x\| \leq \kappa_{n+1} \} \) for \( n < \omega \). By 2.3(b), there is \( N < \omega \) such that \( p_n \notin \text{cl}_{\beta(\alpha)} A \) for every \( n > N \). Hence, without loss of generality, we may suppose that \( D \cap \text{cl}_{\beta(\alpha)} A = \emptyset \). If \( p \in U(\alpha) \), \( M \subseteq W(\kappa) \alpha < \kappa < \kappa \), and \( p \in \text{cl}_{\beta(\alpha)} M \), then \( |M| = \alpha \); hence, \( \text{cl}_{\beta(\alpha)} D \cap \text{cl}_{\beta(\alpha)} A_n = \emptyset \) for all \( n < \omega \). By Lemma 3.6, we can find two disjoint cozero sets \( S \) and \( T \) of \( \beta(\alpha) \) such that \( A \subseteq S \), \( D \subseteq T \). Since \( \alpha^* \) is an \( F \)-space [2, 14.9], \( \text{cl}_p(\beta(\alpha)) D \cap \text{cl}_{\beta(\alpha)} A = \emptyset \).

The next corollary is an immediate consequence of 3.5 and 3.7.

3.8. Corollary. If \( \text{cf}(\alpha) = \omega \), then \( B^3(\alpha) - B^2(\alpha) \neq \emptyset \).

4. The cardinality of \( B(\alpha) \)

We have mentioned (0.1(c)) that \( |B(\alpha)| \leq 2^\alpha \) when \( \alpha \) satisfies some additional properties. In this section we improve this result by calculating \( |B(\alpha)| \) for every \( \alpha \) (Theorems 4.9, 4.13, and 4.18). We will also establish the relations among \( |B(\alpha)| \), \( |\beta(\alpha)| \), and \( |N(\alpha)| \).

The following concept is basic in this section. For other properties of ultra-products not considered here and historical notes see [2, Chapter 12].

4.1. Definition. Let \( p \in \beta(\alpha) \), and let \( \kappa \) be a cardinal. We define the binary relation \( \equiv \) on \( \kappa^\alpha \) by

\[
 f \equiv g \quad \text{if} \quad \{ \xi < \alpha : f(\xi) = g(\xi) \} \in p .
\]

It is easy to see that \( \equiv \) is an equivalence relation on \( \kappa^\alpha \). We let \( \kappa^\alpha/p \) be the set of \( \equiv \)-equivalence classes. \( \kappa^\alpha/p \) is called the ultraproduct of \( \kappa^\alpha \) modulo \( p \).

The next theorem follows from Lemma 2 of [13] (see [2, 12.22]).

4.2. Theorem. Let \( p \in \beta(\alpha) \) be countably incomplete with \( G(p) = \alpha^+ \). If \( \kappa \) is an infinite cardinal, then \( |\kappa^\alpha/p| = \kappa^\alpha \).

The proof of the following lemma is straightforward.
4.3. Lemma [6]. Let \( \omega \leq \kappa \leq \alpha \) be cardinals, \( p \in U(\kappa) \), and \( \{ A_\xi : \xi < \kappa \} \) be a partition of \( \alpha \). If \( f, g : \kappa \to \alpha^* \) are functions such that \( f(\xi), g(\xi) \in \widetilde{A}_\xi \) for every \( \xi < \kappa \), then \( \widetilde{f}(p) = \widetilde{g}(p) \) if and only if \( \{ \xi < \kappa : f(\xi) = g(\xi) \} \in p \).

In the following two results we calculate the cardinality of \( W_\kappa(\alpha) \) and \( N(\alpha) \) that will allow us to estimate \( |B(\alpha)| \).

4.4. Lemma. For \( \omega \leq \kappa \leq \alpha \) we have that \( |W_\kappa(\alpha)| = \alpha^* \cdot 2^{2^\kappa} \).

Proof. It is evident that \( 2^{2^\kappa} \leq |W_\kappa(\alpha)| \leq 2^{2^* \cdot \alpha^\kappa} \), so we only need to show the inequality \( \alpha^\kappa \leq |W_\kappa(\alpha)| \). Let \( \{ A_\xi : \xi < \kappa \} \) be an \( \alpha \)-partition of \( \alpha \). For each \( \xi < \kappa \), let \( \{ p_\xi, \zeta : \zeta < \alpha \} \) be a strongly discrete subset of ultrafilters contained in \( W_\kappa(\alpha) \cap A_\xi \). Fix \( q \in U(\kappa) \) countably incomplete \( \kappa^+ \)-good and, for each \( f \in \alpha^\kappa \), we define \( \phi_f : \kappa \to \alpha^* \) by \( \phi_f(\xi) = p_\xi, f(\xi) \) for \( \xi < \kappa \). Let \( p_\xi = \widetilde{\phi_f}(q) \).

Clearly, for every \( f \in \alpha^\kappa \), \( p_\xi \in W_\kappa(\alpha) \). Since \( \{ \xi < \kappa : \phi_f(\xi) = \phi_g(\xi) \} = \{ \xi < \kappa : f(\xi) = g(\xi) \} \), and by Lemma 4.3, we have that \( p_\xi = \phi_f(p) = \phi_g(p) = p_\xi \) if and only if \( f \equiv g \). Using Theorem 4.2, we have that \( \alpha^\kappa = |\alpha^\kappa / q| \leq |W_\kappa(\alpha)| \).

The next result appears in [3] without proof.

4.5. Corollary. For every \( \alpha \), the following equality holds:

\[
|N(\alpha)| = \alpha^{< \alpha} \cdot \sum_{\gamma < \alpha} 2^{2^\gamma}.
\]

Proof. We have that \( N(\alpha) = \bigcup \{ W_\gamma(\alpha) : \gamma < \alpha \} \). By virtue of 4.4, it follows that \( |N(\alpha)| = \sum_{\gamma < \alpha} 2^{2^\gamma} \cdot \alpha^{< \alpha} = \alpha^{< \alpha} \cdot \sum_{\gamma < \alpha} 2^{2^\gamma} \).

4.6. Observe that \( |N(\alpha)| = \alpha^{< \alpha} \) if \( \alpha \) is a strong limit cardinal; otherwise, \( |N(\alpha)| = \sum_{\gamma < \alpha} 2^{2^\gamma} = \sup_{\gamma < \alpha} 2^{2^\gamma} \).

The following two lemmas are needed in order to calculate the cardinality of \( B(\alpha) \). For a proof of Lemma 4.7 see [9, Lemma 6.5 and exercise 6.14].

4.7. Lemma. If \( \alpha \) is a strong limit singular original, then

\[
\alpha^{< \alpha} = \alpha^{\text{cf}(\alpha)} = 2^\alpha.
\]

4.8. Lemma. Let \( \alpha \) be a nonstrong limit cardinal such that, for some cardinal \( \theta < \alpha \), \( \sum_{\gamma < \alpha} 2^{2^\gamma} = 2^{2^\theta} \). Then

(a) \( \text{cf}(|N(\alpha)|) \geq \alpha^+ \), and

(b) \( |N(\alpha)| = |N(\alpha)|^\gamma \) for every \( \gamma < \alpha \).

Proof. Set \( \theta < \alpha \) such that \( |N(\alpha)| = 2^{2^\theta} \) and \( 2^{\theta} \geq \alpha \).

(a) \( \text{cf}(|N(\alpha)|) = \text{cf}(2^{2^\theta}) > 2^{\theta} \geq \alpha \).

(b) If \( \gamma \) is a cardinal less than \( \alpha \), then

\[
|N(\alpha)|^\gamma = 2^{(2^\theta) \cdot \gamma} = 2^{2^\theta} = |N(\alpha)|.
\]

4.9. Theorem. Let \( \omega \leq \alpha \). Then \( |B(\alpha)| = |N(\alpha)| \) whenever \( \alpha \) satisfies one of the following properties:

(a) \( \alpha \) is a regular cardinal.

(b) \( \alpha \) is a singular cardinal which is not a strong limit and \( \sup_{\gamma < \alpha} 2^{2^\gamma} = 2^{2^\theta} \) for some \( \theta < \alpha \). In this case, \( |B(\alpha)| = 2^{2^\theta} \).

(c) \( \alpha \) is a singular strong limit. In this case we have \( |B(\alpha)| = 2^\alpha \).
Proof. When $\alpha$ is regular, the conclusion is a consequence of 0.1(a). Let $\alpha$ be a singular cardinal. Suppose that $|B^\xi(\alpha)| = |N(\alpha)|$ for every $\xi < \eta < \alpha^+$. If $\eta$ is a limit ordinal, then

$$|N(\alpha)| \leq |B^{\eta}(\alpha)| = \bigcup_{\xi < \eta} B^{\xi}(\alpha) \leq \sum_{\xi < \eta} |B^{\xi}(\alpha)| = |\eta| \cdot |N(\alpha)| = |N(\alpha)|.$$  

If $\eta = \xi + 1$, then

$$|B^{\eta}(\alpha)| \leq \sum_{\gamma < \eta} |B^{\xi}(\alpha)|^{\gamma} \cdot 2^{2^\gamma} : \gamma < \alpha = \sum_{\gamma < \eta} |N(\alpha)|^{\gamma} \cdot 2^{2^\gamma} : \gamma < \alpha.$$  

Hence, if $\alpha$ satisfies (b) (resp. (c)), then we obtain the equality $|B^\eta(\alpha)| = |N(\alpha)|$ because of Lemma 4.8 (resp. Lemma 4.7). Therefore, in these two cases, $|N(\alpha)| \leq |B(\alpha)| \leq \alpha^+ \cdot |N(\alpha)| = |N(\alpha)|$.

Note that if $\alpha$ is a strong limit cardinal, then $|B(\alpha)| = |N(\alpha)| = \alpha^{<\alpha}$.

In 4.13 we shall have $|B(\alpha)|$ for those cardinals not considered in the previous theorem. We need the following definition and lemma.

4.10. Definition. Let $\omega < \kappa < \alpha$. A collection $\mathcal{G}$ of subsets of $\alpha$ is $\kappa$-almost disjoint if $|G| > \kappa$ for $G \in \mathcal{G}$ and $|C_0 \cap C_1| < \kappa$ for $C_0, C_1 \in \mathcal{G}$ and $C_0 \neq C_1$.

A proof of the following lemma can be found in [2, 12.2].

4.11. Lemma. Let $\kappa, \gamma$ be two cardinal numbers with $\omega \leq \kappa$ and $2 < \gamma$. Then there is a $\kappa$-almost disjoint family $\mathcal{G} \subseteq \mathcal{P}(\gamma^{<\kappa})$ on $\gamma^{<\kappa}$ of cardinality $\gamma^\kappa$.

4.12. We will denote by $L$ the set of cardinals that do not satisfy any properties considered in 4.9; that is, $L = \{\alpha : \alpha$ is a singular nonstrong limit cardinal such that $\sup_{\gamma < \alpha} 2^{2^\gamma} > 2^{2^\nu}$ for every $\nu < \alpha\}$. Observe that (see 4.6) if $\alpha$ is not a strong limit and $\{2^{2^\nu}\}_{\nu < \alpha}$ is not eventually constant (in particular, if $\alpha \in L$), then $\text{cf}(\alpha) = \text{cf}(|A(\alpha)|)$.

4.13. Theorem. If $\alpha \in L$, then $|B(\alpha)| = |N(\alpha)|^{\text{cf}(\alpha)} = 2^K$ where $K = 2^{<\alpha}$. Moreover, if $\omega \leq \gamma < \text{cf}(\alpha)$, then $|N(\alpha)| = |N(\alpha)|^{\gamma} \leq |B(\alpha)|$.

Proof. Let $\mu$ be a cardinal less than $|N(\alpha)|$. We choose $\gamma < \alpha$ such that $2^{2^\gamma} \geq 2^{2^\mu}$. If $\nu < \alpha$, then $2^{2^{\nu}} = 2^{(2^{\gamma})^\nu} = 2^{2^\gamma} < |N(\alpha)|$; therefore (see Theorem 19 in [9]),

(*) for every $\text{cf}(\alpha) \leq \nu < \alpha$ we obtain $|N(\alpha)|^{\nu} = |N(\alpha)|^{\text{cf}(\alpha)}$, and

(**) $|N(\alpha)|^{<\text{cf}(\alpha)} = |N(\alpha)|$.

By using inductively the equality in (*) we obtain that $|B^\xi(\alpha)| \leq |N(\alpha)|^{\text{cf}(\alpha)}$ for every $\xi < \alpha^+$. Hence,

$$|B(\alpha)| \leq |N(\alpha)|^{\text{cf}(\alpha)} \cdot \alpha^+ = |N(\alpha)|^{\text{cf}(\alpha)}.$$  

We are now going to prove that $|N(\alpha)|^{\text{cf}(\alpha)} \leq |B^2(\alpha) \setminus B^1(\alpha)|$. Let $\mathcal{G} = \{G_f : f \in |N(\alpha)|^{\text{cf}(\alpha)}\}$ be a $\text{cf}(\alpha)$-almost disjoint family on $|N(\alpha)|$ of cardinality $|N(\alpha)|^{\text{cf}(\alpha)}$ (see Lemma 4.11 and (**)); let $G_f = \{\lambda_f, \xi : \xi < \text{cf}(\alpha)\}$ be a faithful indexing of $G_f$ for each $f \in |N(\alpha)|^{\text{cf}(\alpha)}$; and let $\mathcal{A} = \{A_\delta : \delta < \text{cf}(\alpha)\}$ be an $\alpha$-partition of $\alpha$ and $\alpha_\delta \not= \alpha$.

Since $|A_\delta \cap B^1(\alpha)| = |N(\alpha)|$ for each $\delta < \text{cf}(\alpha)$, we can take $B_\delta = \{p_\delta, \xi : \xi < |N(\alpha)| \subseteq A_\delta \cap B^1(\alpha)$ such that $\|p_\delta, \xi\| = \alpha_\delta$ for $\xi < \text{cf}(\alpha)$ and $p_\delta, \xi \neq p_\delta, \zeta$ for
\( \xi < \zeta < \text{cf}(\alpha) \). For each \( f \in |N(\alpha)|^{\text{cf}(\alpha)} \), we consider the function \( \phi_f : \text{cf}(\alpha) \rightarrow B^1(\alpha) \) defined by \( \phi_f(\xi) = p_{\xi, \lambda_f, \zeta} \) for \( \xi < \text{cf}(\alpha) \). Fix \( q \in U(\text{cf}(\alpha)) \). Then \( \phi_f(\xi) \in \text{cl}_{\beta(\alpha)} \phi_f(\text{cf}(\alpha)) \subseteq B^2(\alpha) \). It suffices to prove that the relation \( f \rightarrow \phi_f(\xi) \) from \( |N(\alpha)|^{\text{cf}(\alpha)} \) to \( B^2(\alpha) \) is one-to-one. Indeed, let \( f, g \in |N(\alpha)|^{\text{cf}(\alpha)} \) such that \( f \neq g \). It is evident that \( \phi_f(\xi) = \phi_g(\xi) \) iff \( \lambda_f, \zeta = \lambda_g, \zeta \) and so \( |\{ \xi < \text{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi) \}| \leq |G_f \cap G_g| < \text{cf}(\alpha) \). Hence, \( \{ \xi < \text{cf}(\alpha) : \phi_f(\xi) = \phi_g(\xi) \} \notin q \). From Lemma 4.3, it follows that \( \phi_f(q) \neq \phi_g(q) \). Reasoning as in 3.2, we can prove that \( \phi_f(q) \in B^2(\alpha) \setminus B^1(\alpha) \) for each \( f \in |N(\alpha)|^{\text{cf}(\alpha)} \); therefore, \( |N(\alpha)|^{\text{cf}(\alpha)} \leq |B^2(\alpha) \setminus B^1(\alpha)| \leq |B(\alpha)| \). Thus, we have that \( |B(\alpha)| = |N(\alpha)|^{\text{cf}(\alpha)} \).

It remains to show that \( |B(\alpha)| = 2^\kappa \). Let \( \theta = \sup_{\gamma < \alpha} 2^{2^\gamma} = |N(\alpha)| \). Since \( \alpha \in L, \kappa \) is a limit cardinal, \( \text{cf}(\alpha) = \text{cf}(\theta) = \text{cf}(\kappa) \), and \( \theta = \sup_{\mu < \kappa} 2^\mu = 2^{2^\kappa} \); therefore, \( |B(\alpha)| = \theta^{\text{cf}(\theta)} = (2^{2^\kappa})^{\text{cf}(\kappa)} \). Because of Lemma 6.5 in [9], we conclude that \( |B(\alpha)| = 2^\kappa \).

The last assertion of Theorem 4.13 is implied from the following inequality which is a consequence of \((**)\):

\[
|N(\alpha)|^\tau = |N(\alpha)| < |N(\alpha)|^{\text{cf}(\text{cf}(\alpha))} = |N(\alpha)|^{\text{cf}(\alpha)} = |B(\alpha)|.
\]

We have finished the proof of Theorem 4.13. \( \square \)

The following result was already shown in [6]. Here we give an alternative proof (see the definition of \( L \) in 4.12).

4.14. **Corollary.** If \( \omega < \alpha \), then

\[
\alpha^{<\alpha} \cdot \sum_{\gamma < \alpha} 2^{2^\gamma} \leq |B(\alpha)| \leq \left( \sum_{\gamma < \alpha} 2^{2^\gamma} \right)^{\text{cf}(\alpha)}.
\]

**Proof.** If \( \alpha \) is a strong limit, then (see 4.5, 4.7, and 4.9)

\[
\alpha^{<\alpha} \cdot \sum_{\gamma < \alpha} 2^{2^\gamma} = |B(\alpha)| = 2^\alpha = \alpha^{\text{cf}(\alpha)} = \left( \sum_{\gamma < \alpha} 2^{2^\gamma} \right)^{\text{cf}(\alpha)}.
\]

If \( \alpha \) is not a strong limit and either \( \alpha \) is a regular cardinal or \( \sum_{\gamma < \alpha} 2^{2^\gamma} = 2^{\alpha} \) for some \( \theta < \alpha \), then

\[
\alpha^{<\alpha} \cdot \sum_{\gamma < \alpha} 2^{2^\gamma} = \sum_{\gamma < \alpha} 2^{2^\gamma} = |B(\alpha)| \leq \left( \sum_{\gamma < \alpha} 2^{2^\gamma} \right)^{\text{cf}(\alpha)} \quad \text{(see 4.9)}.
\]

Finally, when \( \alpha \in L \), we have

\[
|N(\alpha)| < |B(\alpha)| = |N(\alpha)|^{\text{cf}(\alpha)} = \left( \sum_{\gamma < \alpha} 2^{2^\gamma} \right)^{\text{cf}(\alpha)} \quad \text{(see 4.13)}.
\]

The next corollary improves Theorem 3.5 in [6].

4.15. **Corollary.** For every singular cardinal \( \alpha \), we have

\[
|B(\alpha)| = |N(\alpha)|^{\text{cf}(\alpha)} = |N(\alpha)|^\tau = |B(\alpha)|^{\text{cf}(\alpha)} = |B(\alpha)|^\tau.
\]
Proof. If $\alpha \notin L$, then the result follows from 4.9. Suppose that $\alpha \in L$. In this case we have $\text{cf}(|N(\alpha)|) = \text{cf}(\alpha) < \alpha < |N(\alpha)|$ and $\gamma^{\text{cf}(\alpha)} < |N(\alpha)|$ for every $\gamma < |N(\alpha)| = \sup_{\gamma < \alpha} 2^{\gamma}$. Then $|N(\alpha)|^\alpha = |N(\alpha)|^{\text{cf}(\alpha)}$. Now all the equalities in (#) follow from Theorem 4.13.

4.16. Corollary. Let $\omega \leq \alpha$. Then $|N(\alpha)| = |B(\alpha)|$ if and only if $\alpha \notin L$.

It is possible to construct a model $M$ of ZFC in which $|N(\omega)| < |B(\omega)|$ (see [6]). In this model, $\aleph_\omega \in L$.

4.17. Corollary. Let $\omega \leq \alpha$. Then, for every $1 < \xi < \alpha^+$, we have that $|B^\xi(\alpha)| = |B(\alpha)|$.

Proof. We have that $|N(\alpha)| \leq |B^\xi(\alpha)| \leq |B(\alpha)|$. If $\alpha \notin L$, then $|N(\alpha)| = |B(\alpha)|$ (Corollary 4.16). We have proved in 4.13 that $|B(\alpha)| = |N(\alpha)|^{\text{cf}(\alpha)} \leq |B^2(\alpha)|$ whenever $\alpha \in L$. This completes the proof.

In the following theorem we summarize the results regarding all the possible values of $|B(\alpha)|$.

4.18. Theorem. Let $\omega \leq \alpha$, $\kappa = 2^{<\alpha}$, and $\theta = \sup_{\gamma < \alpha} 2^{\gamma}$.

(a) If $\alpha$ is a strong limit, then

(i) $|B(\alpha)| = \alpha$ if and only if $\alpha$ is regular;
(ii) $|B(\alpha)| = 2^\alpha$ if and only if $\alpha$ is singular.

(b) If $\alpha$ is not a strong limit, then

(i) $|B(\alpha)| = 2^\mu$ for some $\mu < \alpha$ if and only if either $\alpha$ is a successor cardinal or $\{2^{2^\gamma}\}_{\gamma < \alpha}$ is eventually constant;
(ii) $|B(\alpha)| = 2^\kappa$ whenever $\alpha \in L$;
(iii) $2^\kappa < |B(\alpha)| = \theta = 2^{<\kappa} < 2^\kappa < 2^{2^\kappa}$ for every $\mu < \alpha$ whenever $\alpha$ is a regular limit and $\{2^{2^\gamma}\}_{\gamma < \alpha}$ is not eventually constant.

Proof. We obtain (a) as a consequence of 4.6 and 4.9(c) and 1.27 in [2]. The necessity in (b)(i) is trivial (see 4.9), and (b)(ii) is proved in 4.13. We only have to prove (b)(i)(\Rightarrow) and (b)(iii).

(b)(i)(\Rightarrow) In this case, $\alpha$ does not belong to $L$ because $\theta^{\text{cf}(\theta)} > \theta \geq 2^{2^\gamma}$ for every $\gamma < \alpha$ (if $\alpha \in L$, then $\text{cf}(\alpha) = \text{cf}(\theta)$ and $|B(\alpha)| = \theta^{\text{cf}(\theta)}$; see 4.13). Thus, $|B(\alpha)| = |N(\alpha)| = \theta$. So, if $|B(\alpha)| = 2^{2\mu}$ for some $\mu < \alpha$ and $\{2^{2^\gamma}\}_{\gamma < \alpha}$ is not eventually constant, then $\mu^+ = \alpha$.

(b)(iii) Since $\alpha$ is a regular nonstrong limit and $\{2^{2^\gamma}\}_{\gamma < \alpha}$ is not eventually constant, $2^{2^\mu} < \theta = |B(\alpha)|$ for every $\mu < \alpha$. It is also clear that $\{2^{2^\gamma}\}_{\gamma < \alpha}$ is not eventually constant, hence, neither is $\{2^{2^\nu}\}_{\nu < \kappa}$ and so $\sup_{\nu < \kappa} 2^{2^\nu} < 2^\kappa$. The inequality $2^\kappa \leq 2^{2^\kappa}$ always holds, and $\theta = \sup_{\nu < \kappa} 2^{2^\nu}$ follows from the properties of $\alpha$.

As immediate consequences of the previous theorem we have the following corollaries (in the first one we determine the conditions under which $B(\alpha)$ has the same cardinality as $\beta(\alpha)$).

4.19. Corollary. (a) If $\alpha \in L$, then $|B(\alpha)| = 2^{2^\kappa}$ if and only if $2^{2^\kappa} = 2^\kappa$ where $\kappa = 2^{<\alpha}$.

(b) If $\alpha \notin L$, then $|B(\alpha)| = 2^{2^\kappa}$ if and only if $2^{2^\kappa} = 2^{2^\mu}$ for some $\mu < \alpha$. 
4.20. **Corollary.** If GCH holds, then for every infinite cardinal $\alpha$ we have that $|N(\alpha)| = |B(\alpha)| < |B(\alpha)|$.

4.21. **Corollary.** If $\alpha$ is a singular cardinal, then $|B(\alpha)| = 2^\mu$ for some cardinal $\mu$.

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