Can. J. Math.Vol. 43 (5), 1991 pp. 985–997

A CRITERION OF CONVERGENCE OF GENERALIZED PROCESSES AND AN APPLICATION TO A SUPERCRITICAL BRANCHING PARTICLE SYSTEM

BEGOÑA FERNÁNDEZ AND LUIS G. GOROSTIZA

1. **Introduction.** The problem of convergence in distribution of a large class of generalized semimartingales to a continuous process is considerably simplified by a recent theorem of Aldous [1], in conjunction with a result of Cremers and Kadelka [3] on convergence of integral functionals, and the results of Mitoma [15] and Fouque [8] for generalized processes. We will give a convenient convergence criterion in this setting. The proof amounts to a direct combination of the results of the abovementioned authors, requiring only a minor extension (of a special case) of the theorem of Cremers and Kadelka.

The processes we will consider are Φ'-valued, where Φ' is the strong dual of Φ, a Fréchet nuclear space or a strict inductive limit of a sequence of Fréchet nuclear spaces. The duality between Φ' and Φ will be denoted by $\langle \cdot, \cdot \rangle$. In the application Φ' will be $\mathcal{S}'(R^d)$, the Schwartz space of tempered distributions, i.e., the dual of the space $\mathcal{S}(R^d)$ of rapidly decreasing infinitely differentiable real functions on R^d.

We will apply the criterion to prove convergence of the fluctuation process of a supercritical branching particle system with immigration. In order to appreciate the power of the new convergence criterion, which is essentially the idea of Aldous [1], we will show briefly the difficulties that arise in trying to prove convergence of this process by the usual approach involving the increasing process of the associated martingale. Obtaining the increasing process requires major computations, and we had not succeeded in establishing tightness by this approach. The new criterion does not use the increasing process.

The type of process we will consider was introduced in [10] in connection with a system of supercritical branching Brownian motions with immigration in R^d. The objective of [10] (and the previous papers [9,4]) was to define and study a process containing some useful genealogical information about the system, which cannot be detected if one looks only at the spatial distribution of particles at each time. The idea is to count the number of particles at a final time T (the present) whose ancestors at time $t \in [0, T]$ had positions in given Borel subsets of R^d. Equivalently, a counting measure is defined

Research partially supported by CONACyT grant PCEXCNA - 040319.
Received by the editors October 30, 1989; revised March 30, 1990.
AMS subject classification: 60F17, 60G20, 60J80.

whose atoms are the locations of the particles present at time t, each atom having a random weight equal to the number of descendants of the corresponding particle at time T. A counting measure-valued process on $[0, T]$ is thus defined. The main result in [10] is a fluctuation limit theorem as $T \longrightarrow \infty$ for this process appropriately rescaled; however, only convergence of finite-dimensional distributions was proved. One of the problems in establishing tightness by the usual semimartingale approach is that it requires estimates based on the increasing process of the martingale part. In this case the jumps caused by the immigration appear in the increasing process in a way that makes it very difficult to handle. We will prove tightness using the new criterion. In addition, the model we will consider here is more general than the one called *special model* in [10], and is not covered by the computations in that paper.

In Section 2 we will state and prove the convergence criterion. In Section 3 we will describe the particle system, introduce the scaling, and state the fluctuation theorem, and for completeness we will include also the law of large numbers for the system and the Langevin equation satisfied by the fluctuation limit process. In Section 4 we will prove the limit theorems. Section 5 is devoted to a brief discussion of the difficulties involved in proving tightness by the usual approach.

The necessary background on branching processes is contained in the Preliminaries of [10].

We end the Introduction with some notation we will use.

The space of right-continuous with left limits functions from R_+ into Φ' is denoted by $D(R_+, \Phi')$ and is endowed with a Skorokhod-type topology [15,8,12]. The space of continuous elements of $D(R_+, \Phi')$ is denoted by $C(R_+, \Phi')$. For each n let $X^n \equiv \{X^n_t,\ t \in R_+\}$ be processes with values in E ($E = \Phi'$ or R^k). We write $X^n \longrightarrow X$ f.d.d. as $n \longrightarrow \infty$, for weak convergence of finite-dimensional distributions, and $X^n \Rightarrow X$ as $n \longrightarrow \infty$, for weak convergence on $D(R_+, E)$. Similarly for $D([0,1], E)$.

We denote by $\{\mathcal{T}_t,\ t \in R_+\}$ the Brownian semigroup

$$\mathcal{T}_t \phi(x) = \int_{R^d} \phi(y) p_t(x,y)\, dy, \quad t > 0, \quad \mathcal{T}_0 = I,$$

where

$$p_t(x,y) = (2\pi t)^{-d/2} \exp\{-|y-x|^2/2t\}.$$

2. Convergence criterion.

THEOREM 2.1. *Let $\{X^n\}_{n\geq 1}$ be a sequence of processes with paths in $D(R_+, \Phi')$ and X^0 a process with paths in $C(R_+, \Phi')$. Assume that*

(a) *For each $\phi \in \Phi$ there exists $\psi_\phi \in \Phi$ such that for every $n \geq 0$ the process*

$$M^n_t(\phi) \equiv \langle X^n_t, \phi \rangle - \int_0^t \langle X^n_s, \psi_\phi \rangle\, ds, \quad t \in R_+, \tag{2.1}$$

is a martingale.

(b) *$X^n \longrightarrow X^0$ f.d.d. as $n \longrightarrow \infty$.*

(c) *For each* $K > 0$ *and* $\phi \in \Phi$ *there exists* $\eta > 0$ *such that*

$$\sup_{n\geq 1} \int_0^K E|\langle X_t^n, \phi\rangle|^{1+\eta} dt < \infty.$$

(d) *For each* $t \in R_+$ *and* $\phi \in \Phi$ *the sequence* $\{M_t^n(\phi)\}_{n\geq 1}$ *is uniformly integrable.*

Then $X^n \Rightarrow X^0$ *as* $n \longrightarrow \infty$ *in* $D(R_+, \Phi')$.

REMARKS. (1) Conditions (c) and (d) of the theorem are satisfied if for each $K > 0$ and $\phi \in \Phi$,

$$\sup_{n\geq 1} \sup_{t\in[0,K]} E\langle X_t^n, \phi\rangle^2 < \infty. \tag{2.2}$$

(2) The convergence criterion also holds for R^k-valued processes, since $(X_1, \ldots, X_k) \in R^k$ can be viewed as an element X of $\mathcal{S}'(R^d)$ of the form $X = \sum_{i=1}^k X_i\delta_{x_i}$, where the δ_{x_i} are Dirac distributions at different fixed points $x_1, \ldots, x_k \in R$.
(3) When Φ is a countably Hilbert nuclear space, the convergence criterion is extended in [7] to cover a wider range of applications in the following way: the function ψ_ϕ in (2.1) is allowed to depend on n and s.

The proof of this theorem will be an immediate consequence of the following lemmas.

LEMMA 2.2 (ALDOUS [1]). *For each* $n \geq 0$, *let* M^n *be a real martingale,* A^n *and* B^n *an increasing and a decreasing process, respectively, such that*

(a) $(M^n, A^n, B^n) \longrightarrow (M^0, A^0, B^0)$ *f.d.d. as* $n \longrightarrow \infty$.

(b) M^0, A^0, *and* B^0 *have continuous paths.*

(c) *For each* $t \in R_+$, $\{M^n(t)\}_{n\geq 1}$ *is uniformily integrable.*

Then $X^n \equiv M^n + A^n + B^n \Rightarrow X^0 \equiv M^0 + A^0 + B^0$ *in* $D(R_+, R)$.

LEMMA 2.3 (CREMERS AND KADELKA [3]). *For each* $n \geq 0$, *let* X^n *be a process with paths in* $D(R_+, R^k)$. *Assume that*

(a) $X^n \longrightarrow X^0$ *f.d.d. as* $n \longrightarrow \infty$.

(b) *For each* $K > 0$ *and some* $\eta > 0$

$$\sup_{n\geq 1} \int_0^K E|X_i^n(t)|^{1+\eta}\, dt < \infty, \quad i = 1, \ldots, k.$$

Then

$$\left(X^n(\cdot), \int_0^{\cdot} \left(X^n(s)\right)^+ ds, \int_0^{\cdot} \left(X^n(s)\right)^- ds\right) \longrightarrow \left(X^0(\cdot), \int_0^{\cdot} \left(X^0(s)\right)^+ ds, \int_0^{\cdot} \left(X^0(s)\right)^- ds\right)$$
$$f.d.d. \text{ as } n \longrightarrow \infty,$$

where $x^+ = (x_1^+, \ldots, x_k^+)$, $x^- = (x_1^-, \ldots, x_k^-)$.

REMARK. Cremers and Kadelka [3] proved Lemma 2.3 more generally for a class of functionals of the form $\int \phi\left(s, x(s)\right) d\mu(s)$. In our case μ is Lebesgue measure on $[0, K]$ and the functions $\phi: R_+ \times R^k \longrightarrow R$ are $\phi\left(s, (x_1, \ldots, x_k)\right) = 1_{[0,t]}(s) x_i^+$, and

$\phi\big(s,(x_1,\dots,x_k)\big) = 1_{[0,t]}(s)x_i^-$, for $i = 1,\dots,k$, $t = t_1,\dots,t_m$. The result in [3] refers only to the convergence of the functionals, not including the process itself, but the proof is completely similar.

PROOF OF THEOREM 2.1. Since by hypothesis $X^n \longrightarrow X^0$ f.d.d., we only have to show tightness of $\{X^n\}_{n\geq 1}$ to prove the theorem [15,8]. For each $\phi \in \Phi$ and $n \geq 0$ we write $\langle X^n,\phi\rangle \equiv \{\langle X_t^n,\phi\rangle,\ t\in R_+\}$. By [15,8] it suffices to prove tightness of $\{\langle X^n,\phi\rangle\}_{n\geq 1}$ for each $\phi \in \Phi$.

By (2.1) the process $\langle X^n,\phi\rangle$ is expressed as

$$\langle X^n,\phi\rangle = M^n(\phi)+A^n(\phi)+B^n(\phi), \quad n\geq 0,$$

where $A^n(\phi)$ and $B^n(\phi)$ are given by

$$A_t^n(\phi) = \int_0^t \langle X_r^n,\psi_\phi\rangle^+\,dr,$$

$$B_t^n(\phi) = -\int_0^t \langle X_r^n,\psi_\phi\rangle^-\,dr.$$

Note that $M^0(\phi)$, $A^0(\phi)$, $B^0(\phi)$ are continuous since $\langle X^0,\phi\rangle$ is continuous.

Then, in order to obtain tightness of $\{\langle X^n,\phi\rangle\}_{n\geq 1}$ (in fact convergence in $D(R_+,R)$) it suffices to verify condition (a) of Lemma 2.2, since condition (b) holds, and condition (c) is hypothesis (d) of the theorem. But condition (a) of Lemma 2.2 follows from Lemma 2.3 due to hypotheses (b) and (c) of the theorem and the continuity of the map $(x,y,z)\mapsto(x-y-z,y,z)$. ■

3. **Particle system.** The particle system in the Euclidean space R^d evolves as follows: Particles are distributed at time 0 and immigrate at later times according to independent Poisson random fields with intensities $\gamma\geq 0$ in a set $B\in\mathcal{B}(R^d)$ and $\beta\geq 0$ in a set $C\in\mathcal{B}(R^d\times R_+)$, respectively. Each initial and each immigrant particle independently migrates according to a standard Brownian motion, and after an exponential lifetime with parameter V branches according to a law $\{\pi_n, n = 0,1,\dots\}$ with finite third moment, whose mean, second and third factorial moments we denote by m_1, m_2 and m_3, respectively. The offspring particles obey the same rules starting their migrations from the locations where their parents branched. We consider the supercritical case, i.e. $m_1 > 1$, and assume for simplicity $\pi_0 = 0$, hence all descendence lines are infinite. We assume the sets B and C satisfy $KB = B$ and $KC_t = C_t$, where $C_t = \{x\in R^d \mid (x,t)\in C\}$ is the t-section of C, for all $K > 0$ and $t > 0$. Furthermore we assume that C_t converges as $t\longrightarrow\infty$ and denote the limit by Q.

We count the particles at a final time T (considered as the present), and denote by $\hat{N}_{t,T}(A)$ the number of particles living at time T such that their ancestors at time $t\in[0,T]$ had positions in $A\in\mathcal{B}(R^d)$. Hence $\hat{N}_{t,T}$ is a random counting measure whose atoms are the positions of the particles present at time t, each atom having a random weight equal to the number of descendants of the corresponding particle at time T. Additionally we weight each atom of $\hat{N}_{t,T}$ by $e^{-\alpha(T-s)}$, where $\alpha = V(m_1 - 1) > 0$ is the Malthusian parameter of the branching and s is the time of birth of the first ancestor of the corresponding particle. Replacing the immigration intensity β by β / T and introducing the space-time scaling $(x,t) \mapsto (T^{1/2}x, Tt)$, we designate by $N^T \equiv \{N_t^T, t \in [0,1]\}$ the random counting measure-valued process so obtained from $\hat{N}_{t,T}$.

Let $X^T \equiv \{X_t^T, t \in [0,1]\}$ denote the fluctuation process defined by

$$X_t^T = T^{-d/4}(N_t^T - EN_t^T), \quad t \in [0,1].$$

The asymptotic behaviour of the process N^T is contained in the following two theorems.

THEOREM 3.1 (LAW OF LARGE NUMBERS). *For each $t \in [0,1]$ and $\phi \in \mathcal{S}(R^d)$,*

$$T^{-d/2}\langle N_t^T, \phi \rangle \longrightarrow \gamma \int_B \mathcal{T}_t\phi(y)\,dy + \beta \int_0^t \int_Q \mathcal{T}_{t-s}\phi(y)\,dy\,ds,$$

in L^2 as $T \longrightarrow \infty$.

THEOREM 3.2 (FLUCTUATION LIMIT). *(a) $X^T \Rightarrow X$ in $D\big([0,1], \mathcal{S}'(R^d)\big)$ as $T \longrightarrow \infty$, where $X \equiv \{X_t, t \in [0,1]\}$ is a centered Gaussian $\mathcal{S}'(R^d)$-valued process with covariance functional*

$$\begin{aligned}\operatorname{Cov}(\langle X_s, \phi \rangle, \langle X_t, \psi \rangle) = m_2(m_1 - 1)^{-1}\{\gamma \int_B \mathcal{T}_s\phi(x)\mathcal{T}_t\psi(x)\,dx \\ + \beta \int_0^s \int_Q \mathcal{T}_{s-r}\phi(x)\mathcal{T}_{t-r}\psi(x)\,dx\,dr\},\end{aligned} \tag{3.1}$$

for each $s,t \in [0,1]$, $s \le t$, $\phi, \psi \in \mathcal{S}(R^d)$.

(b) *X is a (Markovian) generalized Ornstein-Uhlenbeck process which satisfies the generalized Langevin equation*

$$dX_t = \frac{1}{2}\Delta X_t\,dt + [m_2(m_1 - 1)^{-1}\beta]^{1/2}\,dW_t^Q, \quad t \in [0,1],$$

$$X_0 = [m_2(m_1 - 1)^{-1}\gamma]^{1/2}W^B,$$

where W^B is a standard white noise on B, and $W^Q \equiv \{W_t^Q, t \in [0,1]\}$ is an $\mathcal{S}'(R^d)$-valued Wiener process with covariance functional

$$\operatorname{Cov}(\langle W_s^Q, \phi \rangle, \langle W_t^Q, \psi \rangle) = (s \wedge t) \int_Q \phi(x)\psi(x)\,dx,$$

(i.e., dW^Q is a standard space-time white noise on $Q \times [0,1]$). Hence for each $\phi \in \mathcal{S}(R^d)$,

$$\langle X_t, \phi \rangle - \int_0^t \langle X_s, \frac{1}{2}\Delta\phi \rangle\,ds, \quad t \in [0,1], \tag{3.2}$$

is a martingale.

(c) *For each $t \in (0,1]$, X_t is induced by an ordinary Gaussian random field.*

REMARKS. (1) The nature of the fluctuation limit process X can be explained intuitively as follows. The particle system can be viewed as the sum of a compound Poisson field on $B \subset R^d$ with intensity γ at time 0 and an independent compound Poisson field on $C \subset R^d \times R_+$ with intensity β, every point of each of these two Poisson fields originating an independent supercritical branching Brownian motion. Each one of these supercritical Brownian motions is scaled and weighted so that its associated measure-valued process converges almost surely to the process $\{Z\mathcal{T}_t\}$, where Z is a certain random variable related only to the branching and $\mathcal{T}_t$ is the Brownian semigroup (this follows from an almost-sure invariance principle [11]). Hence, asymptotically the system behaves as a sum of two independent compound Poisson fields on B and C, with an independent copy of the process $\{Z\mathcal{T}_t\}$ starting from each point of each of these two Poisson fields. (Note that this is different from the system with an independent Brownian motion stemming from each Poisson point, which has another type of fluctuation limit, see [2], Section 4, Example 2). As a consequence, in the limit the branching and the migration have deterministic effects represented by the constant $EZ^2 = m_2(m_1 - 1)^{-1}$ and the operator $\frac{1}{2}\Delta$, respectively, and all the randomness comes from the two Poisson fields, whose fluctuations converge to the initial spatial white noise $\gamma^{1/2}W^B$ and to the immigration space-time white noise $\beta^{1/2}dW^{Q}$. Therefore the fluctuation limit process X evolves according to the heat equation perturbed by the space-time white noise $[m_2(m_1 - 1)^{-1}\beta]^{1/2}dW^{Q}$, with random initial condition $[m_2(m_1 - 1)^{-1}\gamma]^{1/2}W^B$. This is precisely what the generalized Langevin equation for X represents. On the other hand, this heuristic argument explains also why for each $t > 0$ the generalized random field X_t is induced by an ordinary random field, although X_0 is not [10]. The smoothing effect of the heat semigroup is not affected by the space-time white noise perturbation.

(2) The assumptions $KB = B$, $KC_t = C_t$, $C_t \longrightarrow Q$ as $t \longrightarrow \infty$ should also have been made in [10], where in the limit results C should be $Q \times [0, 1]$.

(3) The asymptotic behaviour described by the previous theorem is analogous if, instead of Brownian motion, the particles migrate according to a symmetric stable process with exponent $\alpha \in (0, 2)$. The operator $\frac{1}{2}\Delta$ is replaced by $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$, and $\mathcal{T}_t$ by the corresponding semigroup, but the fluctuation limit process is still continuous. The only difference is that the Langevin equation is interpreted in a generalized sense, explained in [5], because Δ_α does not map $\mathcal{S}'(R^d)$ into itself.

4. **Proofs of the limit theorems.**

We recall the following notation introduced above. For $t \in [0, T]$ and $A \in \mathcal{B}(R^d)$,

$\hat{N}_{t,T}(A) =$ number of particles present at time T whose ancestors at time t had positions in A.

Similarly, for $x \in R^d$, $s \le t \le T$ and $A \in \mathcal{B}(R^d)$, we denote

$\hat{N}_{x,s,t,T}(A) =$number of particles present at time T whose ancestors at time t had positions in A, starting with a single particle born at the point x at time s.

($\hat{N}_{x,s,t,T} = 0$ for $s > t$). Then

$$\hat{N}_{t,T} = \sum_i \hat{N}_{p_i,0,t,T} + \sum_i \hat{N}_{q_i,u_i,t,T}, \tag{4.1}$$

where $\{p_i\}_{i=1,2,\ldots}$ are the points of the initial Poisson field (with intensity γ in B) and $\{(q_i, u_i)\}_{i=1,2,\ldots}$ are the points of the immigration Poisson field (with intensity β in C).

Each atom of $\hat{N}_{t,T}$ is weighted by multiplying $\hat{N}_{x,s,t,T}$ by $e^{-\alpha(T-s)}$, and we write

$$N_{x,s,t,T} = e^{-\alpha(T-s)}\hat{N}_{x,s,t,T}, \tag{4.2}$$

and $N_{t,T}$ for the random measure so defined; hence we have from (4.1),

$$N_{t,T} = \sum_i N_{p_i,0,t,T} + \sum_i N_{q_i,u_i,t,T}. \tag{4.3}$$

For simplicity of notation we may write

$$N_{t,T} = \sum_i N_{q_i,u_i,t,T}, \tag{4.4}$$

where $\{(q_i, u_i)\}_i$ includes the immigrant as well as the initial particles.

Given times $s \leq t \leq T$ and a particle located at x_i at time s, we designate by x_{ik} the location of the k-th descendant of the particle at time t, and we write $k(t)$ to refer explicitly to the time t. We denote by $Z^x_{t,T}$ the total number of descendants at time T of a particle located at x at time t.

Considering $N_{t,T}$ as an $\mathcal{S}'(R^d)$-valued process, we have from (4.4) , for each $\phi \in \mathcal{S}(R^d)$,

$$\begin{aligned} \langle N_{t,T}, \phi \rangle &= \sum_{i,u_i \leq t} \langle N_{q_i,u_i,t,T}, \phi \rangle \\ &= \sum_{i,u_i \leq t} \sum_{k(t)} \phi(q_{ik}) e^{-\alpha(T-u_i)} Z^{q_{ik}}_{t,T}. \end{aligned} \tag{4.5}$$

Under the immigration intensity β / T and the space-time scaling $(x, t) \longmapsto (T^{1/2}x, Tt)$, we denote by $N^T_{x,s,t}$ and N^T_t the scaled random measures corresponding to $N_{x,s,t,T}$ and $N_{t,T}$, respectively. Thus

$$\langle N^T_t, \phi \rangle = \langle N_{Tt,T}, \phi^T \rangle, \tag{4.6}$$

where $\phi^T = \phi(\cdot / T^{1/2})$ and $N_{Tt,T}$ is given by (4.3) with β / T instead of β.

For $\phi, \psi \in \mathcal{S}(R^d)$ and $s \leq t \leq T$ the following moment expressions are obtained as in [10], using (4.2) and (4.5):

$$E\langle N_{t,T}, \phi \rangle = \gamma \int_B E\langle N_{x,0,t,T}, \phi \rangle \, dx + \beta \int_0^t \int_{C_r} E\langle N_{x,r,t,T}, \phi \rangle \, dx\, dr, \tag{4.7}$$

and

$$\begin{aligned} \mathrm{Cov}(\langle N_{s,T}, \phi \rangle, \langle N_{t,T}, \psi \rangle) = {} & \gamma \int_B E\langle N_{x,0,s,T}, \phi \rangle \langle N_{x,0,t,T}, \psi \rangle \, dx \\ & + \beta \int_0^s \int_{C_r} E\langle N_{x,r,s,T}, \phi \rangle \langle N_{x,r,t,T}, \psi \rangle \, dx\, dr. \end{aligned} \tag{4.8}$$

The next lemma gives the explicit expressions for (4.7) and (4.8), already scaled.

LEMMA 4.1. *For each* $T > 0$, $t \in [0, 1]$, $s \le t$ *and* $\phi, \psi \in \mathcal{S}(R^d)$,

$$E\langle N_t^T, \phi \rangle = T^{d/2}\{\gamma \int_B \mathcal{T}_t\phi(x)\,dx + \beta \int_0^t \int_{C_{Tr}} \mathcal{T}_{t-r}\phi(x)\,dx\,dr\} \tag{4.9}$$

and

(4.10)

$$\begin{aligned}
&\mathrm{Cov}(\langle N_s^T, \phi\rangle, \langle N_t^T, \psi\rangle) \\
&= T^{d/2}\gamma[m_2 e^{-\alpha Ts} + (m_2 - m_1 + 1)e^{-\alpha T}](m_1 - 1)^{-1} \int_B \mathcal{T}_s(\phi\,\mathcal{T}_{t-s}\psi)(x)\,dx \\
&+ T^{d/2}\gamma m_2(m_1 - 1)^{-1} \int_B \int_0^s \alpha T e^{-\alpha Tu} \mathcal{T}_u[(\mathcal{T}_{s-u}\phi)(\mathcal{T}_{t-u}\psi)](x)\,du\,dx \\
&+ T^{d/2}\beta(m_1 - 1)^{-1} \int_0^s \int_{C_{Tr}} [m_2 e^{-\alpha T(s-r)} + (m_2 - m_1 + 1)e^{-\alpha T(1-r)}] \\
&\qquad\qquad \mathcal{T}_{s-r}(\phi\,\mathcal{T}_{t-s}\psi)(x)\,dx\,dr \\
&+ T^{d/2}\beta m_2(m_1 - 1)^{-1} \int_0^s \int_{C_{Tw}} \int_0^{s-w} \alpha T e^{-\alpha Tv} \mathcal{T}_v[(\mathcal{T}_{s-w-v}\phi)(\mathcal{T}_{t-w-v}\psi)](x)\,dv\,dx\,dw.
\end{aligned}$$

PROOF. By (4.2), (4.5) and (4.6),

$$E\langle N_t^T, \phi \rangle = \gamma \int_B E\langle N_{x,0,Tt,T}, \phi^T\rangle\,dx + \beta/T \int_0^{Tt} \int_{C_r} E\langle N_{x,r,Tt,T}, \phi^T\rangle\,dx.$$

In [10] it is shown that

$$E\langle N_{x,r,t,T}, \phi\rangle = E\langle N_{x,r,t,t}, \phi\rangle, \tag{4.11}$$

and the right-hand side of (4.11) is computed explicitly in [13], formula (5.6) ([13] deals with a multitype model. This formula is given below in connection with the second moment). Using these results we obtain

$$E\langle N_t^T, \phi \rangle = \gamma \int_B \mathcal{T}_{Tt}\phi^T(x)\,dx + \beta/T \int_0^{Tt} \int_{C_r} \mathcal{T}_{Tt-r}\phi^T(x)\,dx\,dr.$$

The self-similarity of Brownian motion implies $\mathcal{T}_{Ts}\phi^T = (\mathcal{T}_s\phi)^T$, and with the change of variable $u = r/T$ in the second integral we have

$$E\langle N_t^T, \phi \rangle = \gamma \int_B (\mathcal{T}_t\phi)^T(x)\,dx + \beta \int_0^t \int_{C_{Tu}} (\mathcal{T}_{t-u}\phi)^T(x)\,dx\,du,$$

and since $T^{-d/2}B = B$ and $T^{-d/2}C_r = C_r$ for all $r > 0$,

$$E\langle N_t^T, \phi \rangle = \gamma T^{d/2} \int_B \mathcal{T}_t\phi(x)\,dx + \beta T^{d/2} \int_0^t \int_{C_{Tu}} \mathcal{T}_{t-u}\phi(x)\,dx\,du.$$

Hence (4.9) is proved.

Expression (4.10) can be obtained in a similar way, using (4.2), (4.5), (4.6), (4.8), the fact [10] (formulas (e),(f),(g)):

$$\begin{aligned}
&E\langle N_{x,r,s,T}, \phi\rangle\langle N_{x,r,t,T}, \psi\rangle \\
&= e^{-2\alpha(T-r)}\{(m_2 - m_1 + 1)(m_1 - 1)^{-1}(e^{2\alpha(T-s)} - e^{\alpha(T-s)})E\langle \hat{N}_{x,r,s,s}, \phi\,\mathcal{T}_{t-s}\psi\rangle \\
&\qquad + e^{2\alpha(T-s)}E\langle \hat{N}_{x,r,s,s}, \phi\rangle\langle \hat{N}_{x,r,s,s}, \mathcal{T}_{t-s}\psi\rangle\}, \quad s \le t,
\end{aligned}$$

and formulas (5.6) and (5.7) of [13] (reduced to the single type case), namely,

$$E\langle \hat{N}_{x,r,s,s}, \xi \rangle = e^{\alpha(s-r)}\mathcal{T}_{s-r}\xi(x),$$

and

$$E\langle \hat{N}_{x,r,s,s}, \phi \rangle \langle \hat{N}_{x,r,s,s}, \xi \rangle$$
$$= e^{\alpha(s-r)}\mathcal{T}_{s-r}(\phi\xi)(x) + m_2 V e^{2\alpha(s-r)} \int_0^{s-r} e^{-\alpha u}\mathcal{T}_u[(\mathcal{T}_{s-r-u}\phi)(\mathcal{T}_{s-r-u}\xi)](x)\,du.$$

We omit this calculation since it is rather tedious. ∎

The following consequence is immediate.

COROLLARY 4.2. *For $s \le t$, and $\phi, \psi \in \mathcal{S}(R^d)$,*

$$T^{-d/2}E\langle N_t^T, \phi \rangle \longrightarrow \gamma \int_B \mathcal{T}_t\phi(x)\,dx + \beta \int_0^t \int_Q \mathcal{T}_{t-r}\phi(x)\,dx\,dr, \tag{4.12}$$

and

$$T^{-d/2}\operatorname{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle)$$
$$\longrightarrow m_2(m_1-1)^{-1}\left\{\gamma \int_B \mathcal{T}_s\phi(x)\mathcal{T}_t\psi(x)\,dx + \beta \int_0^s \int_Q \mathcal{T}_{s-r}\phi(x)\mathcal{T}_{t-r}\psi(x)\,dx\,dr\right\}, \tag{4.13}$$

as $T \longrightarrow \infty$.

PROOF. These results follow from (4.9) and (4.10) since $\alpha T e^{-\alpha T v} \longrightarrow \delta_0$ as $T \longrightarrow \infty$ and $C_{Tr} \longrightarrow Q$ as $T \longrightarrow \infty$ $(r > 0)$. ∎

The proof of Theorem 3.1 follows from (4.12) and (4.13) the same way as the proof of the law of large numbers in [10].

The following lemma will be used in the proof of tightness in Theorem 3.2.

LEMMA 4.3. *For each $\phi \in \mathcal{S}(R^d)$,*

$$M_t^T(\phi) \equiv \langle X_t^T, \phi \rangle - \int_0^t \langle X_r^T, \frac{1}{2}\Delta\phi \rangle\,dr, \quad t \in [0,1], \tag{4.14}$$

is a martingale.

PROOF. In the technical report version of [10] it is proved that

$$Z_t(\phi) \equiv \langle N_{t,T}, \phi \rangle - \int_0^t \langle N_{r,T}, \frac{1}{2}\Delta\phi \rangle\,dr - H_t(\phi), \quad t \in [0,T], \tag{4.15}$$

is a martingale with respect to the filtration $\mathcal{F}_t^T = \mathcal{F}\{\langle N_{s,T}, \psi \rangle, s \le t, \psi \in \mathcal{S}(R^d)\}$, where

$$H_t(\phi) \equiv \sum_{i, u_i \le t} \phi(q_i) e^{-\alpha(T-u_i)} Z_{u_i,T}^{q_i},$$

and $\{(u_i, q_i)\}_i$ are the times and places of the initial and immigrant particles, $\{x_i(r)\}_i$ are the locations of the atoms of $N_{r,T}$, $\{s_i(r)\}_i$ are the times of birth of their corresponding first ancestors, and $Z_{t,T}^x$ is the total number of descendants at time T of a particle located at

x at time t. Since $\{H_t(\phi), t \in [0,T]\}$ is a process such that $H_t(\phi) - H_s(\phi)$ is independent of $\mathcal{F}_s^T$ for $s < t$, then

$$Y_t^T(\phi) \equiv \langle N_{t,T}, \phi \rangle - \int_0^t \langle N_{r,T}, \frac{1}{2}\Delta\phi \rangle\, dr - EH_t(\phi), \quad t \in [0,T], \tag{4.16}$$

is a martingale. Introducing the scaling (4.6) and using $\Delta(\phi^T) = T^{-1}(\Delta\phi)^T$, it follows that

$$\langle N_t^T, \phi \rangle - \int_0^t \langle N_r^T, \frac{1}{2}\Delta\phi \rangle\, dr - EH_t^T(\phi), \quad t \in [0,1], \tag{4.17}$$

is a martingale, where $H_t^T(\phi) \equiv H_{Tt}(\phi^T)$ is obtained from the process $\{H_t(\phi)\}$ with immigration intensity β / T and the space-time scaling. Finally, the martingale (4.14) is obtained from the definition of X_t^T and the martingale (4.17). ■

PROOF OF THEOREM 3.2. The $\mathcal{S}'(R^d)$-valued Gaussian process X with covariance functional given by (3.1) exists by Kolmogorov's extension theorem, and it can be shown in the usual way [14] that it has a continuous version. The Langevin equation for X is obtained from the covariance functional by the result of [2], and therefore we have the martingale (3.2) associated with X.

We also have the martingale (4.14) associated with X^T given by Lemma 4.3.

The proof of $X^T \longrightarrow X$ f.d.d. as $T \longrightarrow \infty$ is done the same way as in Theorem 4.6 of [10], using the assumption $m_3 < \infty$.

Hence, in order to prove part (a) of the theorem it remains only to verify condition (2.2), and the conclusion then follows from Theorem 2.1.

From (4.10) it can be shown that for each $K > 0$ and $\phi \in \mathcal{S}(R^d)$there is a constant $M_{K,\phi}$ depending on K and ϕ, such that

$$E(\langle X_t^T, \phi \rangle)^2 \leq M_{K,\phi} \text{ for all } t \in [0,1] \text{ and } T > 0,$$

which implies condition (2.2).

The fact that X_t is induced by an ordinary Gaussian random field for $t \in (0,1]$ is proved by standard methods. ■

5. **Comparison with the usual approach.** By the *usual approach* to proving tightness we mean the application of Theorem 8.6 and Remark 8.7 of Chapter 3 of [6], where the bound is obtained by means of estimates based in the increasing processes of the martingales associated with the processes. In our case the increasing processes are given in the following lemma.

LEMMA 5.1. *For each $\phi \in \mathcal{S}(R^d)$ and $T > 0$, the increasing processes of the martingales (4.16) and (4.14) are given by*

$$\langle Y^T(\phi) \rangle_t \equiv T^{-1} \int_0^t \sum_i \left| \nabla \phi \left(x_i(Tr) T^{-1/2} \right) \right|^2 e^{-2\alpha(T - s_i(Tr))} \left(Z_{Tr,T}^{x_i(Tr)/T^{1/2}} \right)^2 dr$$
$$+ EJ_t^T(\phi)^2, \quad t \in [0,1],$$

and

$$\langle M^T(\phi)\rangle_t = T^{-d/2}\langle Y^T(\phi)\rangle_t, \quad t\in[0,1], \tag{5.1}$$

respectively, where $J_t^T(\phi) = H_t^T(\phi) - EH_t^T(\phi)$.

PROOF. In the preliminary version of [10] it is proved that the increasing process of the martingale (4.15) is given by

$$\langle Z(\phi)\rangle_t = \int_0^t \sum_i \left|\nabla\phi\left(x_i(r)\right)\right|^2 e^{-2\alpha\left(T-s_i(r)\right)}(Z_{r,T}^{x_i(r)})^2\,dr, \quad t\in[0,T],$$

The result is then obtained similarly as in the proof of Lemma 4.3, using $\nabla\phi^T = T^{-1/2}(\nabla\phi)^T$. ∎

REMARK. The proof that the increasing process of $Z(\phi)$ is given as above [10], although it can be simplified, requires a significant amount of heavy computations.

In order to apply the sufficient condition for tightness of $\{\langle X^T,\phi\rangle\}_{T>1}$ given by [6] we need the existence of a random variable $\Gamma_{T,\delta}\ge 0$ such that

$$E[(\langle X_{t+\delta}^T,\phi\rangle - \langle X_t^T,\phi\rangle)^2 \mid \mathcal{F}_t^T] \le E[\Gamma_{T,\delta} \mid \mathcal{F}_t^T], \quad t\in[0,1),\ \delta\in(0,1-t].$$

and

$$\lim_{\delta\to 0}\limsup_{T\ge 1} E\Gamma_{T,\delta} = 0. \tag{5.2}$$

Using the martingale (4.14), its increasing process (5.1) and elementary inequalities, we have

$$\begin{aligned}
&E[(\langle X_{t+\delta}^T,\phi\rangle - \langle X_t^T,\phi\rangle)^2 \mid \mathcal{F}_t^T]\\
&\quad\le 2E[(M_{t+\delta}^T(\phi) - M_t^T(\phi))^2 \mid \mathcal{F}_t^T] + 2E[\delta\int_t^{t+\delta}\langle X_s^T,\tfrac{1}{2}\Delta\phi\rangle^2\,ds]\\
&\quad\le 2T^{-(d/2+1)}E\int_t^{t+\delta}\sum_i\left|\nabla\phi\left(x_i(Tr)T^{-1/2}\right)\right|^2 e^{-2\alpha\left(T-s_i(Tr)\right)}(Z_{Tr,T}^{x_i(Tr)/T^{1/2}})^2\,dr\\
&\qquad + 2T^{-d/2}E[J_{t+\delta}^T(\phi)^2 - J_t^T(\phi)^2] + 2\delta^2 E\sup_{0\le s\le 1}\langle X_s^T,\tfrac{1}{2}\Delta\phi\rangle^2 \qquad (5.3)\\
&\quad\le 2T^{-(d/2+1)}E\int_t^{t+\delta}\sum_i\left|\nabla\phi\left(x_i(Tr)T^{-1/2}\right)\right|^2 e^{-2\alpha\left(T-s_i(Tr)\right)}\left(Z_{Tr,T}^{x_i(Tr)/T^{1/2}}\right)^2\,dr\\
&\qquad + K\beta\delta\int_{R^d}\phi^2(y)\,dy + 2\delta^2 E\sup_{0\le s\le 1}\langle X_s^T,\tfrac{1}{2}\Delta\phi\rangle^2,
\end{aligned}$$

for some constant K, where we have used the fact that

$$EJ_t^T(\phi)^2 = T^{d/2}\beta\int_0^t\int_{C_{Tr}}\phi^2(y)\,dy\,g\big(T(1-r)\big)\,dr,$$

with

$$g(t) = e^{-2\alpha t}EZ_t^2 = (m_1-1)^{-1}[m_2-(m_2-m_1+1)e^{-\alpha t}].$$

Using Doob's inequality for the martingale (4.14) we find a constant K_ϕ depending on ϕ, such that

$$E \sup_{0\leq s\leq 1} \langle X_s^T, \frac{1}{2}\Delta\phi \rangle^2 \leq K_\phi .$$

Hence the last two terms in (5.3) are well-behaved for condition (5.2). The difficult term is

$$R_t^T(\phi) = T^{-(d/2+1)} E \int_t^{t+\delta} \sum_i \left|\nabla\phi\left(x_i(Tr)T^{-1/2}\right)\right|^2 e^{-2\alpha\left(T-s_i(Tr)\right)} (Z_{Tr,T}^{x_i(Tr)/T^{1/2}})^2\, dr,$$

for which we could not find a suitable estimate. A simple estimate would be

$$R_t^T(\phi) \leq K T^{-(d/2+1)} \delta E \sup_{0\leq s\leq 1} \langle N_s^T, |\nabla\phi|\rangle^2,$$

for some constant K. From the definition of X^T and (4.9) we obtain

$$\begin{aligned} E \sup_{0\leq s\leq 1} \langle N_s^T, |\nabla\phi|\rangle^2 &\leq 2T^{d/2} E \sup_{0\leq s\leq 1} \langle X_s^T, |\nabla\phi|\rangle^2 + 2 \sup_{0\leq s\leq 1} (E\langle N_s^T, |\nabla\phi|\rangle)^2 \\ &\leq T^{d/2} K_{1,\phi} + T^d K_{2,\phi} . \end{aligned}$$

Then

$$R_t^T(\phi) \leq \delta\, [T^{-1} K'_{1,\phi} + T^{d/2-1} K'_{2,\phi}],$$

which does not allow us to conclude that

$$\lim_{\delta\to 0} \limsup_{T\to\infty} R_t^T(\phi) = 0.$$

for $d > 2$.

REFERENCES

1. D. Aldous. *Stopping times and tightness, II*, Ann. Probab. **17**(1989),586–595.

2. T. Bojdecki and L. G. Gorostiza, *Langevin equations for $\mathcal{S}'$-valued Gaussian processes and fluctuation limits of infinite particle systems*, Probab. Th. Rel. Fields **73**(1986), 227–244.

3. H. Cremers and D. Kadelka, *On weak convergence of integral functions of stochastic processes with applications to processes taking paths in L_p^E*, Stoch. Proc. Appl. **21**(1986), 305–317.

4. D. A. Dawson and L. G. Gorostiza, *Limit theorems for supercritical branching random fields*, Math. Nachr. **118**(1984), 19–46.

5. ———, *Generalized solutions of a class of nuclear space valued stochastic evolution equations*, Appl. Math. Optim. **22**(1990), 241–263.

6. S. N. Ethier and T. G. Kurtz, *Markov Processes. Characterization and Convergence*. Wiley, New York, 1986.

7. B. Fernández and L. G. Gorostiza, *Convergence of generalized semimartingales to a continuous process*. (Preprint).

8. J. P. Fouque, *La convergence en loi pour les processus a valeurs dans un espace nucléaire*, Ann. Inst. H. Poincaré, Sect. B,(3)**22**(1984), 225–245.

9. L. G. Gorostiza, *Limites gaussiennes pour les champs aléatoires ramifies supercritiques*, in "Aspects statistiques et aspects physiques des processus gaussiens", 385–398, Editions du CNRS, Paris, 1981.

10. ———, *Limit theorems for supercritical branching random fields with immigration*, (Tech. Rep. 64, Lab. Res. Stat. Probab. Carleton Univ., 1985), Adv. Appl. Math. **9**(1988), 56–86.

11. L. G. Gorostiza and N. Kaplan, *Invariance principle for branching random motions*, Bol. Soc. Mat. Mex., **25**(1980), 63–86.
12. A. Jakubowski, *On the Skorokhod topology*, Ann. Inst. H. Poincaré, Sect. B(3)**22**(1986), 263–285.
13. J. A. López-Mimbela, *Teoremas límites para campos aleatorios ramificados multitipo*. Doctoral thesis, Departamento de Matemáticas, CINVESTAV, 1989.
14. I. Mitoma, *On the norm continuity of* $\mathcal{S}'$*-valued Gaussian processes*, Nagoya Math. J. **82**(1981), 209–220.
15. ———, *Tightness of probabilities in* $C([0,1],\mathcal{S}')$ *and* $D([0,1],\mathcal{S}')$, Ann. Probab. **11**(1983), 989–999.

Facultad de Ciencias
UNAM
México 04510, D.F.
México

Centro de Investigación y de Estudios Avanzados
México 07300, D.F.
México